ASYMPTOTIC BEHAVIOR OF STATIC AND STATIONARY VACUUM SPACE-TIMES

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Let (M,g) be an Einstein vacuum chronological space-time. The space-time is stationary if it admits a time-like Killing field $X = \partial/\partial t$, and static if the Killing field is hypersurface orthogonal. Let $u = \sqrt{-g(X,X)} > 0$ and let $\pi : (M,g) \to (S, g_S)$ be the projection to the orbit space of the $\mathbf{R}$ -action generated by X. We recall the following classical and well-known results of Lichnerowicz [1,§85, §90], c.f. however [2] for an earlier version.

Lichnerowicz Theorems. (I). Suppose (M,g) is a geodesically complete static vacuum space-time, with $u(x) \to 1$, as $x \to \infty$ in S. Then (M,g) is flat.

(II). Suppose (M,g) is a geodesically complete stationary vacuum space-time which is asymptotically flat, (AF). Then (M,g) is flat.

Physically, these results appear quite satisfying. If the space-time is vacuum and geodesically complete, then there are no "matter or energy sources" contained in the space-time. Under the asymptotic conditions at infinity, one then concludes that the gravitational field is empty. However one might question why it is necessary to impose *any* asymptotic conditions on the field. (If there is no "source", how could a non-trivial stationary gravitational field be produced under any conditions?).

The assumption that a space-time be AF occurs throughout general relativity and is understood to model the asymptotic behavior of a non-trivial field far from compact energy sources. While physically perhaps reasonable, this assumption is mathematically very strong, imposing stringent requirements on both the topology and the metric at infinity.

Further, the physical reasoning above, if rigorous, would make the Lichnerowicz theorems above tautological. In fact, it would be invalid if one could produce a geodesically complete stationary vacuum space-time (M_∞, g_∞) which is non-empty, (i.e. not flat). Then the condition that a space-time with non-trivial sources be asymptotic to (M_∞, g_∞) at infinity would be equally as valid as the condition that the space-time be AF. Apriori, it is not at all clear why the curvature of a stationary vacuum space-time with compact source region should have curvature decaying *at all*, or anywhere, at infinity.

It is often thought that in order to obtain unique solutions to the vacuum Einstein equations, one must impose boundary conditions at infinity, (as in the Lichnerowicz theorems). This is clearly the case for many elementary physical theories, e.g. electrostatics. However, such boundary conditions at infinity are essentially adhoc, and at a more fundamental level should be derived from the theory itself, and not imposed. This issue was in fact of concern to Einstein, c.f. [3, pp98-108], and is related to versions of Mach's Principle. This discussion serves as some motivation for the following result.

Theorem 1. If (M,g) is a geodesically complete chronological stationary vacuum space-time, then (M,g) is flat.

Thus, the asymptotic conditions in the Lichnerowicz theorems are in fact unnecessary. Given such a global result, one may then use it to obtain apriori estimates for the local behavior of stationary vacuum solutions. Thus, suppose (M,g) and (S,g_S) as above are maximal and let ∂S denote the metric or Cauchy boundary of S w.r.t. g_S. Theorem 1 implies that $\partial S \neq \emptyset$; at ∂S either the metric g_S degenerates or the Killing field X turns null, (or both).

Theorem 2. Let (M,g) be a chronological stationary vacuum space-time. Then there is a $K < \infty$, independent of (M,g), such that for $r(x) = dist(\pi(x), \partial S)$,

$$|R_M|(x) \leq K/r^2(x), \quad \text{and} \quad |\nabla logu| \leq K/r(x).$$

In turn, this result may be used to analyse the apriori *possible* asymptotic behavior of (incomplete) stationary vacuum solutions. Define ∂S to be *pseudo-compact* if $r^{-1}(s) \subset S$ is compact, for some $s > 0$. We point out that there are numerous static or stationary vacuum solutions (for example, the Curzon solution) which are pseudo-compact in this sense, but for which ∂S is *not* compact.

Theorem 3. Let (M,g) be a static vacuum space-time over (S,g_S), with pseudo-compact boundary. Then S has a finite number of ends E_i. Each end E of S on which $liminf_{x\to\infty} u(x) > 0$ is either AF or *small*, in the sense that

$$\int^{\infty} (areaS_E(r))^{-1} dr = \infty,$$

where $S_E(r)$ is the geodesic r-sphere in (E, g_S) about some base point. Further, E is necessarily AF under the (physically reasonable) conditions that $sup_E u < \infty$ and

$$m_E = lim_{r\to\infty} m_E(r) = lim_{r\to\infty} \frac{1}{4\pi} \int_{S_E(r)} < \nabla logu, \nabla r > dA \neq 0.$$

Static vacuum solutions with small ends do exist, for instance the Kasner static solutions, but they have strongly restricted geometry. Their area growth of spheres is on the order of at most $r \cdot (logr)^{1+\varepsilon}$, $\varepsilon > 0$, which is markedly different from that of AF ends. Typically at infinity, they are topologically $\mathbf{R}^2 \times S^1$, (in place of $\mathbf{R}^3$), and the asymptotic geometry is that of (collapsed) Weyl solutions.

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