SINGULARITIES OF THE RICCI FLOW

MICHAEL T. ANDERSON

Department of Mathematics
S.U.N.Y. at Stony Brook
Stony Brook, N.Y. 11794-3651
E-mail: anderson@math.sunysb.edu

Introduction.

Fix a closed n-dimensional manifold M, and let $\mathbb{M}$ be the space of Riemannian metrics on M. Similar to the reasoning leading to the Einstein equations in general relativity, there is basically a unique simple and natural vector field on the space $\mathbb{M}$. Namely, the tangent space $T_g\mathbb{M}$ consists of symmetric bilinear forms; besides multiples of the metric itself, the Ricci curvature Ric_g of g is the only symmetric form depending on at most the 2nd derivatives of the metric, and invariant under coordinate changes, i.e. a $(0,2)$ tensor formed from the metric. Thus, consider

$$X_g = \mu Ric_g + \lambda g,$$

where μ, λ are scalars. Setting $\mu = -2$, the corresponding equation for the flow of X is

$$\frac{d}{dt}g(t) = -2Ric_{g(t)} + \lambda g(t). \tag{1}$$

The Ricci flow, introduced by R. Hamilton [5], is obtained by setting $\lambda = 0$:

$$\frac{d}{dt}g(t) = -2Ric_{g(t)}. \tag{2}$$

Rescaling the metric and time variable t transforms (2) into (1), with $\lambda = \lambda(t)$. For example, rescaling the Ricci flow (2) so that the volume of $(M, g(t))$ is preserved leads to the flow equation (1) with $\lambda = 2\oint R$, twice the mean value of the scalar curvature R

The Ricci flow (2) bears some relation with the metric part of the beta function or renormalization group flow equation

$$\frac{d}{dt}g(t) = \beta(g(t)),$$

for the 2-dimensional sigma model of maps $\Sigma^2 \to M$. The beta function is a vector field on $\mathbb{M}$, invariant under diffeomorphisms, which has an expansion of the form

$$-\beta(g) = Ric_g + \varepsilon Riem^2 + ...,$$

where $Riem^2$ is quadratic in the Riemann curvature tensor. The Ricci flow corresponds to the one loop term or semi-classical limit in the RG flow, cf. [3], [4].

Recently, G. Perelman [9]-[11] has developed new insights into the geometry of the Ricci flow which has led to a solution of long-standing mathematical conjectures on the structure of 3-manifolds, namely the Thurston Geometrization Conjecture [12], and hence the Poincaré Conjecture.

Basic Properties of the Ricci Flow.

In charts where the coordinate functions are locally defined harmonic functions in the metric $g(t)$, (2) takes the form

$$\frac{d}{dt}g_{ij} = \Delta g_{ij} + Q_{ij}(g, \partial g),$$

where Δ is the Laplace operator on functions with respect to the metric $g = g(t)$ and Q is a lower order term quadratic in g and its first order partial derivatives. This is a nonlinear heat-type equation for g_{ij} and leads to the existence and uniqueness of solutions to the Ricci flow on some time interval starting at any smooth initial metric. This is the reason for the minus sign in (2); a plus sign gives a backwards heat-type equation, which has no solutions in general.

The flow (2) gives a natural method to try to construct canonical metrics on the manifold M. Stationary points of the flow (2) are Ricci-flat metrics, while stationary points of the flow (1) are (Riemannian) Einstein metrics, where $Ric_g = \frac{R}{n}g$, with R the scalar curvature of g. One of Hamilton's motivations for studying the Ricci flow were results on an analogous question for nonlinear sigma models. Consider maps f between Riemannian manifolds M, N with Lagrangian given by the Dirichlet energy. Eells-Sampson studied the heat equation for this action and proved that when the target N has non-positive curvature, the flow exists for all time and converges to a stationary point of the action, i.e. a harmonic map $f_\infty : M \to N$. The idea is to see if an analogous program can be developed on the space of metrics $\mathbb{M}$.

