

MAT 401: Undergraduate Seminar
Introduction to Enumerative Geometry
Fall 2018

Homework Assignment VI

Written Assignment due on Tuesday, 12/4, at 1pm in ESS 181

Problem L

Let $F = F(X, Y, Z)$ be a homogeneous polynomial of degree 2 which is not a product of linear factors. Thus,

$$C \equiv Z(F) \equiv \{[X, Y, Z] \in \mathbb{P}^2 : F(X, Y, Z) = 0\}$$

is a smooth curve of degree 2 in \mathbb{P}^2 . Show that there are homogeneous polynomials $P_i = P_i(u, v)$ of degree 2 so that the image of the map

$$f: \mathbb{P}^1 \longrightarrow \mathbb{P}^2, \quad [u, v] \longrightarrow [P_0(u, v), P_1(u, v), P_2(u, v)]$$

is the curve C .

There are at least two ways of going about this. The homogeneous polynomials P_0, P_1, P_2 are determined by 3 coefficients each; the homogeneous polynomial F is given by 6 coefficients. The requirement $f(\mathbb{P}^1) \subset Z(F)$ is equivalent to

$$F(P_0(u, v), P_1(u, v), P_2(u, v)) = 0 \quad \forall u, v.$$

The left-hand side of this equation is a homogeneous polynomial of degree $2 \cdot 2$ in u and v . Collecting the coefficients of the various terms $u^a v^{4-a}$, one obtains 5 equations in 9 unknowns. The extra $9-5$ degrees of freedom correspond to the fact that if P_0, P_1, P_2 work, so do the polynomials $P_i(au+bv, cu+dv)$ for any fixed $a, b, c, d \in \mathbb{C}$. This approach is direct, but would be very messy.

Here is another approach. It is based on the following observation. Let $M \in \text{GL}_3\mathbb{C}$ be an invertible 3×3 -matrix; it determines a bijective linear map $M: \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ and induces a bijective map

$$\bar{M}: \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \quad [v] \longrightarrow [Mv].$$

If $F = F(X, Y, Z)$ is homogeneous polynomial of degree 2, then so is $F \circ M$. Furthermore, F does not split into linear factors if and only if $F \circ M$ does not (you can prove this either directly or using the approach of Chapter 2, #8). If $Z(F) = f(\mathbb{P}^1)$, then $Z(F \circ M) = \{\bar{M}^{-1} \circ f\}(\mathbb{P}^1)$, and $\bar{M}^{-1} \circ f$ is given by the polynomials $M^{-1}(P_0 P_1 P_2)^{tr}$. Thus, it is sufficient to prove the statement with F replaced by $F \circ M$ for some $M \in \text{GL}_3\mathbb{C}$, perhaps repeating the replacement process several times.

For example, if $F(X, Y, Z) = X^2 + Y^2 + Z^2$, we could take

$$M = \begin{pmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & i \end{pmatrix}^{-1}$$

This replaces $X+iY$ with X , $X-iY$ with Y , and Z with iZ , so that F is replaced with $XY-Z^2$. So it is sufficient to do the following steps.

(a) Find f as above that works for $F(X, Y, Z) = XY - Z^2$.

(b) If F does not split into linear factors, show that there exists $M \in GL_n \mathbb{C}$ so that $F \circ M$ is $X^2 + Y^2 + Z^2$ or $XY - Z^2$.

Discussion Problems for 12/4

Counting plane rational curves

Please read the attached note, *even if you are not presenting*, and make sure to actively participate in the discussion, with questions or comments.

If you are presenting,

(1) State formula (1), recalling what n_d is.

(2) Describe how you are going to prove it; this is essentially Sections 1 and 2.

(3) Prove the formula; this is Section 3 plus you need to derive formula (1) from (6). If time permits, use (1) to compute a few of the numbers n_d . What is the analogue of this for \mathbb{P}^3 ?

Please draw pictures, more of them than in the note, and do not just copy the formulas!

Some of this material is related to some of the material in Chapter 3 of the book.

Please prepare your presentation ahead of time so that it fits in 1 hour and 10 minutes.

