ON UNIQUENESS AND DIFFERENTIABILITY IN THE SPACE OF YAMABE METRICS

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Abstract. It is shown that there is a unique Yamabe representative for a generic set of conformal classes in the space of metrics on any manifold. At such classes, the scalar curvature functional is shown to be differentiable on the space of Yamabe metrics. In addition, some sufficient conditions are given which imply that a Yamabe metric of locally maximal scalar curvature is necessarily Einstein.

1. Introduction.

Let M be a closed n-dimensional manifold. For a given smooth metric g on M, let $[g]$ denote the conformal class of g, consisting of smooth metrics on M pointwise conformal to g. By the solution to the Yamabe problem [1], [6], in each conformal class $[g]$ there is a Yamabe metric γ of constant scalar curvature s_γ; the metric γ minimizes the total scalar curvature (or Einstein-Hilbert action)

$$\mathcal{S}(\bar{g}) = v^{-(n-2)/n}\int_M s_{\bar{g}} dV_{\bar{g}}.$$

when $\mathcal{S}$ is restricted to the class of conformal metrics $\bar{g} \in [g]$. Here s_g denotes the scalar curvature of g, dV_g the volume form, and v the total volume of (M, g). The sign of s_γ, i.e $s_\gamma < 0$, $s_\gamma = 0$ or $s_\gamma > 0$ depends only on the conformal class $[\gamma]$.

A number of general features of the class of Yamabe metrics of non-positive scalar curvature are well understood, cf. [2], [7] for example. Thus, negative conformal classes have a unique unit volume Yamabe metric. The space $\mathcal{Y}^-$ of all unit volume negative Yamabe metrics forms a smooth infinite dimensional manifold $\mathcal{Y}^-$, transverse to the space of conformal classes, in the space of all unit volume metrics on M. The scalar curvature s defines a smooth function $s : \mathcal{Y}^- \to \mathbb{R}$, whose critical points are exactly Einstein metrics (of negative scalar curvature) on M. Similar results hold for the space of non-positive Yamabe metrics. All of these results essentially derive from the fact that Yamabe metrics satisfy an elliptic equation, the Yamabe equation, whose solutions satisfy a maximum principle when $s_\gamma \leq 0$.

It has been an open issue for some time to what extent such general features continue to hold for the space $\mathcal{Y}^+$ of positive unit volume Yamabe metrics on M where the corresponding Yamabe equation does not satisfy a maximum principle. Thus, in general, uniqueness of Yamabe metrics fails for positive conformal classes. For example, the conformal class $[\gamma_0]$ of the round metric γ_0 on S^n admits a large, non-compact, family of Yamabe metrics. On $S^{n-1} \times S^1$, there are 1-parameter families of Yamabe metrics in the conformal class of a product metric, for infinitely many values of radii of S^{n-1} and S^1, cf. [7], [8].

The purpose of this note is to establish some partial answers to these issues. Some of the main results are as follows:

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Theorem 1.1. *Generically, Yamabe metrics are unique in their conformal class. Thus, there is an open and dense set $\mathcal{U}$ in the space of positive conformal classes such that each $[g] \in \mathcal{U}$ has a unique unit volume Yamabe metric $\gamma \in [g]$.*

We also prove that the scalar curvature function $s : \mathcal{Y}^+ \to \mathbb{R}$ is differentiable at any Yamabe metric γ which is unique in its conformal class $[\gamma]$, for $[\gamma] \neq [\gamma_0]$, cf. Proposition 2.2. Moreover, if s is differentiable at a metric γ, and γ is a critical point of s on $\mathcal{Y}^+$, then γ must be an Einstein metric, (and hence unique by Obata's theorem [5]).

The scalar curvature $s : \mathcal{Y}^+ \to \mathbb{R}$ is continuous, cf. [2]. It is unknown if s is differentiable everywhere; at present there are no solid reasons to substantiate this. However, we show that away from $[\gamma_0]$, positive directional derivatives of s always exist, i.e. the derivative

$$s'(h) = \frac{d}{dt}^{+} s_{[\gamma+th]}|_{t=0}$$

exists, for any symmetric bilinear form h; here $s_{[g]}$ denotes the scalar curvature of (any) unit volume Yamabe metric in the conformal class $[g] \neq [\gamma_0]$ and $t > 0$.

It has long been an open problem whether a Yamabe metric γ which is a local maximum of s on $\mathcal{Y}^+$ is necessarily an Einstein metric, cf. [2]. Some recent progress on this question has been made in dimension 3 in [3]. The following result gives a partial answer in any dimension.

