BOUNDARY VALUE PROBLEMS FOR METRICS ON 3-MANIFOLDS

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Abstract. We discuss the problem of prescribing the mean curvature and conformal class as boundary data for Einstein metrics on 3-manifolds, in the context of natural elliptic boundary value problems for Riemannian metrics.

1. Introduction

A question long of basic interest to geometers is the existence of complete Einstein metrics on manifolds. Any kind of theory for the existence or uniqueness of such metrics on compact manifolds is still far from sight. The only exception to this is the remarkable work of Perelman and Hamilton, which essentially gives a complete theory for closed 3-manifolds.

Instead of considering closed manifolds, it might be somewhat simpler to consider manifolds with boundary and look for a theory providing existence (and uniqueness) for geometrically natural boundary value problems. This has recently met with some success, in the context of complete conformally compact Einstein metrics, where one prescribes a conformal metric at conformal infinity [3], and in the context of a natural exterior boundary value problem for the static vacuum Einstein equations, [4].

In this note, we consider the simplest situation, namely boundary value problems for Einstein metrics in dimension 3, where the metrics are of constant curvature. Seemingly the simplest or most naive question one could ask in this context is the following:

Question. Given a metric γ on a boundary surface $\partial M = S^2$ for instance, is there an Einstein metric (flat or constant curvature) on the 3-ball $M = B^3$ inducing γ on ∂M?

However, this is basically the isometric immersion problem for surfaces in $\mathbb{R}^3$, (or other space-forms), and is a notoriously difficult problem, also far from any current resolution. Note however that there are examples of smooth metrics on S^2 which do not isometrically immerse in $\mathbb{R}^3$, cf. [15], so the answer to the question is no in general.

The main difficulty here is that although the Einstein equations form an elliptic system of equations in a suitable gauge, Dirichlet boundary data for such a system never give rise to an elliptic boundary value problem. The Gauss constraint equation, (Gauss' Theorema Egregium), is an obstruction to such ellipticity. Thus, one should first consider what are the natural boundary value problems for the Einstein equations.

To describe this, let M be any 3-manifold with boundary ∂M which admits a metric of constant sectional curvature κ. We assume that

$$\pi_1(M, \partial M) = 0;$$

by elementary covering space arguments, this means that ∂M is connected and any loop in M is homotopic to a loop in ∂M, so that M is a 3-dimensional handlebody.

Let $\mathcal{M}_\kappa = \mathcal{M}_\kappa^{m,\alpha}$ be the moduli space of metrics of constant curvature κ on M which are $C^{m,\alpha}$ up to ∂M, $m \geq 2$, $\alpha \in (0,1)$. This is the space of all such constant curvature metrics $\mathbb{M}_\kappa$ modulo

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the action of $\mathrm{Diff}_1^{m+1,\alpha}$ of diffeomorphisms of M equal to the identity on ∂M. In the case of $\kappa = 0$ for instance, the developing map gives an isometric immersion

$$D : (M, g) \to \mathbb{R}^3,$$

which induces an isometric Alexandrov immersion of $(\partial M, \gamma)$ into $\mathbb{R}^3$, where $\gamma = g_{\partial M}$. (An immersion of a surface in $\mathbb{R}^3$ is Alexandrov if it extends to an immersion of the bounding handlebody M). Similar remarks hold for all $\kappa \in \mathbb{R}$.

Let $\mathcal{C}^{m.\alpha}$ denote the space of conformal classes $[\gamma]$ of $C^{m,\alpha}$ metrics γ on ∂M, and let H denote the mean curvature of $\partial M \subset (M, g)$, with respect to the outward unit normal. It is proved in [5] that the moduli space $\mathcal{M}_\kappa = \mathcal{M}_\kappa^{m,\alpha}$ is a (C^∞) smooth Banach manifold. Moreover, setting as above $\gamma = g|_{T(\partial M)}$, the map

$$\Pi : \mathcal{M}_\kappa \to \mathcal{C}^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M), \tag{1.1}$$
$$\Pi([g]) = ([\gamma],\ H),$$

is a (C^∞) smooth Fredholm map, of Fredholm index 0. In fact the boundary data in (1.1) form elliptic boundary data for the Einstein equations. There are other elliptic boundary value problems for Einstein metrics, some of which are discussed in [5]. However, the data in (1.1) is geometrically the most natural so we restrict the discussion to this case.

It follows in particular that $Im\Pi$ is a variety of finite codimension in $\mathcal{C}^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M)$. One would expect that generic metrics in $\mathcal{M}_\kappa$ are regular points for Π, in which case $Im\Pi$ would at least contain open domains in the target $\mathcal{C}^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M)$; (a proof of this is still lacking however).

The result in (1.1) shows that one has a good local existence theory for this boundary value problem and it raises the global problem:

Question. Given $([\gamma], H) \in \mathcal{C}^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M)$, (possibly with some restrictions), does there exist a unique metric, (up to isometry), g on M such that

$$\Pi(g) = ([\gamma], H). \tag{1.2}$$

To the author's knowledge, it does not seem that this question, although clearly quite natural, has been studied previously. There are many previous works on the existence of surfaces of prescribed mean curvature in $\mathbb{R}^3$ for instance, cf. [18] for example, and [19] or [7] for further discussion and references. However, in these situations H is a given function on $\mathbb{R}^3$; moreover, there is no prescription of the conformal class. In the case of $\kappa = 0$ for example, the question can be rephrased as the question of the existence and uniqueness of an Alexandrov immersion of a surface $F : \Sigma = \partial M \to \mathbb{R}^3$ with prescribed conformal class $[\gamma]$ and prescribed mean curvature H, i.e.

$$[F^*(g_{Eucl})] = [\gamma], \quad H(F(x)) = H(x). \tag{1.3}$$

Note that the diffeomorphism group of ∂M acts non-trivially on both parts $[\gamma]$ and H of the boundary data: if $\varphi \in \mathrm{Diff}(\partial M)$, then the immersion $F \circ \varphi$ has the same image as F, but is a reparametrization of F. For different but related studies on surfaces of prescribed mean curvature, see for example [9], [11] and [12].