Counting Plane Rational Curves: a modern approach

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Enumerative geometry of algebraic varieties is a field of mathematics that dates back to the nineteenth century. The general goal of this subject is to determine the number of geometric objects that satisfy pre-specified geometric conditions. The objects are typically (complex) curves in a smooth algebraic manifold. Such curves are usually required to represent the given homology class, to have certain singularities, and to satisfy various contact conditions with respect to a collection of subvarieties. One of the most well-known examples of an enumerative problem is

Question 1 *If d is a positive integer, what is the number n_d of degree d rational curves that pass through $3d-1$ points in general position in the complex projective plane \mathbb{P}^2 ?*

Since the number of (complex) lines through any two distinct points is one, $n_1 = 1$. A little bit of algebraic geometry and topology gives $n_2 = 1$ and $n_3 = 12$. It is far harder to find that $n_4 = 620$, but this number was computed as early as the middle of the nineteenth century; see [5, p378].

The higher-degree numbers n_d remained unknown until the early 1990s, when a recursive formula for the numbers n_d was announced in [2] and [4]:

$$n_d = \frac{1}{6(d-1)} \sum_{d_1+d_2=d} \left(d_1 d_2 - 2 \frac{(d_1 - d_2)^2}{3d-2} \right) \binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}. \quad (1)$$

The argument of the latter paper is described below. It can also be used to solve the natural generalization of Question 1 to the higher-dimensional projective spaces; see [4, Section 10].

We will define an invariant that counts holomorphic maps into \mathbb{P}^2 . A priori, the number we describe depends on the cross ratio of the chosen four points on a sphere. However, it turns out that this number is well-defined. We use its independence to express this invariant in terms of the numbers n_d in two different ways. By comparing the two expressions, we obtain (1).

1 The moduli space of four marked points on a sphere

Let x_0, x_1, x_2 and x_3 be the four points in \mathbb{P}^2 given by

$$x_0 = [1, 0, 0], \quad x_1 = [0, 1, 0], \quad x_2 = [0, 0, 1], \quad x_3 = [1, 1, 1].$$

We denote by $H^0(\mathbb{P}^2; \gamma^{*\otimes 2})$ the space of holomorphic sections of the holomorphic line bundle $\gamma^{*\otimes 2} \rightarrow \mathbb{P}^2$, or equivalently of the degree 2 homogeneous polynomials in three variables. Let

$$\begin{aligned} \mathcal{U} &= \{([s], x) \in \mathbb{P}H^0(\mathbb{P}^2; \gamma^{*\otimes 2}) \times \mathbb{P}^2 : s(x_i) = 0 \ \forall i = 0, 1, 2, 3; \ s(x) = 0\} \\ &\approx \{([A, B]; [z_0, z_1, z_2]) \in \mathbb{P}^1 \times \mathbb{P}^2 : (A-B)z_0z_1 - Az_1z_2 + Bz_0z_2 = 0\}. \end{aligned}$$

The space \mathcal{U} is a compact complex manifold of dimension 2.

Let $\pi : \mathcal{U} \rightarrow \overline{\mathcal{M}}_{0,4} \equiv \mathbb{P}^1$ denote the projection onto the first component. If $[A, B] \in \overline{\mathcal{M}}_{0,4}$, the fiber $\pi^{-1}([A, B])$ is the conic

$$\mathcal{C}_{A,B} = \{[z_0, z_1, z_2] \in \mathbb{P}^2 : (A-B)z_0z_1 - Az_1z_2 + Bz_0z_2 = 0\}.$$

If $[A, B] \neq [1, 0], [0, 1], [1, 1]$, $\mathcal{C}_{A,B}$ is a smooth complex curve of genus zero; it is a sphere with four distinct marked points. If $[A, B] = [1, 0], [0, 1], [1, 1]$, $\mathcal{C}_{A,B}$ is a union of two lines. One of the lines contains two of the four points x_0, \dots, x_3 , and the other line passes through the remaining two points. The two lines intersect in a single point. Figure 1 shows the three singular fibers of the projection map $\pi : \mathcal{U} \rightarrow \overline{\mathcal{M}}_{0,4}$. The other fibers are smooth conics. The fibers should be viewed as lying in planes orthogonal to the horizontal line in the figure.