There are a number of well-known obstructions to the existence of Einstein metrics on manifolds, in particular in dimensions 3 and 4. Thus, the Ricci flow will not exist for all time on a general manifold. Hence, it must develop singularities. A fundamental issue is to try to relate the structure of the singularities of the flow with the topology of the underlying manifold M.

A few simple qualitative features of the Ricci flow (2) are as follows: if $Ric(x,t) > 0$, then the flow contracts the metric $g(t)$ near x, to the future, while if $Ric(x,t) < 0$, then the flow expands $g(t)$ near x. At a general point, there will be directions of positive and negative Ricci curvature, along which the metric locally contracts or expands. The flow preserves product structures of metrics, and preserves the isometry group of the initial metric.

The form of (2) shows that the Ricci flow continues as long as Ricci curvature remains bounded. On a bounded time interval where $Ric_{g(t)}$ is bounded, the metrics $g(t)$ are quasi-isometric, i.e. have bounded distortion compared with the initial metric $g(0)$. Thus, one needs to consider evolution equations for the curvature, induced by the flow for the metric. The simplest of these is the evolution equation for the scalar curvature R:

$$\frac{d}{dt}R = \Delta R + 2|Ric|^2. \tag{3}$$

Evaluating (3) at a point realizing the minimum R_{min} of R on M shows that R_{min} is monotone non-decreasing along the flow. In particular, the Ricci flow preserves positive scalar curvature. Moreover, if $R_{min}(0) > 0$, then

$$t \leq \frac{n}{2R_{min}(0)}. \tag{4}$$

Thus, the Ricci flow exists only up to a maximal time $T \leq n/2R_{min}(0)$ when $R_{min}(0) > 0$. In contrast, in regions where the Ricci curvature stays negative definite, the flow exists for infinite time.

The evolution of the Ricci curvature has the same general form as (3):

$$\frac{d}{dt}R_{ij} = \Delta R_{ij} + \widetilde{Q}_{ij}. \tag{5}$$

The expression for $\widetilde{Q}$ is much more complicated than the Ricci curvature term in (3) but involves only quadratic expressions in the curvature. However, $\widetilde{Q}$ involves the full Riemann curvature tensor $Riem$ of g, and not just the Ricci curvature, (as (3) involves Ricci and not just scalar curvature). An important feature of dimension 3 is that the full Riemann curvature $Riem$ is determined algebraically by the Ricci curvature. So the Ricci flow has a much better chance of "working" in dimension 3. For example, an analysis of $\widetilde{Q}$ shows that the Ricci flow preserves positive Ricci curvature in dimension 3; if $Ric_{g(0)} > 0$, then $Ric_{g(t)} > 0$, for $t > 0$. This is not the case in higher dimensions. On the other hand, in any dimension > 2, the Ricci flow does not preserve negative Ricci curvature, or even a general lower bound $Ric \geq -\lambda$, for $\lambda > 0$. For the remainder of the article, we usually assume then that $dim M = 3$.

The first basic result on the Ricci flow is the following, due to Hamilton [5].

• *Space-form Theorem.* If $g(0)$ is a metric of positive Ricci curvature on a 3-manifold M, then the volume normalized Ricci flow exists for all time, and converges to the round metric on S^3/Γ, where Γ is a finite subgroup of $SO(4)$, acting freely on S^3.

Thus the Ricci flow "geometrizes" 3-manifolds of positive Ricci curvature.

There are two further important structural results on the Ricci flow.