To address a global question as above, the basic issue is whether the boundary map Π is proper. In analogy to the simpler method of continuity commonly used in PDE, this is the closedness issue; one requires apriori estimates or compactness properties for spaces of solutions. For very simple reasons, the map Π is not proper in general, and one first needs to sharpen the problem to account for this. Thus, for example smooth bounded domains in $\mathbb{R}^3$ may degenerate from the "inside", in that the injectivity radius within M may go to 0 near ∂M, causing the boundary to develop self-intersections and the domain M is no longer a manifold. This behavior can be ruled out via the maximum principle, (see also Lemma 2.4 below), under the assumption that $H > 0$. Thus, let

$$\mathcal{M}_\kappa^+ = \Pi^{-1}(\mathcal{C}^{m,\alpha} \times C_+^{m-1,\alpha}),$$

be the space of constant curvature metrics with $H > 0$ at ∂M. Clearly, $\mathcal{M}_\kappa^+$ is an open submanifold of $\mathcal{M}_\kappa$ and one may consider the associated (restricted) boundary map

$$\Pi_+ : \mathcal{M}_\kappa^+ \to \mathcal{C}^{m,\alpha} \times C_+^{m-1,\alpha}. \tag{1.4}$$

Next, recall by the uniformization theorem that the space of metrics $Met(S^2)$ on S^2 equals $\mathrm{Diff}(S^2) \times C_+(S^2)$; any metric γ is of the form $\gamma = \varphi^*(\lambda^2\gamma_{+1})$, where γ_{+1} is the round metric of radius 1. (Similarly for surfaces Σ of higher genus, $Met(\Sigma)$ is a bundle over the Riemann moduli space with fiber $\mathrm{Diff}(\Sigma) \times C_+(\Sigma)$). The group $\mathrm{Diff}(S^2) \times C_+(S^2)$ thus acts transitively on $Met(S^2)$ and has stabilizer at γ_{+1} equal to the group of essential conformal transformations $\mathrm{Conf}(S^2)$ of $S^2(1)$. It follows that one has a natural identification

$$\mathcal{C}^{m,\alpha}(S^2) \simeq \mathrm{Diff}^{m+1,\alpha}(S^2)/\,\mathrm{Conf}(S^2). \tag{1.5}$$

The conformal group also acts on the space $C_+^{m-1,\alpha}$ of mean curvature functions: $H \to H \circ \varphi$, for $\varphi \in \mathrm{Conf}(S^2)$. It is easy to verify that this action is free and proper, except on the functions $H = const$, which are the fixed points of the action; this is because the flow of the conformal vector fields contracts or expands all of $S^2 \setminus \{pt\}$ to a point.

It follows then that at the special values $([\gamma], c)$ where $H = c$, the map Π_+ in (1.4) is not proper. The non-compact conformal group $\mathrm{Conf}(S^2)$ fixes this data, but acts nontrivially (and faithfully) on $\mathcal{M}_\kappa^+$; if $\Pi_+(g) = ([\gamma], c)$, then also $\Pi_+(\varphi^*(g)) = ([\gamma], c)$, for any $\varphi \in \mathrm{Conf}(S^2)$ extended to a diffeomorphism of M. On the other hand, this is the only value where $\mathrm{Conf}(S^2)$ acts non-properly. (Note this issue arises only for S^2, not for boundaries of higher genus).

There are two ways to deal with this issue. First, one may just study the behavior of Π_+ away from the "round" metrics $H = c$, i.e. consider the global behavior of the map

$$\Pi' : \mathcal{M}_\kappa' \to \mathcal{C}^{m,\alpha} \times (C_+^{m-1,\alpha})', \tag{1.6}$$

where $(C_+^{m-1,\alpha})' = C_+^{m-1,\alpha} \setminus \{constants\}$ and $\mathcal{M}_\kappa' = \Pi_+^{-1}((C_+^{m-1,\alpha})')$. This map is again smooth and Fredholm, of index 0.

Alternately, one may include the round metrics, but divide out by the action of $\mathrm{Conf}(S^2)$. Briefly, as is standard, choose a fixed marking to freeze the action of the conformal group on S^2. Thus, fix three points p_i, $i = 1, 2, 3$ on $S^2(1)$ with

$$dist_{\gamma_{+1}}(p_i, p_j) = \pi/2. \tag{1.7}$$

Let $\mathcal{N}_\kappa^+$ be the marked moduli space of constant curvature metrics on $M = B^3$ consisting of metrics g such that (1.7) holds on $\partial M = S^2$ with $\gamma = g|_{\partial M}$ in place of γ_{+1}. The condition (1.7) can always be realized by changing g by a conformal diffeomorphism, so that

$$\mathcal{N}_\kappa^+ \simeq \mathcal{M}_\kappa^+/\,\mathrm{Conf}(S^2),$$

the condition (1.7) giving a slice for the action of $\mathrm{Conf}(S^2)$ on $\mathcal{M}_\kappa$. (Of course this marking is not necessary in case $\chi(\partial M) \leq 0$ so that $\mathcal{N}_\kappa^+ \simeq \mathcal{M}_\kappa^+$ in such cases). The map Π_+ in (1.4) clearly restricts under the slicing (1.7) to a smooth map

$$\Pi_+ : \mathcal{N}_\kappa^+ \to \mathcal{C}^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M). \tag{1.8}$$