Theorem 1.2. *Suppose $\gamma \in \mathcal{Y}^+$ is a local maximum for s. If there are most two Yamabe metrics in the conformal class $[\gamma]$, then γ is Einstein.*

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2. Proofs of the Results.

Let $\mathcal{Y}$ denote the space of all unit volume Yamabe metrics on a given n-manifold M. Henceforth, all metrics will be assumed to have unit volume, for convenience. The discussion and results to follow are all essentially trivial for Yamabe metrics of non-positive scalar curvature. Let $\mathcal{C}$ denote the space of unit volume constant scalar curvature metrics on M, and $\mathbb{M}$ the space of all unit volume metrics on M, so that $\mathcal{Y} \subset \mathcal{C} \subset \mathbb{M}$. Unless stated otherwise, all metrics will be assumed to be (C^∞) smooth.

It is well-known, cf. [2], that generically $\mathcal{C}$ is an infinite dimensional manifold. More precisely, at all metrics g for which $s_g/(n-1)$ is not an eigenvalue of the Laplacian $-\Delta$, (with non-negative spectrum), a neighborhood of g in $\mathcal{C}$ has the structure of a smooth manifold. Observe that $\mathcal{C}$ can be described as the 0-level set of the mapping

$$\Phi : \mathbb{M} \to C^\infty(M, \mathbb{R}), \;\; \Phi(g) = \Delta_g s_g.$$

Since Φ is a real-analytic map, $\mathcal{C}$ thus has the structure of an infinite dimensional real-analytic variety. It would be interesting to know if this structure could lead to further information on the structure of $\mathcal{C}$.

Consider the set $\Lambda = \Lambda[\gamma]$ of all smooth Yamabe metrics in a given conformal class $[\gamma]$, i.e. $\Lambda = \mathcal{Y} \cap [\gamma]$. If g is a fixed representative metric in γ, then all Yamabe metrics $\gamma \in \Lambda[\gamma]$ are of the form

$$\gamma = \phi^{4/(n-2)} g, \tag{2.1}$$

for some positive smooth function $\phi : M \to \mathbb{R}$. By the solution to the Yamabe problem [1], [6], the set $\Lambda[\gamma]$ is compact, i.e. is a compact subset of $C^\infty(M, \mathbb{R})$, for all $[\gamma] \neq [\gamma_0]$, cf. also [7, Prop. 2.1]. In the following, we will always work with conformal classes $[\gamma] \neq [\gamma_0]$. In addition, the sets $\Lambda[\gamma]$ as γ varies are also compact in the following sense; if $[\gamma_i] \to [\gamma] \neq [\gamma_0]$ smoothly, then any sequence of Yamabe metrics $(\gamma^j)_i \in [\gamma_i]$ has a subsequence converging smoothly to a Yamabe metric $\gamma^j \in [\gamma]$.

We begin by discussing the differentiability properties of the scalar curvature function s on $\mathcal{Y}$. Let γ be a unit volume Yamabe metric, with $\gamma \notin [\gamma_0]$, and let g_t be a smooth curve of unit volume metrics of the form

$$g_t = \gamma + th + O(t^2), \tag{2.2}$$

for $h \in T_\gamma \mathbb{M}$. We assume that g_t is orthogonal to the conformal classes in the sense that

$$dV_{g_t} = dV_\gamma. \tag{2.3}$$

In particular, (2.3) implies that $tr_\gamma h = 0$.

Choose a sequence $t_i \to 0$, and let γ_i be a unit volume Yamabe metric in the conformal class of $[g_{t_i}]$. Thus, γ_i is of the form

$$\gamma_i = \phi_i^{4/(n-2)} g_i, \tag{2.4}$$

where $g_i = g_{t_i}$. By passing to a subsequence if necessary, may assume that $\gamma_i \to \gamma'$ smoothly, where γ' is some Yamabe metric in $[\gamma]$. Since $g_i \to g$ smoothly, one has $\phi_i \to \phi$ smoothly, where $\gamma' = \phi^{4/(n-2)}\gamma$. In general, $\phi \neq 1$, since the conformal class $[\gamma]$ may have more than one Yamabe metric. However, by changing the curve g_t to the curve $\widetilde{g}_t = \phi^{4/(n-2)} g_t$, we may and do assume, without loss of generality, that $\phi = 1$. This simplifies some of the computations below.

The functions ϕ_i satisfy the Yamabe equation

$$-c_n \Delta_{g_i} \phi_i + s_{g_i} \phi_i = \phi_i^q s_{\gamma_i}, \tag{2.5}$$

where $c_n = 4(n-1)/(n-2)$ and $q = (n+2)/(n-2)$.

