

MAT 531: Topology&Geometry, II Spring 2011

Solutions to Problem Set 4

Problem 1: Chapter 1, #13ad (10pts)

- (a) Show that $[X, Y]$ is a smooth vector field on M for any two smooth vector fields X and Y on M .
 (d) Show that $[,]$ satisfies the Jacobi identity, i.e.

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for all smooth vector fields on X, Y , and Z on M .

- (a) First, we need to see that for every $p \in M$ the map

$$[X, Y]_p: C^\infty(M) \longrightarrow \mathbb{R}, \quad [X, Y]_p(f) = X_p(Yf) - Y_p(Xf),$$

is well-defined and is an element of T_pM , i.e. it is bilinear, satisfies the product rule, and its value depends only on the germ of f at p . Since X, Y , and f are smooth, Yf and Xf are smooth functions on M by Proposition 1.43. Since X_p and Y_p are linear functionals on $C^\infty(M)$,

$$X_p(Yf), Y_p(Xf) \in \mathbb{R} \quad \implies \quad [X, Y]_p(f) \in \mathbb{R},$$

i.e. the map $[X, Y]_p$ is well-defined. Since it is a composition of linear maps, $[X, Y]_p$ is a linear map as well. Furthermore, if $f, g \in C^\infty(M)$,

$$\begin{aligned} [X, Y]_p(fg) &= X_p(Y(fg)) - Y_p(X(fg)) = X_p(fY(g) + gY(f)) - Y_p(fX(g) + gX(f)) \\ &= (f(p)X_p(Y(g)) + Y_p(g)X_p(f) + g(p)X_p(Y(f)) + Y_p(f)X_p(g)) \\ &\quad - (f(p)Y_p(X(g)) + X_p(g)Y_p(f) + g(p)Y_p(X(f)) + X_p(f)Y_p(g)) \\ &= f(p)(X_p(Y(g)) - Y_p(X(g))) + g(p)(X_p(Y(f)) - Y_p(X(f))) \\ &= f(p)[X, Y]_p(g) + g(p)[X, Y]_p(f), \end{aligned}$$

i.e. the linear map $[X, Y]_p$ satisfies the product rule. Finally, if U is a neighborhood of p in M and $f|_U = g|_U$, then

$$(Xf)|_U = (Xg)|_U \quad \text{and} \quad (Yf)|_U = (Yg)|_U,$$

since for all $q \in U$ the real numbers X_qf and X_qg depend only on the germs of f and g at q . Since the values of X_p and Y_p on $C^\infty(M)$ depend only on the germs of functions at q , we conclude that

$$Y_p(Xf) = Y_p(Xg) \quad \text{and} \quad X_p(Yf) = X_p(Yg) \quad \implies \quad [X, Y]_pf = [X, Y]_pg,$$

i.e. the value of $[X, Y]_p$ on $f \in C^\infty(M)$ depends only on the germ of f at p . Thus, $[X, Y]$ is a vector field on M .

If X, Y , and f are smooth, then Xf and Yf are smooth functions on M by Proposition 1.43. By Proposition 1.43 again, $Y(Xf)$ and $X(Yf)$ are also smooth functions on M . It follows that the function $[X, Y]f$ is smooth for every smooth function f on M . Thus, $[X, Y]$ is a smooth vector

field on M by Proposition 1.43.

(d) We need to show that LHS of the identity is the zero map, i.e. the function obtained by applying LHS to any smooth function f on M is zero. The first summand gives:

$$\begin{aligned} [[X, Y], Z]f &= [X, Y](Zf) - Z([X, Y]f) = (X(Y(Zf)) - Y(X(Zf))) - Z(X(Yf) - Y(Xf)) \\ &= X(Y(Zf)) - Y(X(Zf)) - Z(X(Yf)) + Z(Y(Xf)). \end{aligned}$$

Permuting X , Y , and Z cyclicly, we then obtain

$$\begin{aligned} [[Y, Z], X]f &= Y(Z(Xf)) - Z(Y(Xf)) - X(Y(Zf)) + X(Z(Yf)) \quad \text{and} \\ [[Z, X], Y]f &= Z(X(Yf)) - X(Z(Yf)) - Y(Z(Xf)) + Y(X(Zf)). \end{aligned}$$

The three expressions add up to zero.