However, the index of this map is now -3, since $\mathrm{Conf}(S^2)$ is 3-dimensional and so one must also divide the target space by the remaining action of $\mathrm{Conf}(S^2)$ on the space $C_+^{m-1,\alpha}$ of mean curvature functions. Let then $\mathcal{B}[\mathcal{C}^{m,\alpha} \times C_+^{m-1,\alpha}]$ be the quotient space

$$\mathcal{B}[\mathcal{C}^{m,\alpha} \times C_+^{m-1,\alpha}] = [\mathcal{C}^{m,\alpha} \times C_+^{m-1,\alpha}]/\operatorname{Conf}(S^2) = \mathcal{C}^{m,\alpha} \times (C_+^{m-1,\alpha}/\operatorname{Conf}(S^2)).$$

The map Π_+ in (1.8) then descends to a smooth Fredholm map

$$\Pi_+ : \mathcal{N}_\kappa^+ \to \mathcal{B}[\mathcal{C}^{m,\alpha} \times C_+^{m-1,\alpha}], \tag{1.9}$$

of Fredholm index 0. The two formulations (1.6) and (1.9) are basically equivalent.

Now if Π_+ in (1.9), (or Π' in (1.6)), is proper, then by work of Smale [16], it has a well-defined degree $deg\Pi' \in \mathbb{Z}_2$, (and most likely a $\mathbb{Z}$-valued degree if the spaces can be given an orientation). Elementary degree theory implies that if

$$deg\Pi_+ \neq 0,$$

then Π_+ in (1.6) is surjective, answering at least the existence part of the question above.

In fact, if Π_+ is proper, then one has

$$deg\Pi_+ \neq 0 \ \text{ for } \ \partial M = S^2, \quad \text{but} \ \ deg\Pi_+ = 0 \ \text{ for } \ \partial M \neq S^2. \tag{1.10}$$

This follows from the Alexandrov-Hopf rigidity theorems, [1], [8]. Namely, any metric on a surface $\Sigma = \Sigma_g$ of genus g which is Alexandrov immersed in a space-form with $H = const$ is necessarily a round sphere. This uniqueness also holds infinitesimally, showing that the "round" conformal class $([\gamma_{+1}], H = c)$ is a regular value of Π. Since the Hopf theorem implies that the inverse image of this round regular value is unique, it follows that $deg\Pi_+ \neq 0$ for $\Sigma = S^2$. For Σ_g with $g \neq 0$, the same argument shows that Π_+ in (1.9) is not surjective, which implies $deg\Pi_+ = 0$.

Thus, at least in the case of S^2 the existence question above has been reduced to the properness of Π_+. This issue will be discussed in detail in the next section; we will show however that Π_+ or Π' is in fact not proper, so that further modifications are necessary to understand the global behavior of these boundary maps.

2. Analysis of the Boundary map Π.

We begin by filling in some details from the discussion in §1. Since the full curvature is determined by the Ricci curvature in 3-dimensions, any metric $g \in \mathbb{M}_\kappa$ satisfies the Einstein equation

$$Ric_g - 2\kappa \cdot g = 0. \tag{2.1}$$

We wish to view (2.1) as an elliptic equation for g. This is not possible due to the diffeomorphism invariance of (2.1), and so one needs to choose a gauge to break this invariance. Let $\widetilde{g} \in \mathbb{M}_\kappa$ be a fixed but arbitrary (constant curvature) background metric. The simplest choice of gauge is the Bianchi-gauge, with the associated Bianchi-gauged Einstein operator, given by

$$\Phi_{\widetilde{g}} : Met(M) \to S_2(M), \tag{2.2}$$
$$\Phi_{\widetilde{g}}(g) = Ric_g - 2\kappa g + \delta_g^* \beta_{\widetilde{g}}(g),$$

where $(\delta^* X)(A, B) = \frac{1}{2}(\langle \nabla_A X, B\rangle + \langle \nabla_B X, A\rangle)$ and $\delta X = -tr\delta^* X$ is the divergence and $\beta_{\widetilde{g}}(g) = \delta_{\widetilde{g}} g + \frac{1}{2} dtr_{\widetilde{g}} g$ is the Bianchi operator with respect to $\widetilde{g}$.

Clearly g is Einstein if $\Phi_{\widetilde{g}}(g) = 0$ and $\beta_{\widetilde{g}}(g) = 0$, so that g is in the Bianchi-free gauge with respect to $\widetilde{g}$. Using standard formulas for the linearization of the Ricci and scalar curvatures, cf. [6] for instance, one finds that the linearization of Φ at $\widetilde{g} = g$ is given by

$$L(h) = 2(D\Phi_{\widetilde{g}})_g(h) = D^* D h - 2R(h). \tag{2.3}$$

The zero-set of $\Phi_{\widetilde{g}}$ near $\widetilde{g}$,

$$Z = \{g : \Phi_{\widetilde{g}} = 0\}, \tag{2.4}$$

consists of metrics $g \in Met(M)$ satisfying the equation $Ric_g - 2\kappa g + \delta_g^* \beta_{\widetilde{g}}(g) = 0$.