The following remarks concerning the family $\mathcal{U} \rightarrow \overline{\mathcal{M}}_{0,4}$ are not directly relevant for the purposes of the next two sections and can be omitted. If $[A, B] \in \overline{\mathcal{M}}_{0,4} - \{[1, 0], [0, 1], [1, 1]\}$, $\mathcal{C}_{A,B}$ is a smooth complex curve of genus zero, i.e. it is a sphere holomorphically embedded in \mathbb{P}^2 . Thus, there exists a one-to-one holomorphic map $f : \mathbb{P}^1 \rightarrow \mathcal{C}_{A,B}$. It can be shown directly that if $[u_i, v_i] = f^{-1}(x_i)$,

$$\frac{v_0/u_0 - v_2/u_2}{v_0/u_0 - v_3/u_3} : \frac{v_1/u_1 - v_2/u_2}{v_1/u_1 - v_3/u_3} = \frac{B}{A}.$$

The cross-ratio is the only invariant of four distinct points on \mathbb{P}^1 ; see [1], for example. Thus,

$$\begin{aligned} \mathbb{P}^1 - \{[1, 0], [0, 1], [1, 1]\} &= \mathcal{M}_{0,4} \equiv \{(x_0, x_1, x_2, x_3) \in (\mathbb{P}^1)^4 : x_i \neq x_j \text{ if } i \neq j\} / \sim, \\ \text{where } (x_0, x_1, x_2, x_3) &\sim (\tau(x_0), \tau(x_1), \tau(x_2), \tau(x_3)) \quad \text{if } \tau \in \text{PSL}_2 \equiv \text{Aut}(\mathbb{P}^1). \end{aligned}$$

Furthermore, the restriction of the projection map $\pi : \mathcal{U}|_{\mathcal{M}_{0,4}} \rightarrow \mathcal{M}_{0,4}$ to each fiber $\mathcal{C}_{[A,B]}$ is the cross ratio of the points x_0, \dots, x_3 on $\mathcal{C}_{[A,B]}$, viewed as an element of $\mathbb{P}^1 \supset \mathbb{C}$.

2 Counts of holomorphic maps

If d is an integer and \mathcal{C} is a complex curve, which may be a wedge of spheres, let

$$\mathcal{H}_d(\mathcal{C}) = \{f \in C^\infty(\mathcal{C}; \mathbb{P}^2) : f \text{ is holomorphic, } \deg f = d\}. \quad (2)$$

We give a more explicit description of the space $\mathcal{H}_d(\mathcal{C})$ in the relevant cases below.

Suppose ℓ_0, ℓ_1 and p_2, \dots, p_{3d-1} are two lines and $3d-2$ points in general position in \mathbb{P}^2 . If $\sigma \in \overline{\mathcal{M}}_{0,4}$, let $N_d^\sigma(\ell_0, \ell_1, p_2, \dots, p_{3d-1})$ denote the cardinality of the set

$$\{f \in \mathcal{H}_d(\mathcal{C}_\sigma) : f(x_0) \in \ell_0, f(x_1) \in \ell_1, f(x_2) = p_2, f(x_3) = p_3, p_i \in \text{Im } f \ \forall i\}. \quad (3)$$

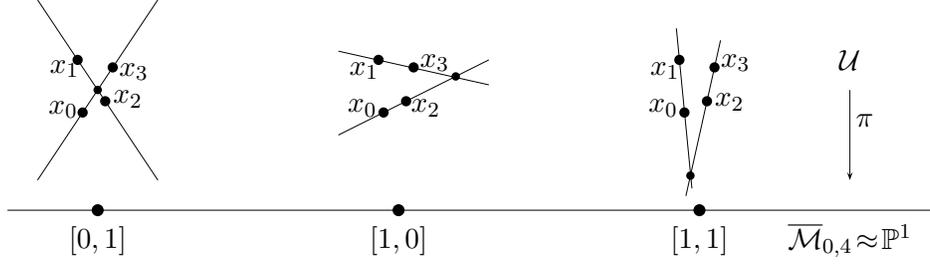


Figure 1: The Family $\mathcal{U} \rightarrow \overline{\mathcal{M}}_{0,4}$

Here \mathcal{C}_σ denotes the rational curve with four marked points, x_0, x_1, x_2 , and x_3 , whose cross ratio is σ ; see Section 1. If $\sigma \neq [1, 0], [0, 1], [1, 1]$, \mathcal{C}_σ is a sphere with four, distinct, marked points. In this case, the condition $f \in \mathcal{H}_d(\mathcal{C}_\sigma)$ means that f has the form

$$f([u, v]) = [P_0(u, v), P_1(u, v), P_2(u, v)] \quad \forall [u, v] \in \mathbb{P}^1,$$

for some degree d homogeneous polynomials P_0, P_1, P_2 that have no common factor. If $\sigma = [1, 0], [0, 1], [1, 1]$, \mathcal{C}_σ is a wedge of two spheres, $\mathcal{C}_{\sigma,1}$ and $\mathcal{C}_{\sigma,2}$, with two marked points each. In this case, the first condition in (2) means that f is continuous and $f|_{\mathcal{C}_{\sigma,1}}$ and $f|_{\mathcal{C}_{\sigma,2}}$ are holomorphic. The second condition in (2) means that $d = d_1 + d_2$ if the degrees of $f|_{\mathcal{C}_{\sigma,1}}$ and $f|_{\mathcal{C}_{\sigma,2}}$ are d_1 and d_2 , respectively.