Problem 2: Chapter 1, #22 (5pts)

Let $\gamma(t)$ be an integral curve for a vector field X on M . Show that if $\gamma'(t) = 0$ for some t , then γ is a constant map.

Suppose $\gamma: (a, b) \rightarrow M$, $\gamma'(t_0) = 0$ for some $t_0 \in (a, b)$, and $\gamma(t_0) = p$. Since γ is an integral curve for X ,

$$X(p) = X(\gamma(t_0)) = \gamma'(t_0) = 0.$$

Let $\beta: (a, b) \rightarrow M$ be the curve defined by $\beta(t) = p$ for all $t \in (a, b)$. Then,

$$\beta(t_0) = p \quad \text{and} \quad \beta'(t) = 0 = X(p) = X(\beta(t)) \quad \forall t \in (a, b).$$

We also have

$$\gamma(t_0) = p \quad \text{and} \quad \gamma'(t) = X(\gamma(t)) \quad \forall t \in (a, b).$$

By the uniqueness theorem for first-order ODEs, or Theorem 1.48, $\beta = \gamma$, i.e. γ is a constant map.

Problem 3: Chapter 1, #17 (5pts)

Show that any smooth vector field on a compact manifold is complete.

Suppose M is a compact m -manifold, X is a smooth vector field on M , and $\gamma: (a, b) \rightarrow M$ is a maximal integral curve for X . Thus, $a < 0$ and $b > 0$. We need to show that $(a, b) = \mathbb{R}$.

Suppose $b \in \mathbb{R}$. Choose a sequence $t_n \in (a, b)$ converging to b . Since M is a compact, a subsequence converges to a point $p \in M$. By (3) of Theorem 1.48, there exists $\epsilon \in (0, |a|)$ and a neighborhood U of p in M such that the flow

$$(-\epsilon, \epsilon) \times U \rightarrow M, \quad (t, q) \rightarrow X_t(q),$$

is well-defined. Choose $t_n \in (a, b)$ such that $b - t_n < \epsilon$ and $\gamma(t_n) \in U$. Let

$$\beta: (-\epsilon, \epsilon) \rightarrow M$$

be the integral curve for X such that $\beta(0) = \gamma(t_n)$. Define

$$\alpha: (a, t_n + \epsilon) \longrightarrow M \quad \text{by} \quad \alpha(t) = \begin{cases} \gamma(t), & \text{if } t \in (a, b); \\ \beta(t - t_n), & \text{if } t \in (t_n - \epsilon, t_n + \epsilon). \end{cases}$$

Let $\tilde{\gamma}(t) = \gamma(t + t_n)$ for $t \in (-\epsilon, b - t_n)$. Since

$$\tilde{\gamma}(0) = \beta(0), \quad \tilde{\gamma}'(t) = X(\tilde{\gamma}(t)), \quad \text{and} \quad \beta'(t) = X(\beta(t)),$$

$\tilde{\gamma} = \beta$ on $(-\epsilon, b - t_n)$ by the uniqueness of integral curves. Thus, α is well-defined. Furthermore,

$$\alpha'(t) = \begin{cases} \gamma'(t), & \text{if } t \in (a, b) \\ \beta'(t - t_n), & \text{if } t \in (t_n - \epsilon, t_n + \epsilon) \end{cases} = \begin{cases} X(\gamma(t)), & \text{if } t \in (a, b) \\ X(\beta(t - t_n)), & \text{if } t \in (t_n - \epsilon, t_n + \epsilon) \end{cases} = X(\alpha(t)),$$

i.e. α is an integral curve for X . Since $t_n + \epsilon > b$ and $\alpha|_{(a,b)} = \gamma$, we conclude that α is an integral curve for X extending γ . Thus, γ is not maximal unless $b = \infty$. The proof that $a = -\infty$ is similar (or apply the conclusion to the vector field $-X$).

Problem 4 (5pts)

Let V be the vector field on \mathbb{R}^3 given by

$$V(x, y, z) = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Explicitly describe and sketch the flow of V .