Next, linearize (2.5) about the limit $(\gamma, 1)$ of (g_i, ϕ_i). The linearization of the scalar curvature is given by

$$L(h) = \frac{d}{dt} s_{g+th}|_{t=0} = \Delta trh + \delta\delta h - \langle Ric, h \rangle,$$

cf. [2] for instance. The adjoint operator L^* to L is given by

$$L^* f = D^2 f - \Delta f \cdot g - f Ric.$$

Since $g_i \to \gamma$ smoothly,

$$s_{g_i} = s_\gamma + t_i L(h) + O(t_i^2). \tag{2.6}$$

Also write

$$\phi_i^q s_{\gamma_i} = (\phi_i^q - \phi_i) s_\gamma + \phi_i^q (s_{\gamma_i} - s_\gamma) + \phi_i s_\gamma.$$

Substituting these expressions in (2.5) gives

$$-c_n \Delta_{g_i}(\phi_i - 1) - (\phi_i^q - \phi_i) s_\gamma = \phi_i^q \left(\frac{s_{\gamma_i} - s_\gamma}{t_i}\right) t_i - \phi_i L(h) t_i + O(t_i^2),$$

so that

$$-c_n \Delta_{g_i}\left(\frac{\phi_i - 1}{t_i}\right) - \psi_i s_\gamma \left(\frac{\phi_i - 1}{t_i}\right) = \phi_i^q \left(\frac{s_{\gamma_i} - s_\gamma}{t_i}\right) - \phi_i L(h) + O(t_i), \tag{2.7}$$

where

$$\psi_i = (\phi_i^4 + ... + \phi_i)/((\phi_i^{q-1})^{n-3} + (\phi_i^{q-1})^{n-4} + ... + \phi_i^{q-1} + 1). \tag{2.8}$$

Here we use the identity that if $y = x^{a/b}$, then $(y-1) = (x-1)(x^{a-1} + ... + 1)/(y^{b-1} + ... + 1)$, with $x = \phi$, $y = \phi^{q-1}$, $a = 4$, $b = n-2$.

Integrating (2.7) over M with respect to g_i then gives the identity

$$-s_\gamma \int (\frac{\phi_i - 1}{t_i})\psi_i dV_{g_i} = (\frac{s_{\gamma_i} - s_\gamma}{t_i}) \int \phi_i^q dV_{g_i} - \int \phi_i L(h) dV_{g_i} + O(t_i). \tag{2.9}$$

To understand the linearization, i.e. the behavior as $t_i \to 0$, the main point is to prove that the left side of (2.9) vanishes in the limit $t_i \to 0$.

Proposition 2.1. *Let g_t be as in (2.2), with $t = t_i \to 0$, $t_i > 0$, and $s_\gamma > 0$. Let γ_i be Yamabe metrics in $[g_i]$, with ϕ_i as in (2.4) satisfying $\phi_i \to 1$ smoothly as $t_i \to 0$. Then*

$$\int (\frac{\phi_i - 1}{t_i})\psi_i dV_{g_i} \to 0, \text{ as } i \to \infty. \tag{2.10}$$

Proof: Note first that since $\phi_i^q \to 1$ and $\psi_i \to \psi = \frac{4}{n-2}$ smoothly, the only terms in (2.9) which may become unbounded are those with t_i in the denominator. Also, the measures dV_{g_i} converge smoothly to the measure dV_g.

Suppose first that $(\phi_i - 1)/t_i$ is bounded in L^2, so that it has a weakly convergent subsequence: $(\phi_i - 1)/t_i \to \phi'$ weakly in L^2. The left side of (2.9) is then bounded, and hence the term $(s_{\gamma_i} - s_\gamma)/t_i$ on the right is also bounded. It then follows from elliptic regularity associated to the equation (2.7) that $(\phi_i - 1)/t_i \to \phi'$ smoothly. Thus, it suffices to show in this case that

$$\int \phi' dV_\gamma = 0, \tag{2.11}$$

since $\psi = \lim \psi_i = 4/(n-2)$. However, since the metrics s_{g_i} and s_{γ_i} have unit volume, one has

$$\int \frac{\phi_i^{2n/(n-2)} - 1}{t_i} dV_{g_i} = 0. \tag{2.12}$$

Taking the limit of (2.12) as $t_i \to 0$ and using the fact that ϕ' exists gives (2.11).

If $(\phi_i - 1)/t_i$ is not bounded in L^2, the proof is more complicated, but based on similar ideas together with the fact that γ_i are Yamabe metrics. To begin, we assume that $(s_{\gamma_i} - s_\gamma)/t_i$ is bounded, and hence converges to a limit

$$s' = \lim_{t_i \to 0} \frac{s_{\gamma_i} - s_\gamma}{t_i}, \tag{2.13}$$

(again in a subsequence). The situation where (2.13) does not hold is dealt with later, based on a simple renormalization argument. The assumption (2.13) implies that the limit of the left side of (2.9) as $t_i \to 0$ also exists.