The requirement that the two lines, ℓ_0 and ℓ_1 , and the $3d-2$ points, p_2, \dots, p_{3d-1} , are in general position means that they lie in a dense open subset \mathcal{U}_σ of the space of all possible tuples $(\ell_0, \ell_1, p_2, \dots, p_{3d-1})$:

$$\mathfrak{X} \equiv \text{Gr}_2\mathbb{C}^3 \times \text{Gr}_2\mathbb{C}^3 \times (\mathbb{P}^2)^{3d-2}.$$

Here $\text{Gr}_2\mathbb{C}^3$ denotes the Grassmanian manifold of two-planes through the origin in \mathbb{C}^3 , or equivalently of lines in \mathbb{P}^2 . The dense open subset \mathcal{U}_σ of \mathfrak{X} consists of tuples $(\ell_0, \ell_1, p_2, \dots, p_{3d-1})$ that satisfy a number of geometric conditions. In particular, $\ell_0 \neq \ell_1$, none of the points p_2, \dots, p_{3d-1} lies on either ℓ_0 or ℓ_1 , the $3d-1$ points $\ell_0 \cap \ell_1, p_2, \dots, p_{3d-1}$ are distinct, no three of them lie on the same line, and so on. In addition, we need to impose certain cross-ratio conditions on the rational curves that pass through ℓ_0, ℓ_1, p_2, p_3 , and a subset of the remaining $3d-4$ points. These conditions can be stated more formally. Define

$$\text{ev}_\sigma: \mathcal{H}_d(\mathcal{C}_\sigma) \times (\mathcal{C}_\sigma)^{3d-4} \rightarrow (\mathbb{P}^2)^{3d} \quad \text{by} \quad \text{ev}_\sigma(f; x_4, \dots, x_{3d-1}) = (f(x_0), f(x_1), \dots, f(x_{3d-1})).$$

The space $\mathcal{H}_d(\mathcal{C}_\sigma)$ is a dense open subset of \mathbb{P}^{3d+2} and the evaluation map ev_σ is holomorphic. There is a natural compactification $\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d)$ of $\mathcal{H}_d(\mathcal{C}_\sigma)$, which consists spaces of holomorphic maps from various wedges of spheres into \mathbb{P}^2 . The complex dimension of each such boundary stratum is less than that of $\mathcal{H}_d(\mathcal{C}_\sigma)$. The evaluation map ev_σ admits a continuous extension over $\partial\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d)$, whose restriction to each stratum is holomorphic. The elements $(\ell_0, \ell_1, p_2, \dots, p_{3d-1})$ of the subspace \mathcal{U}_σ of \mathfrak{X} are characterized by the condition that the restriction of the evaluation map to each stratum of $\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d)$ is transversal to the submanifold

$$\ell_0 \times \ell_1 \times p_2 \times \dots \times p_{3d-1} \subset (\mathbb{P}^2)^{3d}.$$

This condition implies that

$$\text{ev}_\sigma^{-1}(\ell_0 \times \ell_1 \times p_2 \times \dots \times p_{3d-1}) \cap \partial \overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d) = \emptyset$$

and the set in (3) is a finite subset of $\mathcal{H}_d(\mathcal{C}_\sigma)$.

The set \mathcal{U}_σ of “general” tuples $(\ell_0, \ell_1, p_2, \dots, p_{3d-1})$ is path-connected. Indeed, it is the complement of a finite number of proper complex submanifolds in \mathfrak{X} . It follows that the number in (3) is independent of the choice of two lines and $3d-2$ points in general position in \mathbb{P}^2 . We thus may simply denote it by N_d^σ . If $\sigma \neq [1, 0], [0, 1], [1, 1]$, \mathcal{C}_σ is a sphere with four distinct points. In such a case, it is fairly easy to show that the number N_d^σ does not change under small variations of σ , or equivalently of the four points on the sphere. Thus, N_d^σ is independent of

$$\sigma \in \mathcal{M}_{0,4} = \mathbb{P}^1 - \{[1, 0], [0, 1], [1, 1]\} = \overline{\mathcal{M}}_{0,4} - \{[1, 0], [0, 1], [1, 1]\}.$$

It is far harder to prove

Proposition 2 *The function $\sigma \rightarrow N_d^\sigma$ is constant on $\overline{\mathcal{M}}_{0,4}$.*

This proposition is a special case of the gluing theorems first proved in [3] and [4].