• *Curvature pinching estimate*, [7], [8]. For $g(t)$ a solution to the Ricci flow on a closed 3-manifold M, there is a non-increasing function $\phi : (-\infty, \infty) \rightarrow$

$\mathbb{R}$, tending to 0 at ∞, and a constant C, depending only on $g(0)$, such that,

$$Riem(x,t) \geq -C - \phi(R(x,t)) \cdot |R(x,t)|. \tag{6}$$

This estimate does not imply a lower bound on $Riem(x,t)$ uniform in time. However, when combined with the fact that the scalar curvature $R(x,t)$ is uniformly bounded below, (cf. (3)), it implies that $|Riem|(x,t) >> 1$ only where $R(x,t) >> 1$. To control the size of $|Riem|$, it thus suffices to obtain just an upper bound on R. This is remarkable, since the scalar curvature is a much weaker invariant of the metric than the full curvature. Moreover, at points where the curvature is sufficiently large, (6) shows that $Riem(x,t)/R(x,t) \geq -\delta$, for δ small. Thus, if one scales the metric to make $R(x,t) = 1$, then $Riem(x,t) \geq -\delta$. In such a scale, the metric then has almost non-negative curvature near (x,t).

• *Harnack estimate*, [6]. Let $(N, g(t))$ be a solution to the Ricci flow with bounded and non-negative curvature $Riem \geq 0$, and suppose $g(t)$ is a complete Riemannian metric on N. Then for $0 < t_1 \leq t_2$,

$$R(x_2,t_2) \geq \frac{t_1}{t_2} \exp\left(-\frac{d_{t_1}^2(x_1,x_2)}{2(t_2-t_1)}\right) R(x_1,t_1), \tag{7}$$

where d_{t_1} is the distance function on (M, g_{t_1}). This allows one to control the geometry of the solution at different space-time points, given control at an initial point.

Singularity Formation.

The deeper analysis of the Ricci flow is concerned with the singularities that arise in finite time. Equation (3) shows that the Ricci flow will not exist for arbitrarily long time in general. In the case of initial metrics with positive Ricci curvature, this is resolved by rescaling the Ricci flow to constant volume. However, the general situation is necessarily much more complicated. For example, any manifold which is a connected sum of S^3/Γ or $S^2 \times S^1$ factors has metrics of positive scalar curvature. For obvious topological reasons, the volume normalized Ricci flow then cannot converge nicely to a round metric; even the renormalized flow must develop singularities.

The usual method to understand the structure of singularities, particularly in geometric PDE's, is to rescale or renormalize the solution on a sequence converging to the singularity to make the solution bounded, and try to pass to a limit of the renormalization. Such a limit solution models the singularity formation, and one hopes (or expects), that the singularity models have special features making them much simpler than an arbitrary solution of the flow.

A singularity forms for the Ricci flow only where the curvature becomes unbounded. Suppose then that $\lambda_i^2 = |Riem|(x_i,t_i) \to \infty$, on a sequence of points $x_i \in M$, and times $t_i \to T < \infty$. Consider the rescaled or blow-up metrics and times

$$\bar{g}_i(\bar{t}_i) = \lambda_i^2 \phi_i^* g(t), \quad \bar{t}_i = \lambda_i^2 (t - t_i), \tag{8}$$

where ϕ_i are diffeomorphisms giving local dilations of the manifold near x_i by the factor λ_i.

The flow $\bar{g}_i$ is also a solution of the Ricci flow, and has bounded curvature at $(x_i, 0)$. For suitable choices of x_i and t_i, the curvature will be bounded near x_i, and for nearby times to the past, $\bar{t}_i \leq 0$; for example one might choose points (x_i, t_i) where the curvature is maximal on $(M, g(t))$, $t \leq t_i$.

The rescaling (8) expands all distances by the factor λ_i, and time by the factor λ_i^2. Thus, in effect one is studying very small regions, of spatial size on the order of $r_i = \lambda_i^{-1}$ about (x_i, t_i), and "using a microscope" to examine the small-scale features in this region on a scale of size about 1.

