A SURVEY OF EINSTEIN METRICS ON 4-MANIFOLDS

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Abstract. We survey recent results and current issues on the existence and uniqueness of Einstein metrics on 4-manifolds. A number of open problems and conjectures are presented during the course of the discussion.

1. Introduction.

The Einstein equations

$$Ric_g = \lambda g, \quad \lambda \in \mathbb{R}, \tag{1.1}$$

for a Riemannian metric g are the simplest and most natural set of equations for a metric on a given compact manifold M. Historically these equations arose in the context of Einstein's general theory of relativity, where the metric g is of Lorentzian signature. However, over the past several decades there has also been much mathematical interest in Einstein metrics of Riemannian signature on compact manifolds, especially in low dimensions, and in particular in relation to the topology of the underlying manifold.

A strong motivation for this comes from the understanding developed in dimension 2 and more recently in dimension 3. To explain this, there is a complete classification of compact oriented 2-manifolds by the Euler characteristic, originally obtained by purely topological methods through work of Möbius, Dehn, Heegard and Rado. This classification was later reproved via the Poincaré-Koebe uniformization theorem for surfaces, i.e. any compact oriented surface carries a metric of constant curvature. The structure of such metrics then easily gives the full list of possible topological types of such surfaces, (and much more).

It has long been a goal of mathematicians to prove a similar classification of compact oriented 3-manifolds. Thurston [70] realized that the key to this should be in studying the possible geometric stuctures on 3-manifolds; the most important such structures are again the constant curvature, i.e. Einstein, metrics. However, in contrast to surfaces, most 3-manifolds do not admit an Einstein metric (1.1); instead one had a simple, and conjecturally complete, list of well understood obstructions. A general 3-manifold should decompose into a collection of domains, each of which carries a natural geometry, the most important geometry being that of Einstein metrics.

The recent solution of the Thurston geometrization conjecture by Perelman [60]-[62] and Hamilton [35]-[36] has accomplished this goal of completely classifying all 3-manifolds. This has been obtained by understanding how to obtain solutions to the Einstein equations via the parabolic analogue of (1.1), namely the Ricci flow.

Ideally, one would like to carry out a similar program in dimension 4. However, (as with the passage from 2 to 3 dimensions), the world of 4-manifolds is much more complicated than lower dimensions. One encounters a vast variety of exotic smooth structures, there are severe complications in understanding the fundamental group, and so on. Moreover, there are no canonical local models of Einstein metrics, and even when such metrics exist, the tie of the global geometry of these

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metrics with the underlying topology remains currently poorly understood. In fact, as discussed eloquently by Gromov in [33], the "dream" of trying to understand 4-manifolds via Einstein or other canonical metrics may well be impossible to realize. On the other hand, one should keep in mind that the deepest understanding to date of smooth 4-manifolds comes via the geometry of connections or gauge fields, in the theories developed by Yang-Mills, Donaldson and Seiberg-Witten. How far such theories can be carried over to metrics, (the gravitational field), remains to be seen.

This paper is an introductory survey of basic results to date on the existence, uniqueness, and structure of moduli spaces of Einstein metrics on 4-manifolds. Many interesting topics have been omitted or presented only briefly, due partly to the limits of the author's knowledge and taste, and partly to keep the article at a reasonable length. For this reason, we have excluded all discussion of Einstein metrics in higher dimensions. Also, any area of research is only as vital as the interest of significant open questions and problems. Accordingly, we present a number of such open problems throughout the paper, some of which are well-known and others less so.

2. Brief Review: 4-manifolds, complex surfaces and Einstein metrics.

In order to set the stage for the more detailed discussion to follow, in this section we present a very brief overview of the topology of 4-manifolds and the classification of complex surfaces.