Given $\widetilde{g}$, consider the Banach space

$$Met_C(M) = Met_C^{m,\alpha}(M) = \{g \in Met^{m,\alpha}(M) : \beta_{\widetilde{g}}(g) = 0 \text{ on } \partial M\}. \tag{2.5}$$

Clearly the map

$$\Phi : Met_C(M) \to S^2(M),$$

is C^∞ smooth. Let Z_C be the space of metrics $g \in Met_C(M)$ satisfying $\Phi_{\widetilde{g}}(g) = 0$, and let $\mathbb{M}_C = \mathbb{M}_\kappa \cap Z_C$ be the subset of constant curvature metrics g, $Ric_g = 2\kappa g$ in Z_C. It is proved in [5] that Z_C is a smooth Banach manifold and

$$\mathbb{M}_C = Z_C,$$

so that any metric $g \in Z_C$ near $\widetilde{g}$ is necessarily constant curvature, with $Ric_g = 2\kappa g$, and in Bianchi gauge with respect to $\widetilde{g}$. This result also holds at the linearized level. The spaces Z_C are smooth slices for the action of the diffeomorphism group $\mathrm{Diff}_1^{m+1,\alpha}$ on $\mathbb{M}_\kappa$ and it follows that the quotient $\mathcal{M}_\kappa$ is a smooth Banach manifold.

Next consider elliptic boundary data for the operator Φ in (2.2). Dirichlet or Neumann boundary data are not elliptic; this follows by inspection from the Gauss constraint equation (2.12) below, (or from the proof below). The following result is proved in [5]; we give the main details of the proof, since it is useful to compare this with the discussion in §3.

Proposition 2.1. *The Bianchi-gauged Einstein operator Φ with boundary conditions*

$$\beta_{\widetilde{g}}(g) = 0, \quad [g^T] = [\gamma], \quad H_g = h \ \text{ at } \ \partial M, \tag{2.6}$$

is an elliptic boundary value problem of Fredholm index 0.

Proof: It suffices to show that the leading order part of the linearized operators at the Euclidean metric forms an elliptic system. The leading order symbol of $L = D\Phi$ is given by

$$\sigma(L) = -|\xi|^2 I, \tag{2.7}$$

where I is the 3×3 identity matrix. In the following, the subscript 0 represents the direction normal to ∂M in M, and Latin indices run from 1 to 2. The positive roots of (2.7) are $i|\xi|$, with multiplicity 3. Writing $\xi = (z, \xi_i)$, the symbols of the leading order terms in the boundary operators are:

$$-2izh_{0k} - 2i\sum \xi_j h_{jk} + i\xi_k trh = 0,$$
$$-2izh_{00} - 2i\sum \xi_k h_{0k} + iztrh = 0,$$
$$h^T = (\gamma')^T \ \ mod\,\gamma, \quad H'_h = \omega,$$

where h is a 3×3 matrix. Ellipticity requires that the operator defined by the boundary symbols above has trivial kernel when z is set to the root $i|\xi|$. Carrying this out then gives the system

$$2|\xi|h_{0k} - 2i\sum \xi_j h_{jk} + i\xi_k trh = 0, \tag{2.8}$$

$$2|\xi|h_{00} - 2i\sum \xi_k h_{0k} - |\xi|trh = 0, \tag{2.9}$$

$$h_{kl} = \varphi\delta_{kl}, \quad H'_h = 0. \tag{2.10}$$

where φ is an undetermined function.

Multiplying (2.8) by $i\xi_k$ and summing gives

$$2|\xi|i\sum \xi_k h_{0k} = 2i^2\xi_k^2 h_{kk} - i^2\xi_k^2 trh.$$

Substituting (2.9) on the term on the left above then gives

$$2|\xi|^2 h_{00} - |\xi|^2 trh = -2\sum \xi_k^2 h_{kk} + |\xi|^2 trh,$$

so that
$$|\xi|^2 h_{00} - |\xi|^2 trh = -\sum \xi_k^2 h_{kk} = -\varphi|\xi|^2.$$
Using the fact that $trh - h_{00} = \sum h_{kk} = n\varphi$, it follows that $\varphi = 0$ and hence $h^T = 0$.

A simple computation shows that to leading order, $H'_h = tr^T(\nabla_N h - 2\delta^*(h(N)^T))$, which has symbol $iz\sum h_{kk} - 2i\xi_k h_{0k}$. Setting this to 0 at the root $z = i|\xi|$ gives
$$\sum(|\xi|h_{kk} + 2i\xi_k h_{0k}) = 0.$$
Since $h^T = 0$, this gives $\sum \xi_k h_{0k} = 0$, which, via (2.9) gives $h_{00} = 0$ and hence via (2.8), $h = 0$.

This proves that the boundary data (2.6) are elliptic for Φ. The proof that the Fredholm index is 0 is given in [5].

■

We now turn to the main issue, the properness of the map Π_+ in (1.9) or Π' in (1.6). This amounts to proving (apriori) estimates for metrics $g \in \mathcal{M}_\kappa$ in terms of the boundary data $([\gamma], H)$. The main result in this direction is the following:

Proposition 2.2. *Let $\mathcal{K}$ be a compact set in the space of boundary data $\mathcal{C}^{m,\alpha} \times C_+^{m-1,a}$. Then for any $K < \infty$, the space of metrics $g \in \mathcal{N}_\kappa^+$ such that*

$$\Pi_+(g) \in \mathcal{K} \ \text{ and } \ a = area(\partial M) \leq K, \tag{2.11}$$

is compact.