A limit solution of the Ricci flow, defined at least locally in space and time, will exist provided that the local volumes of the rescalings are bounded below, (Gromov compactness). In terms of the original unscaled flow, this requires that the metric $g(t)$ should not be locally collapsed on the scale of its curvature, i.e.

$$vol B_{x_i}(r_i, t_i) \geq \nu r_i^n, \tag{9}$$

for some fixed but arbitrary $\nu > 0$. A maximal connected limit $(N, \bar{g}(\bar{t}), x)$ containing the base point $x = \lim x_i$, is then called a *singularity model*. Observe that the topology of the limit N may well be distinct from the original manifold M, most of which may have been blown off to infinity in the rescaling.

To see the potential usefulness of this process, suppose one does have local noncollapse on the scale of the curvature, and that base points of maximal curvature in space and time $t \leq t_i$ have been chosen. At least in a subsequence, one then obtains a limit solution to the Ricci flow $(N, \bar{g}(\bar{t}), x)$, based at x, defined at least for times $(-\infty, 0]$, with $\bar{g}(\bar{t})$ a complete Riemannian metric on N. Such solutions are called ancient solutions of the Ricci flow. The estimate (6) shows that the limit has non-negative curvature in dimension 3, and so (7) holds on N. Thus the limit is indeed quite special. The topology of complete manifolds N of non-negative curvature is completely understood in dimension 3. If N is non-compact, then N is diffeomorphic to $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ or a quotient of these spaces. If N is compact, then a slightly stronger form of the space-form theorem implies N is diffeomorphic to S^3/Γ, $S^2 \times S^1$ or $S^2 \times_{\mathbb{Z}_2} S^1$.

The study of the formation of singularities in the Ricci flow was initiated by Hamilton in [7]. Recently, Perelman has obtained an essentially complete understanding of the singularity behavior of the Ricci flow, at least in dimension 3.

Perelman's Work.
I. Non-Collapse. Consider the Einstein-Hilbert action

$$\mathcal{R}(g) = \int_M R(g) dV_g, \tag{10}$$

as a functional on $\mathbb{M}$. Critical points of $\mathcal{R}$ are Ricci-flat metrics. It is natural and tempting to try to relate the Ricci flow with the gradient flow of $\mathcal{R}$, (with respect to a natural L^2 metric on the space $\mathbb{M}$). However, it has long been recognized that this cannot be done directly. In fact, the gradient flow of $\mathcal{R}$ does not even exist, since it implies a backwards heat-type equation for the scalar curvature R, (similar to (3) but with a minus sign before Δ).

Consider however the following functional extending $\mathcal{R}$,

$$\mathcal{F}(g,f) = \int_M (R + |\nabla f|^2) e^{-f} dV_g. \tag{11}$$

as a functional on the larger space $\mathbb{M} \times C^\infty(M, \mathbb{R})$, or equivalently a family of functionals on $\mathbb{M}$, parametrized by $C^\infty(M, \mathbb{R})$. The functional (11) also arises in string theory as the low energy effective action; the scalar field f is called the dilaton. Fix any smooth measure dm on M and define the Perelman coupling by requiring that (g, f) satisfy

$$e^{-f} dV_g = dm. \tag{12}$$

The resulting functional

$$\mathcal{F}^m(g,f) = \int_M (R + |\nabla f|^2) dm, \tag{13}$$

becomes a functional on $\mathbb{M}$. (This coupling does not appear to have been considered in string theory). The L^2 gradient flow of $\mathcal{F}^m$ is given simply by

$$\frac{d\widetilde{g}}{dt} = -2(Ric_{\widetilde{g}} + \widetilde{D}^2 f), \tag{14}$$

where $\widetilde{D}^2 f$ is the Hessian of f with respect to $\widetilde{g}$. The evolution equation (14) for $\widetilde{g}$ is just the Ricci flow (2) modified by an infinitesimal diffeomorphism: $\widetilde{D}^2 f = (d/dt)(\phi_t^* \widetilde{g})$, where $(d/dt)\phi_t = \widetilde{\nabla} f$. Thus, the gradient flow of $\mathcal{F}^m$ is the Ricci flow, up to diffeomorphisms. The evolution equation for the scalar field f,