I. *Topology of 4-manifolds.*

Throughout the paper, M will denote a compact, oriented 4-manifold unless otherwise indicated. Together with the fundamental group $\pi_1(M)$, the most important topological datum of a 4-manifold is the cup product or intersection pairing,

$$I : H_2(M,\mathbb{Z}) \otimes H_2(M,\mathbb{Z}) \to \mathbb{Z}. \tag{2.1}$$

By Poincaré duality, I is a symmetric, non-degenerate bilinear form when M is simply connected, (so that $H_2(M,\mathbb{Z})$ is torsion-free). The algebraic classification of such pairings (A, I) over $\mathbb{Z}$, where A is an abelian group, starts with a few simple invariants. Thus, the rank $rank(I)$ and index $\tau(I)$ are defined by $rank(I) = p(I) + n(I)$ and $\tau(I) = p(I) - n(I)$, where p and n are the maximal dimensions on which the form $I \otimes \mathbb{R}$ is positive or negative definite. The form I is called even if $I(a, a) \equiv 0$ mod 2, for all $a \in A$; otherwise I is odd. The algebraic classification then states that any indefinite pairing (A, I) is determined up to isomorphism by its rank, index and parity, and has a simple standard normal form. If (A, I) is definite, there is a huge number of possibilities; for example, there are more than 10^{51} inequivalent definite forms of rank 40. There are only finitely many definite pairings of a given rank although a complete classification remains to be determined.

One has the obvious relations $|\tau(I)| \leq rank(I)$, $\tau(I) \equiv rank(I)$ (mod 2). In addition, if I is even, then $\tau(I) \equiv 0$ (mod 8). Modulo these relations, all values of rank, index and parity are possible algebraically.

Now for a given compact oriented 4-manifold M and for I as in (2.1), set $b_2^+(M) = p(I)$, $b_2^-(M) = n(I)$ and $\tau(M) = \tau(I)$; $\tau(M)$ is called the signature of M. A simply connected 4-manifold M is spin if I is even, non-spin otherwise. A first basic result, due to J.H.C. Whitehead, is that the homotopy type of a simply connected 4-manifold is completely determined by the intersection pairing (2.1); thus, if M and N are simply connected, then they are homotopy equivalent if and only if $I(M) \simeq I(N)$.

In a remarkable work, Freedman [26] established the classification at the next level, i.e. up to homeomorphism. Thus, every pairing (A, I) occurs as the intersection form of a simply connected topological 4-manifold. If M is spin, then the homeomorphism type is unique, while if M is non-spin, there are exactly two homeomorphism types; one and only one of these is stably smoothable, in that the product with $\mathbb{R}$ has a smooth structure.

Passing next to the category of smooth manifolds, the Rochlin theorem gives the further restriction $\tau(M) \equiv 0 \pmod{16}$ when M is spin. Shortly after Freedman's work, Donaldson [22] proved the amazing result that if the intersection pairing $I(M)$ of a smooth, simply connected 4-manifold is definite, then $I(M)$ is a diagonal form over $\mathbb{Z}$ and hence, by Freedman's result, M is homeomorphic to a connected sum $\#k\mathbb{CP}^2$. This led to the existence of distinct smooth structures on 4-manifolds, and together with the Seiberg-Witten equations, has led to a vast zoo of possible smooth structures on a given underlying topological 4-manifold. In fact, it is possible that every smoothable 4-manifold always has infinitely many distinct smooth structures, cf. [25] and references therein for further details.

While there have been some generalizations of the results of Freedman and Donaldson to 4-manifolds with non-trivial fundamental group, comparatively little is known when the fundamental group is unrestricted. This is partly due to well-known undecidability issues; for instance, there is no algorithm to determine whether a general compact 4-manifold is simply connected, or whether two 4-manifolds are homeomorphic.

II. *Complex surfaces.*

Throughout this subsection, M will be a compact complex surface. We briefly review the Kodaira classification of surfaces. This derives from the structure of the canonical line bundle K over M, i.e. the bundle of holomorphic 2-forms on M. In local holomorphic coordinates (z^1, z^2), sections of K have the form $f(z_1, z_2)dz^1 \wedge dz^2$ with f holomorphic. Let K^n denote the n-fold tensor product of K, a line bundle with local sections of the form $f(z_1, z_2)(dz^1 \wedge dz^2)^n$. The n^{th} plurigenus of M is the dimension of the space of holomorphic sections of K^n, $P_n(M) = dim_{\mathbb{C}} H^0(M, K^n)$.