This result shows that Π_+ is proper, under the assumption of an upper bound on a. The proof of this result follows below, organized into several lemmas.

To begin, we recall the constraint equations at ∂M, i.e. the Gauss and Gauss-Codazzi equations:

$$|A|^2 - H^2 + 2K_\gamma = s_g - 2Ric_g(N,N) = 2\kappa, \tag{2.12}$$

$$\delta(A - H\gamma) = -Ric(N,\cdot) = 0, \tag{2.13}$$

where A is the second fundamental form and N is the outward unit normal.

One of the most important issues is to obtain a bound on $|A|$.

Lemma 2.3. *There is a constant $C_0 < \infty$, depending only on $\mathcal{K}$ and K in Proposition 2.2, such that*

$$|A| \leq C_0. \tag{2.14}$$

Proof: The proof is by contradiction, by means of a blow-up argument. To begin, integrating the Gauss constraint (2.12) and using the Gauss-Bonnet theorem gives

$$\int_{\partial M} |A|^2 = \int_{\partial M} H^2 - 4\pi\chi(\partial M) + 2\kappa \cdot area(\partial M). \tag{2.15}$$

By assumption, $a = area(\partial M)$ is uniformly bounded,
$$area(\partial M) \leq K < \infty.$$
This and (2.15) give an apriori bound on the scale-invariant quantity $\int |A|^2$:

$$\int_{\partial M} |A|^2 \leq C. \tag{2.16}$$

Now choose a point x on ∂M where $|A|$ is maximal, and rescale the metric so that $|A|(x) = 1$, with $|A|(y) \leq 1$ everywhere, so $\bar{g} = \lambda^2 g$ where $\lambda = |A|(x)$. It follows directly from the constraint equation (2.12) that the intrinsic curvature $K_{\bar{\gamma}}$ of $\bar{\gamma}$ is also uniformly bounded. The family of such metrics is compact in the pointed $C^{1,\alpha}$ topology, by the Cheeger-Gromov compactness theorem for instance; this means that modulo diffeomorphisms of ∂M, the metric γ itself is uniformly controlled in $C^{1,\alpha}$, (in suitable local coordinates and within bounded distance to x).

Now by assumption, the conformal class $[\gamma]$ of γ is uniformly controlled. It follows that the diffeomorphisms above, (in which $\bar{\gamma}$ is uniformly controlled), are themselves controlled modulo the group of conformal diffeomorphisms, cf. (1.5). Thus, passing if necessary from $\bar{\gamma}$ to $\gamma' = \varphi^*(\bar{\gamma})$, where φ is a conformal diffeomorphism, it follows that the metric γ' is uniformly controlled in $C^{1,\alpha}$, (locally, within bounded distance to x). Together with the uniform bound on $|A|$ above, it follows from Proposition 2.1 and elliptic regularity that the metric g' is controlled in the stronger $C^{m,\alpha}$ norm, up to its boundary.

Suppose then g_i is a sequence where $\max |A| \to \infty$. By rescaling as above one may pass to a smoothly convergent subsequence of $\{g_i'\}$ to obtain a smooth limit g'. The smooth ($C^{m,\alpha}$) convergence implies on the one hand that the limit is not flat, since the condition $|A|(x) = 1$ passes continuously to the limit. The estimate (2.16) also holds on the limit. Since $H \to 0$ in the rescalings, it follows that the limit is a complete immersed minimal surface in $\mathbb{R}^3$ with finite total curvature

$$\int_\Sigma |A|^2 < \infty.$$

Moreover, since the conformal classes $[\gamma_i]$ of g_i on ∂M are uniformly controlled, the sequence γ_i' has a uniformly controlled (large scale) atlas of conformal coordinates. Hence the limit is conformally isometric to $\mathbb{R}^2$, i.e. the limit minimal surface is pointwise conformal to $\mathbb{R}^2$. Finally, these minimal surfaces are in fact embedded; this follows from Lemma 2.4 below. However, it is well-known that the only such surfaces are flat planes, (cf. [14] for instance), and hence $A = 0$ in the limit. This contradiction establishes the bound (2.14).

■

Next we show that the normal exponential map has injectivity radius bounded below

$$inj_N \geq i_0$$

where i_0 depends only on an upper bound for $|A|$. This follows from the following Lemma.

Let here N be the inward unit normal to ∂M in M and consider the associated normal exponential map to ∂M, $tN \to \exp_p(tN)$, giving the geodesic normal to ∂M at p. This is defined for t small, and let $D(p)$ be the maximal time interval on which $\exp_p(tN) \in M$, (so that the geodesic does not hit ∂M again before time $D(p)$). Thus, $D : \partial M \to \mathbb{R}^+$.

Lemma 2.4. *Given $H > 0$, suppose $|A| \leq C_0$. Then there is a constant t_0, depending only on C_0 and κ, (and the lower bound on H when $\kappa < 0$), such that*

$$D(p) \geq t_0. \tag{2.17}$$

Proof: This is a well-known result in Riemannian geometry, essentially due to Frankel, and follows from the 2nd variational formula for geodesics. First, given bounds on $|A|$ and κ, by standard comparison geometry one has a lower bound on the distance to the focal locus of the normal exponential map $\exp(tN)$, i.e. a lower bound d_0 on the distance to focal points. Suppose then

$$\min D < d_0.$$

If the minimum is achieved at p, then the normal geodesic to ∂M at p intersects ∂M again at a point p', and the intersection is orthogonal to ∂M at p'. Denoting this geodesic by σ, and letting $\ell = D(p)$ be the length of σ, the 2nd variational formula of energy gives

