

**MAT 401: Undergraduate Seminar**  
*Introduction to Enumerative Geometry*  
**Fall 2018**

**Homework Assignment III**

**Written Assignment due on Tuesday, 9/25, at 1pm in ESS 181**  
(or by 9/25, noon, in Math 3-111)

Please do 6 of the following problems: Chapter 4, #1,2,3,5, and B-D below, with B-D counted as 2 problems each; B-D are also for presentation on 9/25.

**Problem B (9/25, ~15mins)**

Let  $U \subset \mathbb{C}^4 \times \mathbb{C}^4$  be the subspace consisting of pairs of linearly independent vectors and  $A \subset U$  of pairs of vectors that are orthonormal with respect to the standard Hermitian inner-product on  $\mathbb{C}^4$ . Thus, each element of  $U$  and  $A$  determines a two-dimensional linear subspace of  $\mathbb{C}^4$ ; this induces surjective maps

$$\pi: U \longrightarrow G(2, 4), \quad \pi': A \longrightarrow G(2, 4).$$

Show that these maps induce the same topology on  $G(2, 4)$ .

**Problem C (9/25, ~20mins)**

(a) For each  $i=0, 1, \dots, n$ , let

$$U_i = \{[X_0, X_1, \dots, X_n] \in \mathbb{C}P^n : X_i \neq 0\},$$
$$\phi_i: U_i \longrightarrow \mathbb{C}^n, \quad [X_0, X_1, \dots, X_n] \longrightarrow (X_0/X_i, X_1/X_i, \dots, X_{i-1}/X_i, X_{i+1}/X_i, \dots, X_n/X_i).$$

Show that the maps  $\phi_i$  are homeomorphisms, while

$$\phi_i \circ \phi_j^{-1}: \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$$

are analytic/holomorphic maps between open subspaces of  $\mathbb{C}^n$ .

(b) Describe manifold charts for  $G(2, 4)$  (for the quotient topology of Problem B) and show that  $G(2, 4)$  is also a complex manifold (the overlap maps are analytic/holomorphic).

**Problem D (9/25, ~15mins)**

Let  $F = F(X_0, \dots, X_n)$  be a homogeneous polynomial of degree  $d \in \mathbb{Z}^+$  and  $(Y_0, \dots, Y_n) \in \mathbb{C}^{n+1} - 0$ . If  $F(Y_0, \dots, Y_n) = 0$ , but  $\frac{\partial F}{\partial X_i} \Big|_{(Y_0, \dots, Y_n)} \neq 0$  for some  $i=0, 1, \dots, n$ , show that the hypersurface  $Z(F) \subset \mathbb{C}P^n$  is a complex manifold in a neighborhood of  $[Y_0, \dots, Y_n] \in Z(F)$ . Show that the tangent hyperplane to  $Z(F)$  at  $[Y_0, \dots, Y_n]$  is given by the equation

$$G(X_0, \dots, X_n) = \frac{\partial F}{\partial X_0} \Big|_{(Y_0, \dots, Y_n)} X_0 + \dots + \frac{\partial F}{\partial X_n} \Big|_{(Y_0, \dots, Y_n)} X_n = 0.$$

## Further Discussion Problems for 9/25, 10/2

*Bezout's Theorem for  $\mathbb{C}P^2$* : If  $C, D \subset \mathbb{C}P^2$  are curves of degrees  $c, d \in \mathbb{Z}^+$  such that  $C \cap D$  is finite, then the cardinality of the set  $C \cap D$  counted with multiplicity  $m_p(C, D) \in \mathbb{Z}^+$  for each point  $p \in C \cap D$  is  $cd$ .

The number  $m_p(C, D)$  is defined so that if the curves  $C$  and  $D$  are deformed slightly and generically (by deforming the homogeneous polynomials defining  $C$  and  $D$ ), then  $m_p(C, D)$  is the number of points in the intersection of the deformed curves that lie near  $p$ . Thus, the weighted cardinality of  $C \cap D$  does not change under small changes in  $(C, D)$ . It thus must be independent of  $C$  and  $D$  provided the space

$$\mathfrak{X}_{c,d} \equiv \{(F, G) \in HP_c(\mathbb{C}^3) \times HP_d(\mathbb{C}^3) : Z(F) \cap Z(G) \text{ is finite}\}$$

is connected (as in class  $HP_c(\mathbb{C}^3)$  is the space of homogeneous polynomials on  $\mathbb{C}^3$  of degree  $c$ ). We can thus determine the weighted cardinality of  $C \cap D$  by determining it for a specific pair in  $\mathfrak{X}_{c,d}$ ; in class,  $C$  and  $D$  were taken to consist of  $c$  and  $d$  lines, respectively, with all  $c+d$  lines being distinct, obtaining Bezout's Theorem. The aim of this discussion problem is to fill in some of the gaps in the argument.