If $P_n(M) = 0$ for all $n > 0$, M is said to have Kodaira dimension $\kappa(M) = -\infty$. Otherwise, one has $P_n(M) = O(n^a)$, for $a = 0$, 1 or 2, and the smallest such a defines $\kappa(M)$. Thus $\kappa(M) \in \{-\infty, 0, 1, 2\}$. (Roughly speaking, $\kappa(M)$ is the dimension of the image of M under the Kodaira map).

Next, recall the process of blowing up and down. Given any M and $p \in M$, the blow-up $\hat{M}$ of M at p is the complex surface obtained by replacing a ball $B \subset \mathbb{C}^2$ near p by $\bar{\mathbb{C}}\mathbb{P}^2 \setminus B$, where $\bar{\mathbb{C}}\mathbb{P}^2$ is $\mathbb{CP}^2$ with the opposite orientation. Thus $\hat{M} = M\#\bar{\mathbb{C}}\mathbb{P}^2$ topologically. There is a holomorphic map $\pi : \hat{M} \to M$ and a rational curve $E = \mathbb{CP}^1 \subset \hat{M}$ such that π restricted to $\hat{M} \setminus E$ is a biholomorphism onto $M \setminus \{p\}$, so that π contracts E to $\{p\}$. One has $[E] \cdot [E] = -1$, i.e. E has self-intersection -1 in $\hat{M}$ and $K(\hat{M}) = \pi^* K(M) + E$, (as divisors on $\hat{M}$). It follows that $\kappa(M)$ is unchanged under blow ups.

Whenever a complex surface M has an exceptional curve E, i.e. a rational curve with self-intersection -1, this process may be inverted, i.e. M may be blown down to remove the divisor E. A surface M is minimal if M has no such exceptional curves. Every surface may be blown down (not necessarily uniquely) to a minimal surface. The Kodaira classification then states the following:

• If $\kappa(M) = -\infty$ and M is simply connected, then M is rational; M can be obtained by blowing up $\mathbb{CP}^2$ a finite number of times and then blowing down a finite number of times. The underlying 4-manifold M is diffeomorphic to $\mathbb{CP}^2\#k\bar{\mathbb{C}}\mathbb{P}^2$, $k \geq 0$, or $\mathbb{CP}^1 \times \mathbb{CP}^1$. If M is not simply connected, then M is the blow-up of a minimal ruled surface; there is a holomorphic map $\pi : M \to S$, where S is a complex curve, with fibers $\mathbb{CP}^1$.

If $\kappa(M) \geq 0$, then M can be uniquely blown down to a minimal surface $M_{\min}$, called the minimal model, and from now on, assume then that M is minimal.

• If $\kappa(M) = 0$, then M is a K3 surface or complex torus, or a finite quotient of one of these spaces. Any K3 surface is diffeomorphic to a quadric in $\mathbb{CP}^3$.

• If $\kappa(M) = 1$, then M is elliptic; there is a holomorphic map $\pi : M \to C$, where C is a complex curve, with fibers a smooth curve of genus 1, (an elliptic curve), for almost all $p \in C$.

• If $\kappa(M) = 2$, then M is of general type. For example, hypersurfaces of degree $k \geq 5$ in $\mathbb{CP}^3$ are of general type.