$$E''(V,V) = \int_0^\ell (|\nabla_T V|^2 - \langle R(T,V)V,T\rangle)dt - \langle \nabla_V T, V\rangle|_0^\ell, \tag{2.18}$$

where $T = \dot{\sigma}$ and V is any variation vector field along σ orthogonal to σ. By the minimizing property, one has $E''(V,V) \geq 0$, for all V. Choose then $V = V_i$ to be parallel vector fields e_i, running over an orthonormal basis at $T_p(\partial M)$. The first term in (2.18) then vanishes, while the second sums to $-Ric(T,T) = -2\kappa$. The boundary terms sum to $\pm H$, at p and p'. Taking into account that T points into M at p while it points out of M at p', this gives

$$0 \leq -2\kappa\ell - (H(p) + H(p')).$$

Since $H > 0$, this gives immediately a contradiction if $\kappa \geq 0$, and also gives a contradiction if $\kappa < 0$, if H is bounded below, depending only on the size of κ (if ℓ is sufficiently small). This proves the estimate (2.17).

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Finally we show that Lemma 2.3 implies that the intrinsic geometry of $(\partial M, \gamma)$ is controlled.

LEMMA 2.5. *There is a constant C_1, depending only on C_0 in* (2.14) *and $\mathcal{K}$, such that*

$$|K_\gamma| \leq C_1, \tag{2.19}$$

Moreover, the metric γ is uniformly controlled, modulo conformal diffeomorphisms, by C_1, (and a).

PROOF: As in the proof of Lemma 2.3, via the Gauss constraint equation (2.12), a bound on $|A|$ implies a bound on K_γ, giving (2.19). A standard simple analytic argument then gives control on the metric γ itself when $\chi(\partial M) \leq 0$. Namely, write $\gamma = \lambda^{-2}\gamma_0$, where γ_0 is the conformal metric with constant curvature σ and σ is chosen so that $area\gamma = area\gamma_0$. The formula for the behavior of Gauss curvature under conformal changes then gives

$$\lambda^2 \Delta_{\gamma_0}(\log \lambda) = -\lambda^2\sigma - K_\gamma.$$

The maximum principle implies an upper bound on λ and hence, by elliptic regularity, one has uniform $C^{1,\alpha}$ control on λ and so, (via standard bootstrap arguments), λ is controlled in $C^{m,\alpha}$.

This argument does not work when $\chi(\partial M) > 0$, (since the minimum or maximum principle does not hold). In this case, one can use the same argument as that given in the proof of Proposition 2.2. Thus, the bound $|K_\gamma|$ implies that the metric is controlled modulo diffeomorphisms, by the Cheeger-Gromov compactness theorem. Here we use the fact that the length of the shortest closed geodesic, and hence the injectivity radius of γ, is bounded below, since $|A|$ is bounded above. Since the conformal class $[\gamma]$ is assumed to be controlled, the diffeomorphisms are themselves controlled, modulo the group of conformal diffeomorphisms.

Note that (2.15) shows that $a = area(\partial M)$ is bounded below, and hence the diameter of $(\partial M, \gamma)$ is also bounded above and below. This proves the result. Note also that since $diam(M,g) \leq diam(\partial M, \gamma)$, this also gives a uniform upper bound on the diameter of (M,g).

■

The results above prove Proposition 2.2. This result implies that the "enhanced" boundary map

$$g \to ([\gamma], H, a) \tag{2.20}$$

is proper. While this map is Fredholm, it is Fredholm of index -1 and so does not have a well-defined degree; for this, one needs the Fredholm index to be non-negative.

There are two ways in which one may try to proceed at this point.

(I). One may try to prove that $a = area(\partial M)$ is controlled by the boundary data $([\gamma], H)$, which would then prove that Π itself, (i.e. Π_+ or Π'), is proper.

However, this is false. It follows from the proof of Proposition 2.2 that counterexamples must closely resemble the helicoid (in a suitable scale), since the helicoid is the unique complete embedded minimal surface in $\mathbb{R}^3$ conformally equivalent to $\mathbb{R}^2$, (besides the plane), cf. [13]. In fact conversely, one may use the helicoid to construct examples of metrics g_i where $([\gamma_i], H_i)$ are uniformly bounded but

$$a_i \to \infty. \tag{2.21}$$

To see this, consider the helicoid $\mathcal{H} = \mathcal{H}_L$,

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = L^{-1}\theta,$$

$L = L(\varepsilon) >> 1$, wrapping around z-axis arbitrarily many times in the interval $z \in [-\varepsilon, \varepsilon]$; assume here $\rho \in [-1, 1]$. Consider an almost horizontal $S^1 \subset \mathcal{H}$ formed by connecting two line segments parallel to the x-axis in $\mathcal{H}$ at height $\pm\varepsilon$ by a circular arc joining their endpoints along a helix in $\mathcal{H}$. This S^1 bounds a disc $D^2 \subset \mathcal{H}$. Now form the vertical z-cylinder over this boundary S^1, and take it to a fixed height, say $z = \frac{1}{2}$ and then cap off the circular boundary at $z = \frac{1}{2}$ by a horizontal disc. This gives first an immersed S^2, which is also Alexandrov immersed, since it may be perturbed to an embedding. This S^2 may also be perturbed so that $H > 0$ everywhere. Namely, one may first deform the helicoid very slightly to a surface with $H > 0$, and in fact with H uniformly bounded away from 0 and ∞, cf. [17] for instance. The vertical cylinder has $H > 0$ and one can bend the top flat disc to $H > 0$. Finally, the corners of S^2 may also be smoothed to $H > 0$ everywhere.