First, we claim that

$$\lim_{t_i \to 0} -\int (\frac{\phi_i - 1}{t_i})\psi_i dV_{g_i} \leq 0. \tag{2.14}$$

To see this, since $\phi_i \to 1$ smoothly, one has

$$\lim_{t_i \to 0} \frac{s_{\gamma_i} - s_\gamma}{t_i} \int \phi_i^q dV_{g_i} = s'.$$

On the other hand, by the Yamabe property of γ_i and the fact that g_i is of unit volume,

$$s_{\gamma_i} \leq \int s_{g_i} dV_{g_i}. \tag{2.15}$$

Hence, by (2.13),

$$s' \leq \lim_{i\to\infty} \int \frac{s_{g_i} - s_\gamma}{t_i} dV_{g_i}.$$

However, (2.6) gives

$$\lim_{t_i\to 0} \int \frac{s_{g_i} - s_\gamma}{t_i} dV_{g_i} = \int L(h) dV_\gamma.$$

Via (2.9), this gives the claim (2.14).

We now claim the opposite inequality to (2.14) holds. To see this, consider the sequence of metrics $\widetilde{g}_i = \phi_i^{4/(n-2)}\gamma$ in the conformal class $[\gamma]$. They all have unit volume, by (2.2) and (2.12). Since γ is a Yamabe metric, i.e. it minimizes $\mathcal{S}$ in its conformal class, it follows that

$$\int (c_n|d\phi_i|^2 + s_\gamma\phi_i^2) dV_\gamma = \int [c_n|d(\phi_i - 1)|^2 + s_\gamma(1 + (\phi_i - 1))^2] dV_\gamma \geq s_\gamma. \tag{2.16}$$

Expanding out the term on the right then gives, since $t_i > 0$ and $s_\gamma > 0$,

$$\int [c_n \frac{|d(\phi_i - 1)|^2}{t_i} + s_\gamma(2\frac{(\phi_i - 1)}{t_i} + \frac{(\phi_i - 1)^2}{t_i})] dV_\gamma \geq 0.$$

Also, integration by parts gives

$$\int c_n \frac{|d(\phi_i - 1)|^2}{t_i} dV_\gamma = -c_n \int (\phi_i - 1)\Delta_\gamma(\frac{\phi_i - 1}{t_i}) dV_\gamma = s_\gamma \int \psi_i \frac{(\phi_i - 1)^2}{t_i} dV_\gamma + o(1). \tag{2.17}$$

Here the second equality uses (2.7), together with the fact that $\frac{1}{t_i}(\Delta_{g_i} - \Delta_\gamma)(\phi_i - 1) \to 0$ since $g_i \to \gamma$ smoothly. (The term $\frac{1}{t_i}(\Delta_{g_i} - \Delta_\gamma)$ converges to the derivative $(\Delta)'_h$ of the Laplacian in the direction h; this applied to $(\phi_i - 1)$ tends to 0, since $(\phi_i - 1) \to 0$). Combining these estimates, (and using $s_\gamma > 0$), it follows that

$$\lim_{t_i\to 0} \int 2\frac{(\phi_i - 1)}{t_i} + \frac{(\phi_i - 1)^2}{t_i} + \psi_i \frac{(\phi_i - 1)^2}{t_i} dV_\gamma \geq 0. \tag{2.18}$$

Observe that from the derivation of (2.7), we have $\psi_i(\phi_i - 1)^2 = (\phi_i - 1)(\phi_i^q - \phi_i)$. Now compute $2(\phi_i - 1) + (\phi_i - 1)^2 + (\phi_i - 1)(\phi_i^q - \phi_i) = (\phi_i - 1)(2 + (\phi_i - 1) + \phi_i^q - \phi_i) = (1 + \phi_i^q)(\phi_i - 1) = \phi_i - \phi_i^q + \phi_i^{q+1} - 1$. The last two terms here integrate to 0, by (2.12), since $q + 1 = 2n/(n-2)$. It follows then from (2.18) that

$$\lim_{t_i\to 0} -\int \frac{\phi_i^q - \phi_i}{t_i} dV_\gamma \geq 0,$$

i.e.

$$\lim_{t_i\to 0} -\int \psi_i \frac{\phi_i - 1}{t_i} \geq 0. \tag{2.19}$$

Combining (2.14) and (2.19) proves the result in case (2.13) holds.