III. *Einstein metrics.*

When expressed in local coordinates, the Einstein equations (1.1) form a complicated, quasi-linear system of PDE's for the metric $g = g_{ij}$. This system is not elliptic, due to the diffeomorphism invariance of the equations. However, for a suitable choice of local slice transverse to the action of the diffeomorphism group on the space of metrics, the restricted equations become elliptic. The simplest such choice locally is the harmonic coordinate gauge, where each of the coordinate functions x^i is harmonic with respect to the given metric g, i.e. $\Delta_g x^i = 0$. In such coordinates, the Einstein equations have the pleasant form

$$-\tfrac{1}{2}\Delta_g g_{ij} + Q_{ij}(g, \partial g) = \lambda g_{ij}, \tag{2.2}$$

where Q is quadratic in the metric g and its first derivatives. Thus, the system of Einstein equations locally can be viewed as a non-linear and coupled version of the equations for eigenfunctions of the Laplacian Δ. The ellipticity of the system (2.2) implies that Einstein metrics are C^∞, (in fact real-analytic) in such harmonic coordinates.

In dimensions 2 and 3, Einstein metrics are of constant curvature, and so locally rigid; they are all locally isometric to domains in the space-forms of constant curvature. This is no longer the case in dimensions 4 and above. In fact, the space of local solutions, (i.e. solutions defined on a ball), of the equations (1.1) is infinite dimensional, (analogous to the scalar eigenfunction equation).

In dimension 4, Einstein metrics enjoy a variational characterization, for instance as metrics minimizing the L^2 norm of the curvature tensor R, (cf. (4.2) below), or as critical points of the volume-normalized total scalar curvature functional

$$\mathcal{S}(g) = (vol_g(M))^{-1/2} \int_M s_g dV_g, \tag{2.3}$$

where s_g is the scalar curvature of g. It would be very interesting if Einstein metrics could be constructed variationally, by using methods in the calculus of variations for either of the functionals above. For example, the solution of the Yamabe problem [65, 54] gives the existence of a metric, called a Yamabe metric, minimizing $\mathcal{S}$ in the conformal class $[g]$ of any metric g on M. One would then like to understand the existence of Einstein metrics on M by understanding the limiting behavior of sequences of Yamabe metrics g_i, whose scalar curvature tends to its maximal value. Unfortunately, comparatively little is currently known about this approach.

3. Constructions of Einstein metrics I.

In this section we discuss the primary method of proving the existence of Einstein metrics on 4-manifolds, namely the existence of Kähler-Einstein metrics via the solution of the Calabi conjecture. We also list the remaining currently known Einstein metrics on 4-manifolds; see also §7 for further discussion.

To begin, suppose (M, J) is a compact complex 4-manifold admitting a Kähler metric g, with Kähler form $\omega = g(J, \cdot)$. For Kähler metrics, the Ricci form $\rho = Ric(J,)$ is given, (up to a factor of i), by the curvature form of the canonical line bundle K over M. The Ricci form thus depends only on the complex structure J and the volume form μ_g of the Kähler metric g and

$$[\rho] = \frac{1}{2\pi}[c_1(M)] \in H^2(M, \mathbb{C}). \tag{3.1}$$

Consequently, if g is in addition Einstein, satisfying (1.1), then one has

$$[c_1(M)] = 2\pi\lambda[\omega]. \tag{3.2}$$

In particular $c_1 < 0$, $c_1 = 0$ or $c_1 > 0$ according to whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. This of course gives a strong restriction on the existence of Kähler-Einstein metrics; the first Chern class c_1 must be definite or identically 0, (in the sense that the associated symmetric bilinear form has these properties). Thus, "most" complex surfaces do not admit a Kähler-Einstein metric.

On the other hand, a basic reason why one is able to solve the existence problem for Kähler-Einstein metrics, as opposed to solving the existence problem for general Einstein metrics, is that via (3.1)-(3.2), the existence of Kähler-Einstein metrics is tightly bound to the cohomology of M. In this way, the Kähler-Einstein equation can be reduced to a single scalar PDE, instead of the complicated full system of PDE's in, for instance, (2.2).