The integral curves for this vector fields are the solutions of the system

$$x'(t) = y(t), \quad y'(t) = -x(t), \quad z'(t) = 1.$$

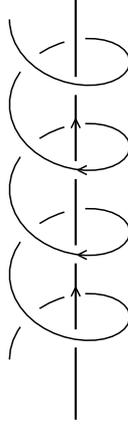
The first pair of equations is independent of the third; its solutions are

$$(x(t), y(t)) = (x_0 + iy_0)e^{-it} \in \mathbb{C}.$$

The corresponding curve goes around a circle centered at the origin *clockwise* at the unit angular speed. The solution of the third equation is $z(t) = z_0 + t$. Thus, the flow for X is given by

$$X_t: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}, \quad X_t(x, y, z) = ((x + iy)e^{-it}, z + t),$$

i.e. the flow rotates clockwise around the vertical axis at the unit angular speed and climbs up at the unit angular speed (the vertical axis itself simply moves up).



Problem 5 (10pts)

Suppose X and Y are smooth vector fields on a manifold M . Show that for every $p \in M$ and $f \in C^\infty(M)$,

$$\lim_{s,t \rightarrow 0} \frac{f(Y_{-s}(X_{-t}(Y_s(X_t(p)))))) - f(p)}{st} = [X, Y]_p f \in \mathbb{R}.$$

Do not forget to explain why the limit exists.

Note: This means that the extent to which the flows $\{X_t\}$ of X and $\{Y_s\}$ of Y do not commute (i.e. the rate of change in the “difference” between $Y_s \circ X_t$ and $X_t \circ Y_s$) is measured by $[X, Y]$.

By (d) of Theorem 1.48, we can choose a neighborhood U of p and $\epsilon > 0$ such that the maps

$$\begin{aligned} (-\epsilon, \epsilon) \times U &\longrightarrow M, & (z, q) &\longrightarrow X_z(q), & (-\epsilon, \epsilon)^2 \times U &\longrightarrow M, & (w, z, q) &\longrightarrow Y_w(X_z(q)), \\ & & & & (-\epsilon, \epsilon)^3 \times U &\longrightarrow M, & (v, w, z, q) &\longrightarrow X_v(Y_w(X_z(q))), & \text{and} \\ & & & & (-\epsilon, \epsilon)^4 \times U &\longrightarrow M, & (u, v, w, z, q) &\longrightarrow Y_u(X_v(Y_w(X_z(q)))) \end{aligned}$$

are defined and smooth. Define

$$K: (-\epsilon, \epsilon)^4 \longrightarrow \mathbb{R} \quad \text{and} \quad H: (-\epsilon, \epsilon)^2 \longrightarrow \mathbb{R} \quad \text{by}$$

$$K(u, v, w, z) = f(Y_u(X_v(Y_w(X_z(p)))))) - f(p) \quad \text{and} \quad H(s, t) = K(-s, -t, s, t).$$

Since K is a composition of smooth functions, K is smooth. Since H is a composition of smooth functions, H is smooth. Furthermore,

$$X_0 = Y_0 = \text{id}_X, \quad Y_{-s} \circ Y_s = \text{id}_{\text{Dom}_s Y}, \quad X_{-t} \circ X_t = \text{id}_{\text{Dom}_t X} \quad \implies \quad H(s, 0) = H(0, t) = 0$$

for all s and t . Thus¹, the limit in the statement of the problem exists and equals to the mixed second partial derivative of H :

$$\lim_{s,t \rightarrow 0} \frac{f(Y_{-s}(X_{-t}(Y_s(X_t(p)))))) - f(p)}{st} = \lim_{s,t \rightarrow 0} \frac{H(s, t) - H(0, 0)}{st} = \frac{\partial^2 H}{\partial s \partial t} \Big|_{(0,0)}.$$

¹this follows, for example, from Lemma 3.5 in *Lecture Notes* applied twice

On the other hand, by the Chain Rule,

$$\frac{\partial^2 H}{\partial s \partial t} \Big|_{(0,0)} = \frac{\partial^2 K}{\partial u \partial v} \Big|_{(0,0,0,0)} - \frac{\partial^2 K}{\partial u \partial z} \Big|_{(0,0,0,0)} - \frac{\partial^2 K}{\partial v \partial w} \Big|_{(0,0,0,0)} + \frac{\partial^2 K}{\partial w \partial z} \Big|_{(0,0,0,0)}.$$