Finally, suppose the ratio $(s_{\gamma_i} - s_\gamma)/t_i$ in (2.13) is unbounded. Then divide each term in (2.9) by C_i, where $C_i \to \infty$ is chosen to make the resulting ratio in (2.13) equal to 1, in absolute value; thus $(s_{\gamma_i} - s_\gamma)/C_i t_i$ remains bounded, and bounded away from 0. Performing exactly the arguments as above following (2.13), dividing by C_i as called for, leads to the conclusion that $\int(\phi_i - 1)/C_i t_i \to 0$, and $\int \phi_i L(h)/C_i \to 0$, but $(s_{\gamma_i} - s_\gamma)/C_i t_i$ is bounded away from 0. This contradicts (2.9), and so completes the proof. ∎

Proposition 2.1 and (2.9) imply that the limit

$$s'(h) = \lim_{t_i \to 0^+} \frac{s_{\gamma_i} - s_\gamma}{t_i}, \tag{2.20}$$

exists, and is given by

$$s'(h) = \int \langle L^*1, h\rangle dV_\gamma = \int \langle -z, h\rangle dV_\gamma, \tag{2.21}$$

where z is the trace-free Ricci curvature, $z = Ric - \frac{s}{n}g$.

The discussion above easily proves the following result.

Proposition 2.2. *Suppose γ is the unique Yamabe metric (of unit volume) in $[\gamma] \neq [\gamma_0]$. Then s is differentiable on $\mathcal{Y}$ at γ, and the derivative is given by (2.21).*

Proof: Let $[g_t] = [\gamma + th]$ be any variation of the conformal class of γ. If γ_t is any Yamabe metric in $[g_t]$ written in the form (2.4), then as discussed following (2.4), $\phi_t \to 1$ as $t \to 0$. It follows from (2.21) that for any h, and for any sequence of Yamabe metrics converging to γ along the curve $[g_t]$, one has

$$s'(h) = \int \langle -z, h\rangle dV_\gamma.$$

The right hand side is linear in h, and hence s is differentiable at γ. ■

Corollary 2.3. *Any critical point γ of s on $\mathcal{Y}$, for which γ is unique in its conformal class is necessarily an Einstein metric.*

Proof: If γ is unique in its conformal class, then Proposition 2.2 implies that s is differentiable on $\mathcal{Y}$ at γ. By (2.21), a critical point γ of s then satisfies $z = 0$, i.e. γ is Einstein.

■

Now suppose there is more than one Yamabe metric in the conformal class $[\gamma]$. Such metrics are of the form $\widetilde{\gamma} = \phi^{4/(n-2)}\gamma$, for some positive function ϕ. A standard formula for conformal changes, cf. [2] gives,

$$\widetilde{z} = z + (n-2)\phi^{2/(n-2)} D_0^2 \phi^{-2/(n-2)},$$

where D_0^2 is the trace-free Hessian with respect to γ. Since h transforms as $\widetilde{h} = \phi^{4/(n-2)}h$ on passing from γ to $\widetilde{\gamma}$, and $dV_{\widetilde{\gamma}} = \phi^{2n/(n-2)}dV_\gamma$, one has

$$\int \langle \widetilde{z}, \widetilde{h}\rangle dV_{\widetilde{\gamma}} = \int \phi^{(2n-4)/(n-2)} \langle z_\gamma + (n-2)\phi^{2/(n-2)} D_0^2 \phi^{-2/(n-2)}, h\rangle dV_\gamma.$$

Set $u = \phi^{2/(n-2)}$, so that

$$\int \langle \widetilde{z}, \widetilde{h}\rangle dV_{\widetilde{\gamma}} = \int u^{n-2} \langle z_\gamma + (n-2)u(D_0^2 u^{-1}), h\rangle dV_\gamma. \tag{2.22}$$

The forms

$$Z_\phi = u^{n-2}(z_\gamma + (n-2)uD_0^2 u^{-1}), \tag{2.23}$$

are viewed as tangent vectors $Z_\phi \in T_\gamma\mathbb{M}$, corresponding to the set of Yamabe metrics $\Lambda[\gamma]$ in $[\gamma]$, cf. the discussion preceding (2.1). Let g_t, $t \geq 0$, be a smooth curve of unit volume metrics as in (2.2) and let γ_i be an associated sequence of Yamabe metrics as in (2.4). It follows from (2.21) and (2.22)-(2.23) that if γ_i converges to a Yamabe metric $\widetilde{\gamma} = \phi^{4/(n-2)}\gamma$ in $[\gamma] = [g(0)]$, then

$$s'(h) = \int \langle -Z_\phi, h \rangle dV_\gamma, \tag{2.24}$$

where $s' = \lim_{t_i \to 0^+}(s_{\gamma_i} - s_\gamma)/t_i$.