It is worth describing this remarkable reduction in more detail. A given Kähler metric g has the form

$$g = \sum g_{k\bar{j}} dz^k d\bar{z}^j,$$

in local complex coordinates. The Ricci tensor is then given by

$$Ric_{k\bar{l}} = -\partial_k \partial_{\bar{l}} \log det(g_{m\bar{n}}),$$

and the equation $[\rho] = \lambda[\omega]$ implies there is a smooth function F such that $Ric_{k\bar{l}} - \lambda g_{k\bar{l}} = \partial_k \partial_{\bar{l}} F$. To study the existence of a Kähler-Einstein metric with the same Kähler class as ω, let $\widetilde{g}_{k\bar{l}} = g_{k\bar{l}} + \partial_k \partial_{\bar{l}} \phi$. Then the Einstein equation $Ric_{\widetilde{g}} = \lambda \widetilde{g}$ is equivalent to the equation

$$\frac{det(g_{k\bar{l}} + \phi_{k\bar{l}})}{det(g_{k\bar{l}})} = e^{-\lambda\phi + F}. \tag{3.3}$$

To see this, taking $\partial_k \partial_{\bar{l}}$ of the logarithm of both sides of (3.3) gives

$$-(\widetilde{Ric}_{k\bar{l}} - Ric_{k\bar{l}}) = -\lambda \partial_k \partial_{\bar{l}} \phi + \partial_k \partial_{\bar{l}} F = -\lambda(\widetilde{g}_{k\bar{l}} - g_{k\bar{l}}) + (Ric_{k\bar{l}} - \lambda g_{k\bar{l}}) = -\lambda \widetilde{g}_{k\bar{l}} + Ric_{k\bar{l}},$$

which implies $\widetilde{Ric}_{k\bar{l}} = \lambda \widetilde{g}_{k\bar{l}}$. Thus a Kähler-Einstein metric exists in the class $[\omega]$ if and only if there is a solution ϕ of the scalar complex Monge-Ampere equation (3.3) with $g_{k\bar{l}} + \phi_{k\bar{l}}$ positive definite.

The existence of a solution of (3.3) is proved by the method of continuity, solving (3.3) on a curve $\phi_t = t\phi$ with a suitable choice of F_t. The initial set-up above shows that a solution exists for $t = 0$, and one proves that the set of $t \in [0, 1]$ for which (3.3) has a solution ϕ_t is both open and closed. The openness result is a straightforward consequence of the inverse function theorem. The main task is to prove closedness of the set of solutions; this requires rather difficult apriori estimates on the behavior of the solutions.

The basic results are as follows:

Theorem 3.1. [8], [75]. *A compact complex surface (M, J) admits a Kähler-Einstein metric with $\lambda < 0$ if and only if $c_1 < 0$. This occurs precisely for (M, J) which are minimal surfaces of general type which contain no (-2)-curves, (rational curves of self-intersection -2). The Kähler-Einstein metric is uniquely determined by (M, J).*

Theorem 3.2. [75]. *A compact complex surface (M, J) admits a Kähler-Einstein metric with $\lambda = 0$ if and only if $c_1 = 0$. This occurs precisely when (M, J) is finitely covered by a K3 surface or a complex torus. The Kähler-Einstein metric is uniquely determined by (M, J) and a choice of Kähler class in the Kähler cone in $H^{1,1}(M, \mathbb{R})$.*

Theorem 3.3. [72]. *A compact complex surface (M, J) admits a Kähler-Einstein metric with $\lambda > 0$ if and only if $c_1 > 0$, and the Lie algebra of holomorphic vector fields is reductive. This occurs exactly on $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1$ or the blow-up $\mathbb{CP}^2 \# k\bar{\mathbb{CP}}^2$, $3 \leq k \leq 8$, of $\mathbb{CP}^2$ at k points in general position. Again, the Kähler-Einstein metric is uniquely determined by (M, J), a result of* [11].

These results give a complete understanding of Kähler-Einstein metrics on a given complex surface, at least concerning formal existence and uniqueness issues. Note that there are relatively few Kähler-Einstein metrics when $\lambda \geq 0$; most all have $\lambda < 0$, in analogy to the case of Riemann surfaces.

