

MAT 562: Symplectic Geometry

Solution to Problem D

- (a) Suppose a compact Lie group G acts smoothly on a symplectic manifold (X, ω) . Show that there exists an ω -compatible almost complex structure J on X preserved by G , i.e.

$$dg \circ J = J \circ dg: TX \longrightarrow g^*TX \quad \forall g \in G.$$

- (b) Suppose S^1 acts smoothly on a compact almost complex manifold (X, J) , i.e. preserving J . Let $\xi \in \Gamma(X; TX)$ be the vector field generating this action, i.e.

$$\xi(x) = \left. \frac{d}{dt} (e^{2\pi i t} \cdot x) \right|_{t=0} \in T_x X \quad \forall x \in X.$$

Show that the flow of $-J\xi$ extends this action to a well-defined smooth action of $\mathbb{C}^* \supset S^1$ on X ,

$$\psi: \mathbb{C}^* \times X \longrightarrow X, \quad \psi(e^{2\pi i t}, x) = e^{2\pi i t} \cdot x, \quad \frac{d}{ds} \psi(e^{2\pi(s+it)}, x) = -J\xi(\psi(e^{2\pi(s+it)}, x)),$$

and $u_x \equiv \psi(\cdot, x): \mathbb{C}^* \longrightarrow X$ is a J -holomorphic map for every $x \in X$. If in addition J is integrable, show that this $\mathbb{C}^*$ -action is J -holomorphic.

- (c) Suppose S^1 acts smoothly on a compact symplectic manifold (X, ω) with an associated Hamiltonian $H: X \longrightarrow \mathbb{R}$, i.e. $\iota_\xi \omega = -dH$, where $\xi \in \Gamma(X; TX)$ is the vector field generating the S^1 -action, and $x \in X$. Let J be an S^1 -invariant ω -tamed almost complex structure on X and u_x be as in (b). Show that there exist S^1 -fixed points $x_-, x_+ \in X$ such that

$$\lim_{s \rightarrow \pm\infty} u_x(e^{2\pi(s+it)}) = x_\pm \quad \text{and} \quad E_{g_J^\omega}(u_x) = \int_{\mathbb{C}^*} u_x^* \omega = H(x_+) - H(x_-),$$

where g_J^ω is the metric on X determined by ω and J .

Note. This corrects and sharpens MS Exercise 5.1.5. By (c), u_x extends to a continuous map $\tilde{u}_x: \mathbb{CP}^1 \longrightarrow X$ with bounded energy on $\mathbb{C}^*$. By (b), the restriction of this map to $\mathbb{C}^*$ is J -holomorphic. The Removal of Singularity Theorem (Proposition 4.8 in the *Notes on J-Holomorphic Maps*) then implies that the extension $\tilde{u}_x$ is J -holomorphic as well. Thus, a compact connected symplectic manifold (X, ω) with a non-trivial Hamiltonian S^1 -action contains a non-constant J -holomorphic sphere through every point $x \in X$ and for every ω -tamed almost complex structure J . This provides the motivation for MR2484280, one of the rare established connections between Gromov-Witten invariants and geometric properties of the symplectic manifold.

- (a) By averaging with respect to the Haar measure on G , there exists a Riemannian metric g on X preserved by G . The almost complex structure $J_{g,\omega} \in \mathcal{J}_{\text{cm}}(\omega)$ constructed in the proof of Proposition 1.20 is then also preserved by G .

(b) Since X is compact, the flow of the vector field $-J\xi$ on X ,

$$\psi_s: X \longrightarrow X, \quad \psi_0 = \text{id}_X, \quad \frac{d}{ds}\psi_s = -J\xi \circ \psi_s,$$

is defined for every $s \in \mathbb{R}$. Thus, the map

$$\psi: \mathbb{C}^* \times X \longrightarrow X, \quad \psi(e^{2\pi(s+it)}, x) = \psi_s(e^{2\pi it} \cdot x),$$

is also well-defined and smooth. The resulting $\mathbb{R}$ -action on X commutes with the original S^1 -action if and only if the Lie bracket of the vector fields generating the two actions vanishes. Since the S^1 -action preserves J , $\psi_t^* J = J$, and thus $\mathcal{L}_\xi J = 0$. It follows that

$$[\xi, -J\xi] = -\mathcal{L}_\xi(J\xi) = -(\mathcal{L}_\xi J)\xi - J(\mathcal{L}_\xi \xi) = 0 - J[\xi, \xi] = 0.$$

Thus, the $\mathbb{R}$ - and S^1 -actions indeed commute and thus correspond to an action of $\mathbb{C}^* \approx \mathbb{R} \times S^1$.

Let $x \in X$. Since the S^1 - and $\mathbb{R}$ -actions above are the flows of the vector fields ξ and $-J\xi$, respectively,

$$\frac{\partial}{\partial s} u_x(e^{2\pi(s+it)}) = -J\xi(u_x(e^{2\pi(s+it)})), \quad \frac{\partial}{\partial t} u_x(e^{2\pi(s+it)}) = \xi(u_x(e^{2\pi(s+it)})) = J \frac{\partial}{\partial s} u_x(e^{2\pi(s+it)}). \quad (1)$$

Since $J_{\mathbb{C}} \frac{\partial}{\partial s} e^{2\pi(s+it)} = \frac{\partial}{\partial t} e^{2\pi(s+it)}$, where $J_{\mathbb{C}}$ is the standard (almost) complex structure on $\mathbb{C}$, it follows that the map u_x is J -holomorphic, i.e.

$$du_x \circ J_{\mathbb{C}} = J \circ du_x: T\mathbb{C}^* \longrightarrow u_x^* TX.$$

Suppose J is integrable. Exercise 1.15(b) then gives

$$\mathcal{L}_{-\xi} J J = -J(\mathcal{L}_\xi J) = 0,$$

i.e. the flow of $-\xi J$ preserves J and so does the $\mathbb{R}$ -action. Thus, the entire $\mathbb{C}^*$ -action defined above preserves J , i.e. this action is J -holomorphic.