Note that

$$\begin{aligned} \frac{\partial^2 K}{\partial u \partial v} \Big|_{(0,0,0,0)} &= \frac{\partial}{\partial v} \left(\frac{\partial}{\partial u} f(Y_u(X_v(p))) \Big|_{u=0} \right) \Big|_{v=0} = \frac{\partial}{\partial v} \left(d_{X_v(p)} f \left(\frac{\partial}{\partial u} Y_u(X_v(p)) \Big|_{u=0} \right) \right) \Big|_{v=0} \\ &= \frac{\partial}{\partial v} \left(d_{X_v(p)} f(Y) \right) \Big|_{v=0} = \frac{\partial}{\partial v} \left(\{Yf\}(X_v(p)) \right) \Big|_{v=0} = d_p \{Yf\}(X) = X_p(Yf). \end{aligned}$$

Similarly,

$$\frac{\partial^2 K}{\partial u \partial z} \Big|_{(0,0,0,0)} = X_p(Yf), \quad \frac{\partial^2 K}{\partial v \partial w} \Big|_{(0,0,0,0)} = Y_p(Xf), \quad \frac{\partial^2 K}{\partial w \partial z} \Big|_{(0,0,0,0)} = X_p(Yf).$$

Putting these together, we conclude that

$$\lim_{s,t \rightarrow 0} \frac{f(Y_{-s}(X_{-t}(Y_s(X_t(p)))))) - f(p)}{st} = \frac{\partial^2 H}{\partial s \partial t} \Big|_{(0,0)} = X_p(Yf) - Y_p(Xf) = [X, Y]_p f.$$

Problem 6 (10pts)

Let U and V be the vector fields on \mathbb{R}^3 given by

$$U(x, y, z) = \frac{\partial}{\partial x} \quad \text{and} \quad V(x, y, z) = F(x, y, z) \frac{\partial}{\partial y} + G(x, y, z) \frac{\partial}{\partial z},$$

where F and G are smooth functions on \mathbb{R}^3 . Show that there exists a proper² foliation of \mathbb{R}^3 by 2-dimensional embedded submanifolds such that the vector fields U and V everywhere span the tangent spaces of these submanifolds if and only if

$$F(x, y, z) = f(y, z) e^{h(x,y,z)} \quad \text{and} \quad G(x, y, z) = g(y, z) e^{h(x,y,z)}$$

for some $f, g \in C^\infty(\mathbb{R}^2)$ and $h \in C^\infty(\mathbb{R}^3)$ such that (f, g) does not vanish on \mathbb{R}^2 .

If at every point of \mathbb{R}^3 the vector fields U and V span the tangent space of a 2-dimensional submanifold, then their span is two-dimensional, i.e. (F, G) does not vanish. If this is the case, by Frobenius Theorem there exists an integral submanifold for the distribution $\mathcal{D} \subset T\mathbb{R}^3$ spanned by U and V through every point of \mathbb{R}^3 if and only if the vector field

$$[U, V] = F_x \frac{\partial}{\partial y} + G_x \frac{\partial}{\partial z}$$

lies in the span of U and V over $C^\infty(\mathbb{R}^3)$. This is the case if and only if there exists $\lambda \in C^\infty(\mathbb{R}^3)$ such that

$$\begin{aligned} [U, V] = \lambda V &\iff F_x = \lambda F, \quad G_x = \lambda G \\ &\iff F(x, y, z) = f(y, z) e^{h(x,y,z)}, \quad G(x, y, z) = g(y, z) e^{h(x,y,z)}, \end{aligned}$$

²in the sense of Definition 10.4 in *Lecture Notes*

where $h \in C^\infty(\mathbb{R}^3)$ is such that $h_x = \lambda$ and $f, g \in C^\infty(\mathbb{R}^2)$ are such that (f, g) does not vanish on \mathbb{R}^2 (so that V does not vanish).