The results above show that the moduli space $\mathcal{E}_{KE}$ of Kähler-Einstein metrics may be naturally identified with the moduli space $\mathcal{M}_C$ of complex structures on a given manifold M when $c_1(J) < 0$ or $c_1(J) > 0$, and similarly with the space of Kähler classes over $\mathcal{M}_C$ when $c_1(J) = 0$. A fundamental issue of interest, raised in particular by Yau, cf. [76] for instance, is then to use this identification to study in more detail the structure of each of these moduli spaces. This is still basically an undeveloped area and much work remains to be done in this direction.

Another well-known problem is to understand these Kähler-Einstein metrics more explicitly; see for instance the recent work of Donaldson [23] and further references therein.

One would also like to answer simple uniqueness questions. For instance, are Kähler-Einstein metrics the unique Einstein metrics on a given (complex) 4-manifold M?

A further collection of Einstein metrics on 4-manifolds are the locally homogeneous metrics, given by left-invariant metrics on a compact locally homogeneous space $\Gamma \setminus G/H$. A complete list of these Einstein metrics, modulo finite covers, is:

$$\mathbb{S}^4, \;\; \mathbb{R}^4/\Gamma, \;\; \mathbb{H}^4/\Gamma, \;\; \mathbb{CP}^2, \;\; \mathbb{S}^2 \times \mathbb{S}^2, \;\; \mathbb{CH}^2/\Gamma, \;\; \mathbb{H}^2/\Gamma_1 \times \mathbb{H}^2/\Gamma_2,$$

all with their canonical metrics. Observe that all such metrics are in fact locally symmetric.

Next, one may consider Einstein metrics which although not (locally) homogeneous, have a non-trivial or rather large isometry group. Thus, recall a Riemannian manifold (M, g) is of cohomogeneity k if the isometry group (M, g) acts on M with principal orbits of codimension k. By a well-known theorem of Bochner, any Einstein metric with a non-trival connected isometry group which does not split a Euclidean factor must have positive scalar curvature, $\lambda > 0$.

A very interesting and explicit Einstein metric of cohomogeneity 1 on $\mathbb{CP}^2 \# \bar{\mathbb{C}}\mathbb{P}^2$ was found by Page [58]; this has the form

$$g = V^{-1}dr^2 + V\sigma_1^2 + f(\sigma_2^2 + \sigma_3^2),$$

where $\{\sigma_i\}$ are the standard coframing of $S^3 = SU(2)$ and $V = V(r)$, $f = f(r)$ are explicit functions. This metric has an isometric $U(2)$ action and, up to finite covers, is the only known cohomogeneity 1 Einstein metric on a compact 4-manifold; it is still an open problem whether there are any other cohomogeneity 1 Einstein metrics on compact 4-manifolds, cf. [21].

Recently, another very interesting Einstein metric was found by Chen-LeBrun-Weber [19] on the 2-point blow-up of $\mathbb{CP}^2$, i.e. $\mathbb{CP}^2 \# 2\bar{\mathbb{C}}\mathbb{P}^2$. This metric is toric, and so has an isometric $S^1 \times S^1$ action Both the Page metric and the Chen-LeBrun-Weber metric are Hermitian-Einstein, and conformal to Kähler metrics, but are not Kähler themselves.

Together with the results on Kähler-Einstein metrics above, we see that the rational surfaces $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2 \# k\bar{\mathbb{C}}\mathbb{P}^2$, $0 \leq k \leq 8$, all admit Einstein metrics. In §4, we will see that $\mathbb{CP}^2 \# k\bar{\mathbb{C}}\mathbb{P}^2$ does not admit an Einstein metric, for $k \geq 9$.