Part I (9/25, ~20mins): A topological space  $\mathfrak{X}$  is called **connected** if it can't be written as a disjoint union of two nonempty open subset,  $\mathfrak{X} \neq U \sqcup V$ ;  $\mathfrak{X}$  is called **path-connected** if for any  $p, q \in \mathfrak{X}$  there exists a continuous map  $f: [0, 1] \rightarrow \mathfrak{X}$  such that  $f(0) = p$  and  $f(1) = q$  (thus every two points in  $\mathfrak{X}$  are connected by a path). Show that

- (a) any continuous map from a connected space to  $\mathbb{Z}$  is constant;
- (b) any path-connected space is connected and thus  $\mathbb{C}^n$  is connected;
- (c) if  $A \subset \mathfrak{X}$  is connected (in the subspace topology) and  $A \subset B \subset \bar{A}$ , then  $B \subset \mathfrak{X}$  is also connected.

Part II (10/2, ~30mins): Show that

- (a) if  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  is a non-constant analytic function (analytic in each variable), then for every  $p \in f^{-1}(0)$  there exists  $r > 0$  such that  $B_r(p) - f^{-1}(0)$ , where  $B_r(p)$  is the  $r$ -ball centered at  $p$ , is path-connected;
- (b) if  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  is a non-constant analytic function, then  $\mathbb{C}^n - f^{-1}(0)$  is path-connected;
- (c)  $\mathfrak{X}_{c,d}$  is connected.

Part III (10/2, ~40mins): Let  $f, g: \mathbb{C}^2 \rightarrow \mathbb{C}$  be two polynomials of degrees at most  $c$  and  $d$  (not necessarily homogeneous) such that  $f(0), g(0) = 0$  and there exists  $r > 0$  such that  $B_r(0) - 0$  contains no points  $f^{-1}(0) \cap g^{-1}(0)$ . For  $p \in \mathbb{C}^2$ , let  $\nabla f|_p: \mathbb{C}^2 \rightarrow \mathbb{C}$  be the gradient of  $f$  at  $p$ . If  $c \in \mathbb{Z}^+$ , denote by  $P_c(\mathbb{C}^2)$  the space of polynomials on  $\mathbb{C}^2$  of degree at most  $c$ ; there is a natural norm on  $P_c(\mathbb{C}^2)$ , since such polynomials correspond to tuples of elements of  $\mathbb{C}$ . Show that

- (a) if  $(\ker \nabla f|_0) \cap (\ker \nabla g|_0) = \{0\}$ , there exists  $\epsilon > 0$  such that the set

$$\{f + \tilde{f}\}^{-1}(0) \cap \{g + \tilde{g}\}^{-1}(0) \cap B_{r/2}(0) \subset \mathbb{C}^2$$

consists of precisely one element whenever  $|\tilde{f}|, |\tilde{g}| < \epsilon$  (in such a case, 0 is said to be a *simple intersection point*, or *point of intersection multiplicity 1*, of  $f^{-1}(0)$  and  $g^{-1}(0)$ );

- (b) the set  $\mathfrak{X}_{c,d}(f, g) \equiv \{(\tilde{f}, \tilde{g}) \in P_c(\mathbb{C}^2) \times P_d(\mathbb{C}^2) : Z(f + \tilde{f}) \cap Z(g + \tilde{g}) \text{ is finite};$

$$(\ker \nabla(f + \tilde{f})|_p) \cap (\ker \nabla(g + \tilde{g})|_p) = \{0\} \quad \forall p \in Z(f + \tilde{f}) \cap Z(g + \tilde{g})\}$$

is connected and non-empty;

- (c) there exists  $\epsilon > 0$  such that the cardinality of the set

$$\{f + \tilde{f}\}^{-1}(0) \cap \{g + \tilde{g}\}^{-1}(0) \cap B_{r/2}(0) \subset \mathbb{C}^2$$

is independent of  $\tilde{f}, \tilde{g} \in \mathfrak{X}_{c,d}(f, g)$  with  $|\tilde{f}|, |\tilde{g}| < \epsilon$ . (This number is called the *multiplicity of  $0 \in \mathbb{C}^2$  as an intersection point of the curves  $Z(f)$  and  $Z(g)$*  and is denoted by  $m_0(f, g)$  or  $m_0(Z(f), Z(g))$ ).