The conformal class of the helicoid is fixed under arbitrary rescalings, (i.e. variations of L), and the gluing process above is also uniformly controlled; hence the conformal class of the collection of surfaces above is uniformly controlled. It is clear from the construction that (2.21) holds as the number of wrappings of the helicoid is taken to infinity.

Recall the Hopf uniqueness theorem: if Σ is a sphere immersed in a space-form of constant curvature with $H = const$, then Σ is umbilic and so locally isometric to a round sphere. The examples above seem to indicate or suggest that the rigidity associated to the Hopf theorem cannot be weakened to an "almost rigidity" theorem; thus we expect given any $\varepsilon > 0$, there exist surfaces $\Sigma_\varepsilon \subset \mathbb{R}^3$ diffeomorphic to S^2 such that

$$2 - \varepsilon \leq H_{\Sigma_\varepsilon} \leq 2 + \varepsilon,$$

which are not close to a round sphere $S^2(1) \subset \mathbb{R}^3$. Of course, one must have $area(\Sigma_\varepsilon) \to \infty$ as $\varepsilon \to 0$. It would be interesting to know the answer to this question.

(II). Instead, refering to the context of (2.20), one may add an extra scalar variable λ to the domain to obtain a map of Fredholm index 0. [Alternately, one may restrict the domain $\mathcal{M}'_\kappa$ in (1.6), or $\mathcal{N}^+_\kappa$ in (1.9) to surfaces where $a = 1$; correspondingly, one must then divide the target space by an $\mathbb{R}^+$ action. There is no essential difference between these so we discuss only the former].

Thus, extend for example the domain $\mathcal{M}'_\kappa$ to $\mathcal{M}'_\kappa \times \mathbb{R}^+$, and consider the following typical examples:

$$(g, \lambda) \to ([\gamma], H, a + \lambda). \tag{2.22}$$

$$(g, \lambda) \to ([\gamma], \frac{H}{H_{\min}} - 1 + \lambda, a), \tag{2.23}$$

where $H_{\min}$ is the minimum value of H.

These maps are Fredholm, of Fredholm index 0. However, neither map is proper; in (2.22), one may have $\lambda \to 0$ while in (2.23) one may have $H_{\max} \to \infty$, both within compact sets of boundary data. Consider next shifting the λ-variable also to the space $\mathcal{C}$. For example, let ψ_λ be a curve of diffeomorphisms of ∂M with $\psi_\lambda \to \infty$ as $\lambda \to 0$ and consider

$$(g, \lambda) \to ([\psi^*_\lambda(\gamma)], H, a + \lambda). \tag{2.24}$$

As before, this behaves well in the second two factors, but it is not clear if there exists a curve ψ_λ for which (2.24) is proper; in any case we have not been able to find a construction to make this map proper.

Next consider enlarging the domain by adding a scale factor, when $\kappa \neq 0$. Thus, let $\mathcal{M}' = \cup_{\kappa<0}\mathcal{M}'_\kappa$, and define

$$\mathcal{M}' \to \mathcal{C} \times C_+, \qquad (2.25)$$
$$g \to ([\gamma], H, a).$$

This gives of course control of both H and a, and by (2.15), one deduces uniform control on $|\kappa|$, when $\kappa < 0$. However, one cannot prevent the possibilty that $\kappa \to 0$, so that again its not clear if this map can be made proper. (Including the spaces $\mathcal{M}'_\kappa$ with $\kappa > 0$ also does not seem to help).

Consider finally the following modification of (2.23):

$$\widetilde{\Pi} : (g, \lambda) \to ([\gamma], \frac{H}{H_{\min}} - 1 + \lambda, a + H_{\max}). \qquad (2.26)$$

The map $\widetilde{\Pi}$ is Fredholm, of index 0, and is now proper by Proposition 2.2, since control of the data in the target space gives control on H, a and the conformal class $[\gamma]$. It thus has a well-defined degree.

However, the Hopf rigidity theorem now shows that

$$deg\widetilde{\Pi} = 0. \qquad (2.27)$$

Namely, consider the case $\partial M = S^2$ and $\kappa = 0$. When restricted to the "round" metrics where $H = const$, by the constraint equation (2.12) one has

$$H^2 a = 16\pi,$$

so that $H = H_{\max} = \sqrt{\frac{16\pi}{a}}$. This gives

$$\beta(a) \equiv a + H_{\max} = a + \sqrt{\frac{16\pi}{a}},$$

which, for a given value of $\beta(a) = c$ has two positive real solutions $a > 0$. The function β is a simple fold map $\mathbb{R}^+ \to [(4\pi)^{1/3}, \infty)$. This implies (2.27). (Although the round metric is not in $\mathcal{M}'_\kappa$, the discussion above remains valid for data near the round metric).