One thus has a map

$$\mathcal{Z} : \Lambda[\gamma] \to \{Z_\phi\}, \quad \mathcal{Z}(\phi^{4/(n-2)}\gamma) = Z_\phi. \tag{2.25}$$

It will be shown below that $\mathcal{Z}$ is 1-1, cf. Lemma 2.5. While of course Z_ϕ *may* be defined for any Yamabe metric in $\Lambda[\gamma]$, we only define Z_ϕ for a Yamabe metric $\widetilde{\gamma}$ in $[\gamma]$ arising as a limit of a sequence γ_i as in (2.24), and call such Z_ϕ admissible. It is not known if all Z_ϕ are admissible, i.e. if $\mathcal{Z}$ is then defined on all of $\Lambda[\gamma]$. There may Yamabe metrics $\bar{\gamma}$ in $[\gamma]$ which are not moved in any perturbation of the conformal class $[\gamma]$; such $\bar{\gamma}$ are thus not near a Yamabe metric in any conformal class close to $[\gamma]$, and so are "isolated" Yamabe metrics.

It is a consequence of (2.24) that s' is independent of the sequence t_i, among sequences for which $\gamma_i \to \widetilde{\gamma}$ as above. However, apriori, even within the fixed curve g_t specified in (2.24), different sequences $t_i \to 0$ may give sequences of Yamabe metrics converging to different Yamabe limits in $[\gamma]$. Thus, apriori, one may have a collection of distinct ϕ's associated to a given h. Of course in general the collection of ϕ's and Z_ϕ's changes with h.

This analysis leads to the following result.

Proposition 2.4. *The scalar curvature function s: $\mathcal{Y} \to \mathbb{R}$ has (positive) directional derivatives in any direction $h \in T_\gamma\mathbb{M}$, $[\gamma] \neq [\gamma_0]$, with derivative given by*

$$s'(h) = \frac{d}{dt}^{+} s_{[\gamma+th]}|_{t=0} = \min_\phi \int \langle -Z_\phi, h \rangle dV_\gamma, \tag{2.26}$$

where the minimum is taken over all admissible Z_ϕ in $[\gamma]$.

Proof: This is an immediate consequence of the discussion above and the fact that Yamabe metrics minimize the scalar curvature functional $\mathcal{S}$ in the conformal class; in particular, s_γ is independent of the representative $\gamma \in [\gamma]$. ∎

As above, let $\{\gamma^\lambda\}_{\lambda \in \Lambda}$ denote the set of unit volume Yamabe metrics in the conformal class $[\gamma]$, $[\gamma] \neq [\gamma_0]$. By the compactness mentioned at the beginning of this section, if $[\gamma_i] \to [\gamma]$ smoothly, then any sequence of Yamabe metrics $(\gamma^j)_i \in [\gamma_i]$ has a subsequence converging smoothly to some Yamabe metric $\gamma^j \in [\gamma]$. However, the cardinality of the sets $\{\gamma^\lambda\}$ may well change in passing to limits. The cardinality may drop, for instance when distinct Yamabe metrics in a sequence of conformal classes merge to a common limit. The cardinality of $\{\gamma^\lambda\}$ may also increase in the limit, due to the "birth" of a new Yamabe metric, not arising as a limit of a given sequence.

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1.

It is well-known, cf. [8] for instance, that the set of conformal classes containing only finitely many unit volume Yamabe (or more generally constant scalar curvature) metrics is generic, i.e. forms an open and dense set in the space of all Yamabe metrics, with respect to the C^2 (or stronger) topology. For simplicity, we assume $[\gamma]$ is generic in this sense. Then nearby conformal classes $[g_t]$, in any direction $g_t = \gamma + th$, are also generic, and so have only finitely many Yamabe metrics.

The proof is by contradiction, and so suppose there exists an open set $\mathcal{V}$ of generic Yamabe metrics such that, for each $\gamma \in \mathcal{V}$ there are at least two distinct Yamabe metrics in $[\gamma]$. Using the compactness mentioned above, by passing to a smaller $\mathcal{V}$ if necessary, we may assume that there is a fixed upper bound on the number of distinct Yamabe metrics in $[\gamma]$, $\gamma \in \mathcal{V}$.