3 Holomorphic maps vs. complex curves

In this subsection, we express the numbers $N_d^{[0,1]}$ and $N_d^{[1,1]}$ of Subsections 2 in terms of the numbers $n_{d'}$, with $d' \leq d$, of Question 1. By Proposition 2, $N_d^{[0,1]} = N_d^{[1,1]}$. We obtain a recursion for the numbers of Question 1 by comparing the expressions for $N_d^{[0,1]}$ and $N_d^{[1,1]}$.

Let \mathcal{C}_1 denote the component of $\mathcal{C}_{[0,1]}$ containing the marked points x_0 and x_3 ; see Figure 1. We denote by \mathcal{C}_2 the other component of $\mathcal{C}_{[0,1]}$. By definition,

$$N_d^{[0,1]} = \sum_{d_1+d_2=d} N_{d_1, d_2}^{[0,1]} \quad \text{where}$$

$$N_{d_1, d_2}^{[0,1]} = |\{f \in \mathcal{H}_d(\mathcal{C}_{[0,1]}; \mathbb{P}^2) : \deg f|_{\mathcal{C}_1} = d_1, \deg f|_{\mathcal{C}_2} = d_2; p_i \in \text{Im } f \forall i; f(x_0) \in \ell_0, f(x_1) \in \ell_1, f(x_2) = p_2, f(x_3) = p_3\}|.$$

Since the group PSL_2 of holomorphic automorphisms acts transitively on triples of distinct points on the sphere,

$$N_{d_1, d_2}^{[0,1]} = |\{(f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : f_1(\infty) = f_2(\infty), p_i \in f_1(S^2) \cup f_2(S^2) \forall i; f_1(0) \in \ell_0, f_1(1) = p_3, f_2(0) \in \ell_1, f_2(1) = p_2\}|.$$

Since the maps f_1 and f_2 above are holomorphic, $d_1, d_2 \geq 0$ if $N_{d_1, d_2}^{[0,1]} \neq 0$. Since every degree 0 holomorphic map is constant and $p_3 \notin \ell_0$, $N_{0, d}^{[0,1]} = 0$. Similarly, $N_{d, 0}^{[0,1]} = 0$. Thus, we assume that $d_1, d_2 > 0$. Since the points p_3, \dots, p_{3d-1} are in general position, $f_1(S^2)$ contains at most $3d_1 - 2$ of

the points p_4, \dots, p_{3d-1} . Similarly, the curve $f_2(S^2)$ passes through at most $3d_2 - 2$ of the points p_4, \dots, p_{3d-1} . Thus, if $I = \{4, \dots, 3d-1\}$,

$$N_{d_1, d_2}^{[0,1]} = \sum_{I=I_1 \sqcup I_2, |I_1|=3d_1-2} N_{d_1, d_2}^{[0,1]}(I_1, I_2),$$

where $N_{d_1, d_2}^{[0,1]}(I_1, I_2)$ is the cardinality of the set

$$\begin{aligned} \mathcal{S}_{d_1, d_2}^{[0,1]}(I_1, I_2) = \{ & (f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : p_i \in f_1(S^2) \ \forall i \in I_1, \ p_i \in f_2(S^2) \ \forall i \in I_2; \\ & f_1(\infty) = f_2(\infty), \ f_1(0) \in \ell_0, \ f_1(1) = p_3, \ f_2(0) \in \ell_1, \ f_2(1) = p_2 \}. \end{aligned}$$