$$f_t = -\widetilde{\Delta} f - \widetilde{R} \tag{15}$$

is a backward heat equation, (balancing the forward evolution of the volume form of $\widetilde{g}(t)$). Thus, this flow will not exist for general f, going forward in t. However, one of the basic points of view is to let the (pure) Ricci flow (2) flow for a time $t_0 > 0$. At t_0, one may then take an arbitrary $f = f(t_0)$ and flow this f backward in time ($\tau = t_0 - t$) to obtain an initial value $f(0)$ for f. The choice of $f(t_0)$ determines, together with the choice of volume form of $g(0)$), (or $g(t_0)$), the measure dm and so the choice of $\mathcal{F}^m$. The process of passing from $\mathcal{F}$ to $\mathcal{F}^m$ corresponds to a reduction of the symmetry group of all diffeomorphisms $\mathcal{D}$ of $\mathcal{F}$ to the group $\mathcal{D}_0$ of volume preserving diffeomorphisms; the quotient space $\mathcal{D}/\mathcal{D}_0$ has been decoupled into a space $C^\infty(M, \mathbb{R})$ of parameters.

The functionals $\mathcal{F}^m$ are not scale-invariant. To achieve scale-invariance, Perelman includes an explicit insertion of the scale parameter, related to time, by setting

$$\mathcal{W}(g,f,\tau) = \int [\tau(|\nabla f|^2 + R) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV, \tag{16}$$

with coupling so that $dm = (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$ is fixed. The entropy functional $\mathcal{W}$ is invariant under simultaneous rescaling of τ and g, and $\tau_t = -1$. Again the gradient flow of $\mathcal{W}$ is the Ricci flow modulo diffeomorphisms and rescalings and the stationary points of the gradient flow are the gradient Ricci solitons,

$$Ric_g + D^2 f - \frac{1}{2\tau} g = 0,$$

for which the metrics evolve by diffeomorphisms and rescalings. Gradient solitons arise naturally as singularity models, due to the rescalings and diffeomorphisms in the blow-up procedure (8). An important example is the cigar soliton on $\mathbb{R}^2 \times \mathbb{R}$, (or $\mathbb{R}^2 \times S^1$),

$$g = (1 + r^2)^{-1} g_{Eucl} + ds^2. \tag{17}$$

Perelman then uses the scalar field f to probe the geometry of $g(t)$. For instance, the collapse or noncollapse of the metric $g(t)$ near a point $x \in M$ can be detected from the size of $\mathcal{W}(g(t))$ by choosing e^{-f} to be an approximation to a delta function centered at (x,t). The more collapsed $g(t)$ is near x, the more negative the value of $\mathcal{W}(g(t))$. The collapse of the metric $g(t)$ on any scale in finite time is then ruled out by combining this with the fact that the entropy functional $\mathcal{W}$ is increasing along the Ricci flow.

Much more detailed information can be obtained by studying the path integral associated to the evolution equation (15) for f, given by

$$\mathcal{L}(\gamma) = \int_\gamma \sqrt{\tau}[|\dot{\gamma}(\tau)|^2 + R(\gamma(\tau))]d\tau,$$

where R and $|\dot{\gamma}(\tau)|$ are computed with respect to the evolving metrics $g(\tau)$. In particular, the study of the geodesics and the associated variational theory of the length functional $\mathcal{L}$ are important in understanding the geometry of the Ricci flow near the singularities.

II. Singularity Models.

A major accomplishment of [9] is essentially a classification of all complete singularity models $(N, g(t))$ that arise in finite time. In the simple case where N is compact, then as noted above, N is diffeomorphic to S^3/Γ, $S^2 \times S^1$ or $S^2 \times_{\mathbb{Z}_2} S^1$.