Now fix some $\gamma_o \in \mathcal{V}$, and consider the class of variations $h \in T_{\gamma_o}\mathbb{M}$ with $||h||_{C^2} = 1$. As discussed above, each such h determines a subset of Yamabe metrics $\{\gamma^{\lambda}(h)\}$ of the full set of Yamabe metrics $\{\gamma^{\lambda}\}$ in $[\gamma_o]$, (namely those which persist under perturbation in the direction h). As h changes, this subset can of course change. However, since there is a fixed bound on the cardinality of $\{\gamma^{\lambda}\}$, there is an open set U of $h \in T_{\gamma}\mathbb{M} \cap \{||h||_{C^2} = 1\}$, with the following property: there is a fixed subset $\{\gamma^{\lambda'}\} \subset \{\gamma^{\lambda}\}$, of cardinality at least two, such that for each $h \in U$, there is a sequence $t_i \to 0^+$ such that $[\gamma_o + t_i h]$ has at least two Yamabe metrics converging in subsequences to at least two distinct elements of $\{\gamma^{\lambda'}\}$. We may assume that $\gamma_o \in \{\gamma^{\lambda'}\}$. Henceforth, set $\gamma_o = \gamma$ to simplify the notation.

Since the scalar curvature of a Yamabe metric depends only on the conformal class of the metric, it then follows that, for all $h \in U$, one has

$$s'(h) = \int \langle -Z_{\phi}, h\rangle dV_{\gamma} = \int \langle -z, h\rangle dV_{\gamma}, \tag{2.27}$$

for all ϕ representing the Yamabe metrics in $\{\gamma^{\lambda'}\}$. In particular, (2.27) holds for at least one $\phi \neq 1$. The condition (2.27) is linear and since it holds for all $h \in U$, it must hold for all h in the linear span of U, and hence it holds for all (trace-free) $h \in T_{\gamma_o}\mathbb{M}$. This implies that one has pointwise

$$Z_{\phi} = z, \tag{2.28}$$

i.e.

$$u^{n-2}(z + (n-2)uD_0^2 u^{-1}) = z, \tag{2.29}$$

for some $\phi \neq 1$.

Thus to prove Theorem 1.1, it suffices to prove that (2.29) has only trivial solutions. This is done by a computation in the following Lemma.

Lemma 2.5. *For $(M, [\gamma]) \neq (S^n, [\gamma_0])$, the only solution to (2.29) is*

$$u = \phi = 1. \tag{2.30}$$

Proof: We first prove this in case $n > 3$ and afterwards prove the case $n = 3$. Write (2.29) as

$$u^{-(n-2)}(1 - u^{n-2})z = (n-2)u(D_0^2 u^{-1}), \tag{2.31}$$

where as above, $u = \phi^{2/(n-2)}$. For $\widetilde{g} = u^2 g$, as noted following Corollary 2.3,

$$\widetilde{z} = z + (n-2)uD_0^2 u^{-1}.$$

Combining this with (2.31) gives

$$z = \phi^2 \widetilde{z}. \tag{2.32}$$

The metrics $\widetilde{g}$ and g are Yamabe, and so by the Bianchi identity,

$$\delta_g(\phi^2 \widetilde{z}) = \phi^2 \delta_g \widetilde{z} - \widetilde{z}(\nabla_g \phi^2) = 0.$$

A simple computation shows that

$$\delta_g \widetilde{z} = u^2 \delta_{\widetilde{g}} \widetilde{z} - u^2 \widetilde{z}(\nabla_{\widetilde{g}} \log u^{-1}).$$

Since $\nabla_g \phi^2 = u^2 \nabla_{\widetilde{g}} \phi^2$, these equations and (2.32) give

$$\widetilde{z}(\nabla_{\widetilde{g}} \log u) = \widetilde{z}(\nabla_{\widetilde{g}} \log \phi^2).$$

Since $\phi^2 = u^{n-2}$, it follows that $\widetilde{z}(\nabla_{\widetilde{g}} u) = 0$, because $n > 3$. Interchanging the roles of g and $\widetilde{g}$ then gives

$$z(\nabla_g u) = 0. \tag{2.33}$$

To complete the proof when $n > 3$, return to (2.31). Let $f(u) = u^{-(n-1)}(1 - u^{n-2})$. Assuming $f(u)$ is not identically 0, divide (2.31) by $f(u)$ and pair it with $D^2 u^{-1}$ to obtain

$$\langle D^2 u^{-1}, z \rangle = \frac{(n-2)}{f(u)} \langle D^2 u^{-1}, D^2 u^{-1} - \frac{\Delta u^{-1}}{n} g \rangle. \tag{2.34}$$

