MAT 401: Undergraduate Seminar
Introduction to Enumerative Geometry
Fall 2008

Homework Assignment III

Reminder: Your grade in this class will be based on class participation and the problem sets. Therefore, you are expected to regularly attend classes and contribute to the discussion (with presentations, questions, and comments), as well as to submit solutions to the written assignments and study the discussion problems ahead of time even if you won't be presenting them. Written solutions must be turned in by the beginning of class on the due date (typed solutions can also be e-mailed by the same time). **Late problem sets will not be accepted**.

Written Assignment due on Thursday, 10/2, at 11:20am in Physics P-117
(or by 10/2, 11am, in Math 3-111)

Please do 6 of the following problems: Chapter 4, #1,2,3,4, and A, B, C below, with A, B, and C counted as 2 problems each.

Problem A

Let $U\subset\mathbb{C}^4\times\mathbb{C}^4$ be the subspace consisting of pairs of linearly independent vectors and $A\subset U$ of pairs of vectors that are orthonormal with respect to the standard hermitian inner-product on $\mathbb{C}^4$. Thus, each element of U and A determines a two-dimensional linear subspace of $\mathbb{C}^4$; this induces surjective maps

$$\pi: U\longrightarrow G(2,4), \qquad \pi': A\longrightarrow G(2,4).$$

Show that these maps induce the same topology on $G(2,4)$.

Problem B

(a) For each $i=0,1,\ldots,n$, let

$$U_i=\big\{[X_0,X_1,\ldots,X_n]\in\mathbb{C}P^n:\ X_i\neq 0\big\},$$
$$\phi_i: U_i\longrightarrow\mathbb{C}^n, \quad [X_0,X_1,\ldots,X_n]\longrightarrow (X_0/X_i,X_1/X_i,\ldots,X_{i-1}/X_i,X_{i+1}/X_i,\ldots,X_n/X_i).$$

Show that the maps ϕ_i are homeomorphisms, while

$$\phi_i\circ\phi_j^{-1}: \phi_j(U_i\cap U_j)\longrightarrow\phi_i(U_i\cap U_j)$$

are analytic maps between open subspaces of $\mathbb{C}^n$.
(b) Describe manifold charts for $G(2,4)$ (for the quotient topology of Problem A) and show that $G(2,4)$ is also a complex manifold (the overlap maps are analytic).

Problem C

Let $F = F(X_0, \ldots, X_n)$ be a homogeneous polynomial of degree $d \in \mathbb{Z}^+$ and $(Y_0, \ldots, Y_n) \in \mathbb{C}^{n+1} - 0$. If $F(Y_0, \ldots, Y_n) = 0$, but $\frac{\partial F}{\partial X_i}|_{(Y_0,\ldots,Y_n)} \neq 0$ for some $i = 0, 1, \ldots, n$, show that the hypersurface $Z(F) \subset \mathbb{C}P^n$ is a complex manifold in a neighborhood of $[Y_0, \ldots, Y_n] \in Z(F)$. Show that the tangent hyperplane to $Z(F)$ at $[Y_0, \ldots, Y_n]$ is given by the equation

$$G(X_0, \ldots, X_n) = \frac{\partial F}{\partial X_0}\bigg|_{(Y_0,\ldots,Y_n)} X_0 + \ldots + \frac{\partial F}{\partial X_n}\bigg|_{(Y_0,\ldots,Y_n)} X_n = 0.$$

Discussion Problem for 10/2

(More on Bezout's Theorem for $\mathbb{C}P^2$)

Bezout's Theorem for $\mathbb{C}P^2$: If $C, D \subset \mathbb{C}P^2$ are curves of degrees $c, d \in \mathbb{Z}^+$ such that $C \cap D$ is finite, then the cardinality of the set $C \cap D$ counted with multiplicity $m_p(C, D) \in \mathbb{Z}^+$ for each point $p \in C \cap D$ is cd.

The number $m_p(C, D)$ is defined so that if the curves C and D are deformed slightly and generically (by deforming the homogeneous polynomials defining C and D), then $m_p(C, D)$ is the number of points in the intersection of the deformed curves that lie near p. Thus, the weighted cardinality of $C \cap D$ does not change under small changes in (C, D). It thus must be independent of C and D provided the space

$$\mathfrak{X}_{c,d} \equiv \big\{(F, G) \in HP_c(\mathbb{C}^3) \times HP_d(\mathbb{C}^3) \colon Z(F) \cap Z(G) \text{ is finite}\big\}$$

is connected (as in class $HP_c(\mathbb{C}^3)$ is the space of homogeneous polynomials on $\mathbb{C}^3$ of degree c). We can thus determined the weighted cardinality of $C \cap D$ by determining it for a specific pair in $\mathfrak{X}_{c,d}$; in class, C and D were taken to consist of c and d lines, respectively, with all $c+d$ lines being distinct, obtaining Bezout's Theorem. The aim of this discussion problem is to fill in some of the gaps in the argument.