There is another, quite different argument showing that $\widetilde{\Pi}$ is not onto, and hence has degree 0. Namely on any $g \in \mathcal{M}_\kappa$ with boundary data (γ, H), one has

$$\int_{S^2} X(H) dV_\gamma = 0, \qquad (2.28)$$

where X is any conformal Killing field on S^2. This follows from the constraint equation (2.13). Namely, pairing (2.13) with a vector field X and integrating over $(\partial M, \gamma)$ gives

$$\int_{\partial M} \langle \delta A, X \rangle = -\int_{\partial M} \langle dH, X \rangle.$$

The left side equals $\int_{\partial M} \langle A, \delta^* X \rangle$, and for X conformal Killing, $\int_{\partial M} \langle A, \delta^* X \rangle = \frac{1}{n} \int_{\partial M} H divX$. On the other hand, $-\int_{\partial M} \langle dH, X \rangle = \int_{\partial M} H divX$, which gives (2.28), since $dim\partial M = n \neq 1$. The result (2.28) is essentially due to [2], although the proof given here is much simpler. The condition (2.28) is of course reminiscent of the Kazdan-Warner type obstruction [10] for the prescribed Gauss curvature problem. As in the Gauss curvature problem, note that the condition (2.28) is not conformally invariant, i.e. it is not a well-defined condition on the target space $\mathcal{C}^{m,\alpha} \times C_+^{m-1,\alpha}$.

The "balancing condition" (2.28) implies for instance that any H which is a monotone function of a height function on $S^2(1)$ is not the mean curvature of a conformal immersion in a space-form.

On the other hand, although (2.28) formally represents 3 independent conditions on H, it does not imply that $Im\Pi$ has codimension 3, (or any other codimension), in $\mathcal{C}^{m,\alpha} \times C_+^{m-1,\alpha}$, again since it is not defined on this target space.

Although we have succeeded in constructing a proper map $\widetilde{\Pi}$, it is not at all clear what $Im\widetilde{\Pi}$ is, or what the images of the closely related maps Π_+ and Π' in (1.9), (1.6) are. For instance, can $Im\Pi_+$ be described as the locus where a finite number of real-valued functions on the target are positive? Can one explicitly identify such functions characterising the boundary values of metrics in $\mathcal{M}_\kappa$?

Finally, regardless of the surjectivity issue, the discussion in (I) above suggests that Π_+ is infinite-to-one, so highly non-unique. In sum, it would be interesting to understand these issues better, which seem on the whole much easier than the existence and uniqueness question for the isometric immersion problem discussed in §1.

3. Generalization

Let (M_κ, g_κ) be any complete Riemannian 3-manifold of constant curvature κ; thus, up to scaling, M_κ is one of $\mathbb{R}^3$, $\mathbb{H}^3$ or $\mathbb{S}^3$ or a quotient of one of these spaces. Let $f : \partial M \to M_\kappa$ be an Alexandrov immersion, and let F denote an extension of f to M. Then since $\pi_1(M, \partial M) = 0$, the metric $F^*(g_\kappa)$ is uniquely determined by the immersion f on ∂M, modulo $\mathrm{Diff}_1(M)$. Thus, the map Π_+ in (1.4) is equivalent to a map

$$\Pi_+ : Imm_A(\partial M) \to \mathcal{C}^{m,\alpha}(\partial M) \times C_+^{m-1,\alpha}(\partial M).$$

This suggests that one could replace the space of metrics $\mathcal{M}_\kappa$ by the space of immersions of $\Sigma = \partial M$ into a space-form M_κ or more generally into an arbitrary complete Riemannian manifold (N, g_N). This is in fact the case:

Proposition 3.1. *Let $\Sigma = \Sigma_g$ be a compact surface of genus g and let (N, g_N) be any complete Riemannian 3-manifold. Let $Imm^{m+1,\alpha}(\Sigma, N)$ be the space of $C^{m+1,\alpha}$ immersions of $\Sigma \to N$. Then the map*

$$\Pi : Imm^{m+1,\alpha}(\Sigma, N) \to \mathcal{C}^{m,\alpha}(\Sigma) \times C^{m-1,\alpha}(\Sigma), \tag{3.1}$$

$$\Pi(f) = ([f^*(g_N)|_{\partial M}], H(f(x))),$$

is a smooth Fredholm map of Fredholm index 0.

Proof: The space $Imm(\Sigma, N)$, is a smooth Banach manifold; the tangent space is given by the space of vector fields v along a given immersion $f : \Sigma \to N$. The differential $D\Pi$ of Π in (3.1) is given by

$$([(\delta^* v)^T]_0, H'_{\delta^* v}), \tag{3.2}$$

where $(\delta^* v)^T$ is the restriction of $\delta^* v$ to $T(\partial M)$. The Fredholm property then follows by showing that the data (3.2) form an elliptic system of equations for v. We do this following the proof of Proposition 2.2.

Thus, write $v = v^T + fN$, where v^T is tangent and N is normal to $T(\partial M)$. Then

$$\delta^* v = \delta^* v^T + fA + df \cdot N, \tag{3.3}$$

so that $(\delta^* v)^T = \delta^* v^T + fA$. The second term is lower order in v and so does not contribute to principal symbol. The principal symbol σ of the first term is thus

$$\sigma([(\delta^* v)^T]_0) = \xi_i v_j - \frac{\xi_i v_i}{2} \delta_{ij}, \tag{3.4}$$

where i, j are indices for ∂M. For the mean curvature, one has $H'_{\delta^* v} = -\Delta f + v(H)$, so that the leading order term is just $-\Delta f$ with symbol $|\xi|^2 f$. Hence one has elliptic data for the normal component f of v.

For the tangential part of v, (3.4) gives

$$\xi_1 v_2 = \xi_2 v_1 = 0 \quad \text{and} \quad \xi_1 v_1 = \xi_2 v_2.$$

Since $(\xi_1, \xi_2) \neq (0, 0)$, it is elementary to see that the only solution of these equations is $v_1 = v_2 = 0$, which proves ellipticity. It is straightforward to verify further that Π has Fredholm index 0.

■

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