(c) Since the $\mathbb{R}$ -action is generated by the vector field $-J\xi$,

$$\begin{aligned} \frac{\partial}{\partial s} H(u_x(e^{2\pi(s+it)})) &= d_{u_x(e^{2\pi(s+it)})} H(-J\xi(u_x(e^{2\pi(s+it)}))) = \omega(\xi, J\xi)|_{u_x(e^{2\pi(s+it)})} \\ &= g_J^\omega(\xi, \xi)|_{u_x(e^{2\pi(s+it)})} \geq 0. \end{aligned} \quad (2)$$

Since X is compact, H is bounded above and below. Thus, $H(u_x(e^{2\pi(s+it)}))$ approaches some $H_\pm \in \mathbb{R}$ and $\xi(u_x(e^{2\pi(s+it)})) \longrightarrow 0$ as $s \longrightarrow \pm\infty$. Since H is S^1 -invariant (being a Hamiltonian for the S^1 -action), $H_\pm$ does not depend on t . Furthermore, $u_x(e^{2\pi(s+it)})$ has at least one accumulation point $x_\pm$ as $s \longrightarrow 0$. We need to show that there is precisely one such point, at least for each t fixed, i.e. that the claimed limit actually exists.

If $x_{\pm} \in X$ is the limit of a sequence $u_x(e^{2\pi(s_k+it)}) \in X$ with $s_k \rightarrow \pm\infty$, then $H(x_{\pm}) = H_{\pm}$ and $\xi(x_{\pm}) = 0$. By the last statement, $x_{\pm}$ is a fixed point of the S^1 -action on X . By the Hamiltonian Darboux Theorem, there exist a finite subset $S \subset \mathbb{Z} - \{0\}$, a decomposition

$$T_{x_{\pm}}X = T_{x_{\pm}}X^{S^1} \oplus \bigoplus_{\alpha \in S} \mathcal{N}^{\alpha}$$

of $(T_{x_{\pm}}X, J_{x_{\pm}})$ into complex vector subspaces, a neighborhood $\mathcal{U} \subset T_{x_{\pm}}X$ of 0, a neighborhood $U \subset X$ of $x_{\pm}$, and a diffeomorphism $\Phi: \mathcal{U} \rightarrow U$ so that

$$\begin{aligned} \Phi(0) &= x_{\pm}, \quad d_0\Phi = \text{id}: T_0(T_{x_{\pm}}X) = T_{x_{\pm}}X \rightarrow T_{x_{\pm}}X, \quad \Phi^*\omega = \omega_{x_{\pm}}|_{\mathcal{U}}, \\ \Phi(w_0, (u^{\alpha}w_{\alpha})_{\alpha \in S}) &= u \cdot \Phi(w_0, (w_{\alpha})_{\alpha \in S}), \\ H(\Phi(w_0, (w_{\alpha})_{\alpha \in S})) &= H(x_{\pm}) + \pi \sum_{\alpha \in S} |w_{\alpha}|^2, \quad \forall (w_0, (w_{\alpha})_{\alpha \in S}) \in \mathcal{U} \subset T_{x_{\pm}}X^{S^1} \oplus \bigoplus_{\alpha \in S} \mathcal{N}^{\alpha}, \quad u \in S^1. \end{aligned}$$

If Φ preserved J in addition to ω , the pullback of $-J\xi$ by Φ would be an integer combination of the radial vector fields in the w_{α} coordinates. The accumulation points of the flows of $-J\xi$ would then be unique. While Φ does not preserve J , it comes close enough for the conclusion to hold.

Let $g_J^{\omega}(w, w') \equiv \omega(w, Jw')$ be the Riemannian metric on X determined by ω and J . For $x \in X$ and $w \in T_xX$,

$$g_J^{\omega}(w, -J\xi(x)) = \omega(w, \xi(x)) = d_x H(w).$$

Thus, $-J\xi$ is the gradient of the smooth function H with respect to the metric g_J^{ω} . By the Hamiltonian Darboux Theorem, this gradient satisfies (A.3) with $r=2$ in the Appendix of the notes. By Exercises A.2(b) there, this implies that the accumulation points of the flows of $-J\xi$ are unique, i.e. the claimed limit exists. Since these flows are S^1 -equivariant and $x_{\pm}$ is a fixed point of the S^1 -action, this limit does not depend on $e^{2\pi it}$. This establishes the first claim.

The first equality in the second equation holds by (1.45) in the notes. By (1) and (2),

$$u_x^* \omega = \omega(-J\xi, \xi)|_{u_x(e^{2\pi(s+it)})} ds \wedge dt = \frac{\partial}{\partial s} H(u_x(e^{2\pi(s+it)})) ds \wedge dt.$$

Thus,

$$\begin{aligned} E_{g_J^{\omega}}(u_x) &= \int_{\mathbb{C}^*} u_x^* \omega = \int_0^1 \left(\int_{-\infty}^{\infty} \frac{\partial}{\partial s} H(u_x(e^{2\pi(s+it)})) ds \right) dt = \int_0^1 (H(x_+) - H(x_-)) dt \\ &= H(x_+) - H(x_-). \end{aligned}$$