In the much more important case where N is complete and non-compact, Perelman proves that the geometry of N near infinity is that of a union of ε-necks. Thus, at time 0, and at points x with $r(x) = dist(x, x_0) >> 1$, for a fixed base point x_0, a region of radius ε^{-1} about x, in the scale where $R(x) = 1$, is ε-close to such a region in the standard round product metric on $S^2 \times \mathbb{R}$; ε may be made arbitrarily small by choosing $r(x)$ sufficiently large. For example, this shows that the cigar soliton (17) cannot arise as a singularity model. Moreover, this structure also holds on a time interval on the order of ε^{-1} to the past, so that on such regions the solution is close to the (backwards) evolving Ricci flow on $S^2 \times \mathbb{R}$.

Perelman shows that this structural result for the singularity models themselves also holds for the solution $g(t)$ very near any singularity time T. Thus, at *any* base point (x,t) where the curvature is sufficiently large, the rescaling as in (8) of the space-time by the curvature is smoothly close, on large compact domains, to corresponding large domains in a complete singularity model. The "ideal" complete singularity models do actually describe the geometry and topology near any singularity. Consequently, one has a detailed understanding of the small scale geometry and topology in a neighborhood of every point where the curvature is large on $(M, g(t))$, for t near T.

The main consequence of this analysis is the existence of canonical, almost round 2-spheres S^2 in any region of $(M, g(t))$ where the curvature is sufficiently large; the radius of the S^2's is on the order of the curvature radius. One then disconnects the manifold M into pieces, by cutting M along a judicious choice of such 2-spheres, and glueing in round 3-balls in a natural way. This surgery process allows one to excise out the regions of $(M, g(t))$ where the Ricci flow is almost singular, and thus leads to a naturally defined Ricci flow with surgery, valid for all times $t \in [0, \infty)$.

The surgery process disconnects the original connected 3-manifold M into a collection of disjoint (connected) 3-manifolds M_i, with the Ricci flow running on each. However, topologically, there is a canonical relation between M and the components M_i; M is the connected sum of $\{M_i\}$. An analysis of the long-time behavior of the volume normalized Ricci flow confirms the expectation that the flow approaches a fixed point, i.e. an Einstein metric, or collapses along 3-manifolds admitting an S^1 fibration. This then leads to the proof of Thurston's Geometrization Conjecture for 3-manifolds and consequently the proof of the Poincaré Conjecture. It gives a full classification of all closed 3-manifolds, much like the classification of surfaces given by the classical uniformization theorem.

Further Reading

[1] Anderson, M. Scalar curvature and geometrization conjectures for 3–manifolds, in: Comparison Geometry, (Ed. K. Grove and P. Petersen), MSRI Publications, **30**, (1997), 49-82.

[2] Collected Papers on Ricci Flow, Ed. H.T. Cao, B. Chow, S.C. Cheng and S.T. Yau, International Press, Boston, (2003).

[3] D'Hoker, E. String theory, in: Quantum Fields and Strings: A Course for Mathematicians, Vol. 2, Amer. Math. Soc., Providence, RI, (1999).

[4] Friedan, D. Nonlinear models in $2+\varepsilon$ dimensions, Annals of Physics, **163**, (1985), 318-419.

[5] Hamilton, R. Three manifolds of positive Ricci curvature, Jour. Diff. Geom. **17**, (1982), 255-306.

[6] Hamilton, R. The Harnack estimate for the Ricci flow, Jour. Diff. Geom. **37**, (1993), 225-243.

[7] Hamilton, R. Formation of singularities in the Ricci flow, in Surveys in Differential Geometry, Vol. 2, International Press, (1995), 7-136.

[8] Ivey, T. Ricci solitons on compact three-manifolds, Diff. Geom. and Appl. **3**, (1993), 301-307.

[9] Perelman, G. The entropy formula for the Ricci flow and its geometric applications, math.DG/0211159.

[10] Perelman, G. Ricci flow with surgery on three-manifolds, math.DG/0303109.

[11] Perelman, G. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, math.DG/0307245.

[12] Thurston, W. Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. **6**, (1982), 357-381.