If the above is the case, the maximal connected integral submanifolds for the distribution \mathcal{D} spanned by U and V partition \mathbb{R}^3 . We will show that all such submanifolds are embedded. Since e^h does not vanish, \mathcal{D} is spanned by the vector fields

$$U(x, y, z) = \frac{\partial}{\partial x} \quad \text{and} \quad W(x, y, z) = f(y, z) \frac{\partial}{\partial y} + g(y, z) \frac{\partial}{\partial z}.$$

Let $\gamma: (a, b) \rightarrow \mathbb{R}^2$ be a maximal integral curve for the vector field

$$\widetilde{W}(y, z) = f(y, z) \frac{\partial}{\partial y} + g(y, z) \frac{\partial}{\partial z}.$$

Since (f, g) does not vanish on \mathbb{R}^2 and $\gamma'(t) = \widetilde{W}(\gamma(t))$ γ is a maximal connected integral submanifold for the distribution $\widetilde{\mathcal{D}}$ on \mathbb{R}^2 spanned by \widetilde{W} . Furthermore,

$$\psi = \text{id} \times \gamma: \mathbb{R} \times (a, b) \rightarrow \mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$$

is a maximal connected integral submanifold for \mathcal{D} and every maximal connected submanifold for \mathcal{D} has this form. It is an integral submanifold for \mathcal{D} because

$$\begin{aligned} d\psi|_{(s,t)} \frac{\partial}{\partial s} &= \frac{\partial}{\partial s} \times 0 = U(\psi(s, t)), \\ d\psi|_{(s,t)} \frac{\partial}{\partial t} &= (0, \gamma'(t)) = (0, \widetilde{W}(\gamma(t))) = W(\psi(s, t)), \end{aligned}$$

if (s, t) are the standard coordinates on $\mathbb{R} \times (a, b)$. Since the maximal integral curves γ for \widetilde{W} partition \mathbb{R}^2 , the images of the maps $\text{id} \times \gamma$ partition \mathbb{R}^3 . Thus, each map $\text{id} \times \gamma$ must be a maximal connected submanifold for \mathcal{D} . In the following paragraph, we show that every integral curve γ for \widetilde{W} must be embedded in \mathbb{R}^2 . This implies that every maximal connected integral submanifold $\text{id} \times \gamma$ for \mathcal{D} is embedded in \mathbb{R}^3 .

Suppose $\gamma: (a, b) \rightarrow \mathbb{R}^2$ is a maximal integral curve for \widetilde{W} and $t_0 \in (a, b)$. By Proposition 1.53, we can choose a coordinate chart

$$\varphi = (x_1, x_2): (\mathcal{U}, \gamma(t_0)) \rightarrow (\mathbb{R}^2, 0)$$

and a neighborhood (c, d) of t_0 in (a, b) such that

$$[-2, 2] \times [-2, 2] \subset \varphi(\mathcal{U}), \quad \widetilde{W}|_{\mathcal{U}} = \frac{\partial}{\partial x_1}|_{\mathcal{U}}, \quad \text{and} \quad \gamma|_{(c,d)}: (c, d) \rightarrow \varphi^{-1}((-2, 2) \times 0) \quad (1)$$

is a diffeomorphism. The middle condition implies that

$$\text{Im } \gamma \cap \varphi^{-1}((-2, 2) \times (-2, 2))$$

is a union of horizontal slices $\varphi^{-1}((-2, 2) \times y)$ with $y \in S_\gamma$, where S_γ is a subset of $(-2, 2)$. To show that γ is an embedding, we show that there exists $\epsilon > 0$ such that

$$S_\gamma \cap (-\epsilon, \epsilon) = \{0\}.$$
³

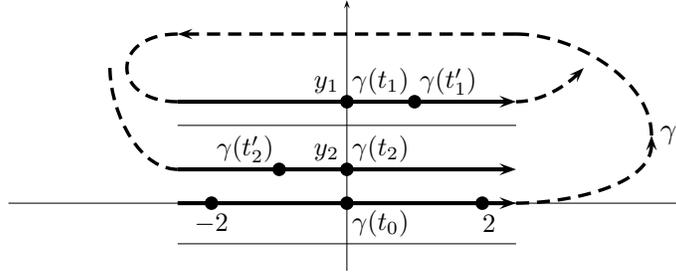
³In such a case, if $(c', d') \subset (c, d)$ is a basis element around $t_0 \in (a, b)$, then

$$\gamma((c', d')) = (\gamma(c'), \gamma(d')) \times 0 = \gamma((c', d')) \cap ((\gamma(c'), \gamma(d')) \times (-\epsilon, \epsilon)),$$