Part I ($\sim$15mins): A topological space $\mathfrak{X}$ is called connected if it can't be written as a disjoint union of two nonempty open subset, $\mathfrak{X} \neq U \sqcup V$; $\mathfrak{X}$ is called path-connected if for any $p, q \in \mathfrak{X}$ there exists a continuous map $f : [0, 1] \longrightarrow \mathfrak{X}$ such that $f(0) = p$ and $f(1) = q$ (thus every two points in $\mathfrak{X}$ are connected by a path). Show that
(a) any continuous map from a connected space to $\mathbb{Z}$ is constant;
(b) any path-connected space is connected and thus $\mathbb{C}^n$ is connected;
(c) if $A \subset \mathfrak{X}$ is connected (in the subspace topology), so is $\bar{A} \subset \mathfrak{X}$.

Part II ($\sim$30mins): Show that
(a) if $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ is an analytic function (analytic in each variable), then for every $p \in f^{-1}(0)$ there exists $r > 0$ such that $B_r(p) - f^{-1}(0)$, where $B_r(p)$ is the r-ball centered at p, is path-connected;
(b) if $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ is an analytic function, then $\mathbb{C}^n - f^{-1}(0)$ is path-connected;
(c) $\mathfrak{X}_{c,d}$ is connected.

Part III ($\sim$25mins): Let $f, g : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be two polynomials of degrees at most c and d (not necessarily homogeneous) such that $f(0), g(0) = 0$ and there exists $r > 0$ such that $B_r(0) - 0$ contains no points

$f^{-1}(0)\cap g^{-1}(0)$. For $p\in\mathbb{C}^2$, let $\nabla f|_p\colon \mathbb{C}^2\longrightarrow\mathbb{C}$ be the gradient of f at p. If $c\in\mathbb{Z}^+$, denote by $H_c(\mathbb{C}^2)$ the space of polynomials on $\mathbb{C}^2$ of degree at most c; there is a natural norm on $H_c(\mathbb{C}^2)$, since such polynomials correspond to tuples of elements of $\mathbb{C}$. Show that
(a) if $(\ker\nabla f|_0)\cap(\ker\nabla g|_0)=\{0\}$, there exists $\epsilon>0$ such that the set

$$\{f+\tilde{f}\}^{-1}(0)\cap\{g+\tilde{g}\}^{-1}(0)\cap B_{r/2}(0)\subset\mathbb{C}^2$$

consists of precisely one element whenever $|\tilde{f}|,|\tilde{g}|<\epsilon$ (*in such a case,* 0 *is said to be a simple intersection point, or point of intersection multiplicity* 1*, of* $f^{-1}(0)$ *and* $g^{-1}(0)$);
(b) the set

$$\mathfrak{X}_{c,d}(f,g)\equiv\Big\{(\tilde{f},\tilde{g})\in P_c(\mathbb{C}^2)\times P_d(\mathbb{C}^2)\colon\ Z(f+\tilde{f})\cap Z(g+\tilde{g}) \text{ is finite};$$
$$(\ker\nabla(f+\tilde{f})|_p)\cap(\ker\nabla(g+\tilde{g})|_p)=\{0\}\ \forall\, p\in Z(f+\tilde{f})\cap Z(g+\tilde{g})\Big\}$$

is connected and non-empty;
(c) there exists $\epsilon>0$ such that the cardinality of the set

$$\{f+\tilde{f}\}^{-1}(0)\cap\{g+\tilde{g}\}^{-1}(0)\cap B_{r/2}(0)\subset\mathbb{C}^2$$

is independent of $\tilde{f},\tilde{g}\in\mathfrak{X}_{c,d}(f,g)$ with $|\tilde{f}|,|\tilde{g}|<\epsilon$. (*This number is called the multiplicity of* $0\in\mathbb{C}^2$ *as an intersection point of the curves* $Z(f)$ *and* $Z(g)$ *and is denoted by* $m_0(f,g)$ *or* $m_0(Z(f),Z(g))$).