If $(f_1, f_2) \in \mathcal{S}_{d_1, d_2}^{[0,1]}(I_1, I_2)$, $f_1(S^2)$ is one of the n_{d_1} curves passing through the points $\{p_i : i \in \{3\} \sqcup I_1\}$. Similarly, $f_2(S^2)$ is one of the n_{d_2} curves passing through the points $\{p_i : i \in \{2\} \sqcup I_2\}$. The point $f_1(\infty) = f_2(\infty)$ must be one of the points of $f_1(S^2) \cap f_2(S^2)$; by Bezout's theorem there are $d_1 d_2$ such points. Finally, $f_1(0)$ must be one of the d_1 points of $f_1(S^2) \cap \ell_0$, while $f_2(0)$ must be one of the d_2 points of $f_2(S^2) \cap \ell_1$. Thus, we conclude that

$$\begin{aligned} N_d^{[0,1]} &= \sum_{d_1+d_2=d} N_{d_1, d_2}^{[0,1]} = \sum_{d_1+d_2=d} \sum_{I=I_1 \sqcup I_2, |I_1|=3d_1-2} N_{d_1, d_2}^{[0,1]}(I_1, I_2) \\ &= \sum_{d_1+d_2=d} \sum_{I_1 \subset I, |I_1|=3d_1-2} (d_1 d_2) (d_1 n_{d_1}) (d_2 n_{d_2}) \\ &= \sum_{d_1+d_2=d} \binom{3d-4}{3d_1-2} d_1^2 d_2^2 n_{d_1} n_{d_2}; \end{aligned} \tag{4}$$

where $I = \{4, \dots, 3d-1\}$.

We compute the number $N_d^{[1,1]}$ similarly. We denote by \mathcal{C}_1 the component of $\mathcal{C}_{[1,1]}$ containing the points x_0 and x_1 and by \mathcal{C}_2 the other component of $\mathcal{C}_{[1,1]}$. By definition,

$$\begin{aligned} N_d^{[1,1]} &= \sum_{d_1+d_2=d} N_{d_1, d_2}^{[1,1]}, \quad \text{where} \\ N_{d_1, d_2}^{[1,1]} &= \left| \{ (f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : f_1(\infty) = f_2(\infty), \ p_i \in f_1(S^2) \cup f_2(S^2) \ \forall i; \right. \\ & \quad \left. f_1(0) \in \ell_0, \ f_1(1) \in \ell_1, \ f_2(0) = p_2, \ f_2(1) = p_3 \} \right|. \end{aligned}$$

Since every degree-zero holomorphic map is constant, $N_{d,0}^{[1,1]} = 0$ as before. However,

$$\begin{aligned} N_{0,d}^{[1,1]} &= \left| \{ f_2 \in \mathcal{H}_d(S^2) : f_2(\infty) \in \ell_0 \cap \ell_1, \ f_2(0) = p_2, \ f_2(1) = p_3; \right. \\ & \quad \left. p_i \in f_2(S^2) \ \forall i = 4, \dots, 3d-1 \} \right|. \end{aligned}$$

Thus, $N_{0,d}^{[1,1]} = n_d$. If $d_1, d_2 > 0$,

$$N_{d_1, d_2}^{[1,1]} = \sum_{I=I_1 \sqcup I_2, |I_1|=3d_1-1} N_{d_1, d_2}^{[1,1]}(I_1, I_2),$$

where $N_{d_1, d_2}^{[1,1]}(I_1, I_2)$ is the cardinality of the set

$$\mathcal{S}_{d_1, d_2}^{[1,1]}(I_1, I_2) = \{(f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : p_i \in f_1(S^2) \forall i \in I_1, p_i \in f_2(S^2) \forall i \in I_2; \\ f_1(\infty) = f_2(\infty), f_1(0) \in \ell_0, f_1(1) \in \ell_1, f_2(0) = p_2, f_2(1) = p_3\}.$$

Proceeding as in the previous paragraph, we conclude that

$$\begin{aligned} N_d^{[1,1]} &= \sum_{d_1+d_2=d} N_{d_1, d_2}^{[1,1]} = n_d + \sum_{d_1+d_2=d} \sum_{I=I_1 \sqcup I_2, |I_1|=3d_1-1} N_{d_1, d_2}^{[1,1]}(I_1, I_2) \\ &= n_d + \sum_{d_1+d_2=d} \sum_{I_1 \subset I, |I_1|=3d_1-1} (d_1 d_2) (d_1^2 n_{d_1}) (n_{d_2}) \\ &= n_d + \sum_{d_1+d_2=d} \binom{3d-4}{3d_1-1} d_1^3 d_2 n_{d_1} n_{d_2}; \end{aligned} \tag{5}$$

Comparing equations (4) and (5), we obtain

$$n_d = \sum_{d_1+d_2=d} \left(\binom{3d-4}{3d_1-2} d_1 d_2 - \binom{3d-4}{3d_1-1} d_1^2 \right) d_1 d_2 n_{d_1} n_{d_2}. \tag{6}$$

The recursive formula (1) is the symmetrized version of (6).

References

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