Suppose not, i.e. there exists a sequence $t_k \in (a, b)$ converging to either a or b such that $\gamma(t_k) = 0 \times y_k$ with $y_k \in S_\gamma$ converging to $0 \in \mathbb{R}$. By taking a subsequence and by symmetry, it is sufficient to assume that $t_k \rightarrow b$ and $y_k \in \mathbb{R}^+$. We can then choose $t_1, t_2 \in (t_0, b)$ with $t_1 < t_2$ so that

$$\begin{aligned} \varphi(\gamma(t_1)) = 0 \times y_1, \quad \varphi(\gamma(t_2)) = 0 \times y_2 \quad \text{s.t.} \\ 0 < y_2 < y_1 < 2, \quad (t_0, t_1) \cap \gamma^{-1}(\varphi^{-1}(0 \times (0, y_1))) = \emptyset, \quad (t_1, t_2) \cap \gamma^{-1}(\varphi^{-1}(0 \times (0, y_1))) = \emptyset. \end{aligned}$$

In other words, $t_1 \in (t_0, b)$ is the smallest number so that $\varphi(\gamma(t_1)) \in 0 \times (0, 2)$ and $t_2 \in (t_1, b)$ is the smallest number so that $\varphi(\gamma(t_2)) \in 0 \times (0, y_1)$.



By the middle condition in eq1,

$$t'_1 \equiv \gamma^{-1}(\varphi^{-1}(1 \times y_1)) \in (t_1, t_2), \quad t'_2 \equiv \gamma^{-1}(\varphi^{-1}((-1) \times y_2)) \in (t'_1, t_2),$$

i.e. the flow is to the right in each slice as indicated in the diagram.

Since γ is an integral curve, it has no self-intersections. Thus,

$$\mathcal{C} \equiv \gamma((t_0, t_1)) \cup \varphi^{-1}(0 \times [0, y_1])$$

is a simple closed curve in \mathbb{R}^2 since

$$(t_0, t_1) \cap \gamma^{-1}(\varphi^{-1}(0 \times (0, y_1))) = \emptyset.$$

Let ℓ be the straight line segment between $\varphi(\gamma(t'_1))$ and $\varphi(\gamma(t'_2))$ in \mathbb{R}^2 . Since the curve $\varphi^{-1}(\ell)$ intersects the simple closed curve \mathcal{C} exactly once, one of its endpoints, i.e. $\gamma(t'_1)$ or $\gamma(t'_2)$, must lie inside of \mathcal{C} and the other outside. Thus, the curve $\gamma((t'_1, t'_2))$ must intersect \mathcal{C} at least once. Since

$$\gamma((t'_1, t'_2)) \cap \varphi^{-1}(0 \times [0, y_1]) \subset \gamma((t_1, t_2)) \cap \varphi^{-1}(0 \times [0, y_1]) = \emptyset$$

and $\gamma((t'_1, t'_2)) \cap \mathcal{C} \neq \emptyset$, we conclude that

$$\gamma((t'_1, t'_2)) \cap \gamma((t_0, t_1)) \neq \emptyset.$$

However, this is impossible as well, since $t_1 < t'_1$ and an integral curve cannot intersect itself.

Remark: The above argument implies that if \mathcal{D} is any distribution on \mathbb{R}^2 , every connected integral submanifold for \mathcal{D} is embedded. This is not the case for other manifolds, including \mathbb{R}^3 and T^2 (see Chapter 1, #21, p51).

i.e. γ takes open subsets of (a, b) to sets in $\text{Im } \gamma$ that are open with respect to the topology $\text{Im } \gamma$ inherits as a subspace of \mathbb{R}^2 . So, γ is an embedding.