The left side of (2.34) is smooth, and hence so is the right side. Let $U^+ = \{u \geq 1\}$. Applying the divergence theorem over the domain U^+ gives

$$\int_{U^+} \langle D^2 u^{-1}, z \rangle = \int_{U^+} \langle du^{-1}, \delta z \rangle + \int_{\partial U^+} z(du^{-1}, \nu) = 0,$$

where we have used the Bianchi identity and (2.33). Hence

$$\int_{U^+} \frac{n-2}{f(u)} \{ |D^2 u^{-1}|^2 - \frac{1}{n} (\Delta u^{-1})^2 \} = 0.$$

Since $f(u) \leq 0$ in U^+, the Cauchy-Schwarz inequality shows that the integrand here is pointwise non-positive, and so one must have

$$D^2 u^{-1} = \frac{\Delta u^{-1}}{n} g, \tag{2.35}$$

on (U^+, γ). The same arguments apply to the complementary region U^-, and so (2.35) holds on (M, γ). By (2.31), this of course implies $z = 0$, at least on the domain where $f(u) \neq 0$. Taking the divergence of (2.35) on this domain and using the identity $div D^2 \psi = d\Delta\psi + Ric(\nabla\psi)$ gives

$$d(\frac{n-1}{n} \Delta u^{-1} + \frac{s u^{-1}}{n}) = 0.$$

Since u is smooth on M, it follows that $\Delta u^{-1} + \frac{s}{n-1} u^{-1} = c$ on M, with $c = \frac{s}{n-1} \int u^{-1}$. Let $v = u^{-1} - \frac{n-1}{s} c$, so that $\int v = 0$, and $\Delta v = -\frac{s}{n-1} v$. It follows then from (2.35) that

$$D^2 v = -\frac{s}{n(n-1)} v g. \tag{2.36}$$

By Obata's theorem [5], it is well-known that the only non-zero solution to (2.36) is with v a 1^{st} eigenfunction of the Laplacian on the round metric γ_0 on S^n, (up to a constant rescaling). Since by assumption $(M, [\gamma]) \neq (S^n, [\gamma_0])$, it follows then that $v = 0$ and hence $u = 1$. This completes the proof in case $n > 3$.

For $n = 3$, a different argument must be used, since the proof of (2.33) does not hold in this case. Instead, return to (2.32), and recall that $\omega = d(r - \frac{s}{4} g)$ is a conformal invariant of weight 0 in dimension 3, cf. [2]; thus if $\widetilde{g} = u^2 g$, then $\widetilde{\omega} = \omega$. The metrics g and $\widetilde{g}$ are Yamabe, and so

$$\widetilde{dz} = dz.$$

On the other hand, by (2.32), one has

$$dz = d\phi^2 \wedge \widetilde{z} + \phi^2 d\widetilde{z},$$

and one easily computes from the definition that $d\widetilde{z} = \widetilde{dz}$. Hence $(1-\phi^2)dz = d\phi^2 \wedge z$. This implies that $d\phi^2 \wedge z = 0$ on the level set $\{\phi = u = 1\}$. Taking the trace of this $(2,1)$ form gives $z(du) = 0$ on $\{u = 1\}$. The remainder of the proof then follows exactly as following (2.34). ■

The proof of Theorem 1.1 now follows easily. Lemma 2.5 implies there are no non-trivial solutions of (2.29). Hence the open set $\mathcal{V}$ must be empty, which proves the result.

■

The proof of Theorem 1.2 is now also very simple.

Proof of Theorem 1.2.

Suppose that the scalar curvature function s has a local maximum on $\mathcal{Y}$ at γ, i.e. $s_{\gamma'} \leq s_{\gamma}$, for any unit volume Yamabe metric γ' near γ. Then by Proposition 2.4, the positive directional derivative $s'(h)$ exists for any h and

$$s'(h) \leq 0.$$

Consider then the collection of tangent vectors Z_ϕ in $T_\gamma\mathbb{M}$ with $Z_\phi = z$ for $\phi = 1$. Any h determines at least one Z_ϕ, (and maybe several), and (2.26) gives

$$\int \langle -Z_\phi, h\rangle dV_\gamma \leq 0.$$

Hence, each h must lie in the positive half-space of its corresponding Z_ϕ. However, if there are only two Yamabe metrics in $[\gamma]$, there are most two such Z_ϕ and the two corresponding half-spaces do not cover all of $T_\gamma\mathbb{M}$ unless

$$z = -Z_\phi.$$

From (2.23), this implies

$$u^{-1}(1 + u^{n-2})z = -(n-2)D_0^2 u^{-1}. \tag{2.37}$$

We now argue as in the proof of Lemma 2.5. Thus, pair (2.37) with z; integrating over M and using the Bianchi identity implies that $z = 0$, i.e γ is Einstein. ■

It would be of interest to know if this method of proof can be generalized to prove that any local maximum of s on $\mathcal{Y}$ is necessarily Einstein.

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