It would be interesting to understand the class of cohomogeneity 2 Einstein metrics on compact 4-manifolds in general. In the case of static $S^1 \times S^1$ actions, where the metric is a double warped product over a 2-dimensional base, a classical procedure due to Weyl, cf. [67], gives a method of constructing local Ricci-flat metrics from an axisymmetric harmonic function on a domain in $\mathbb{R}^3$; these are the so-called static axisymmetric vacuum metrics in general relativity and are governed by a single linear (!) scalar equation. Can this procedure be generalized to situations where $\lambda > 0$ or $\lambda < 0$? Can one construct new global solutions of the Einstein equations by this or related procedures?; see [67], and [16] for instance for further discussion.

There has also been very little study of Einstein metrics having only isometric S^1 actions, although again these are important in general relativity; see [66] for some non-trivial results in this direction.

A further construction of Einstein metrics, analogous to the Thurston theory of Dehn surgery on hyperbolic 3-manifolds will be discussed in §7. Together with the metrics listed above, this constitutes the *complete* list of known Einstein metrics. Clearly, there is much further territory to explore here.

4. Obstructions to Einstein metrics.

In this section, we discuss a number of the known obstructions to the existence of Einstein metrics on 4-manifolds. In tandem with this, we also discuss several rigidity or uniqueness results for such metrics on a given manifold.

The most elementary obstruction comes from a simple observation of Berger.

Theorem 4.1. [12]. *If (M^4, g) is an Einstein metric, then*

$$\chi(M) \geq 0, \tag{4.1}$$

with equality if and only if (M^4, g) is flat.

Proof: The Chern-Gauss-Bonnet formula in dimension 4 reads

$$\chi(M) = \frac{1}{8\pi^2}\int_M |R|^2 - |z|^2 = \frac{1}{8\pi^2}\int_M |W|^2 - \tfrac{1}{2}|z|^2 + \tfrac{1}{24}s^2, \tag{4.2}$$

where $z = Ric - \frac{s}{4}g$ is the trace-free Ricci curvature R is the full Riemann curvature and W is the Weyl curvature. Einstein metrics are characterized by the condition that $z = 0$, and so the result follows immediately. ■

In a certain sense then, at least one-half of 4-manifolds do not admit Einstein metrics. Simple examples include all circle bundles over 3-manifolds $M^3 \neq T^3$, or products of surfaces of genus of non-equal sign, e.g. $S^2 \times \Sigma_g$, $g \geq 1$, or $T^2 \times \Sigma_g$, $g \geq 2$, (or more generally such surface bundles over surfaces). This kind of argument, although of course very simple, is typical of many of the known obstructions to the existence of Einstein metrics. One finds (sharp) inequalities among characteristic numbers of M, for which equality implies a rigidity of Einstein metrics on the given space. Such rigidity then appears as the border between possible existence and non-existence.

For example, a strengthening of this argument was found by Hitchin and Thorpe (independently), by bringing in the Hirzebruch signature formula:

$$\tau(M) = \frac{1}{12\pi^2}\int_M |W_+|^2 - |W_-|^2, \tag{4.3}$$

where $\tau(M)$ is the signature of M and $W_\pm$ are the self-dual and anti-self-dual components of the Weyl tensor. The analysis of the equality case below is due to Hitchin.

Theorem 4.2. [37, 71]. *If (M^4, g) is an Einstein metric, then*

$$\chi(M) \geq \frac{3}{2}|\tau(M)|, \tag{4.4}$$

with equality if and only if (M^4, g) is flat, or (M^4, g) is a Ricci-flat Kähler metric on the $K3$ surface (Calabi-Yau metric), or a quotient of such.

Sketch of Proof: The inequality (4.4) is essentially an immediate consequence of the formulas (4.2) and (4.3). If equality holds, then (M, g) has (anti)-self-dual curvature and zero scalar curvature, and so is Ricci-flat. If M is simply connected, then an examination of the space $\Lambda^+(M)$ of self-dual 2-forms shows that (M, g) is Kähler, so that $c_1 = 0$, which gives the result. ■

For example, for the rational surfaces $M_k = \mathbb{CP}^2 \# k\bar{\mathbb{C}}\mathbb{P}^2$, one has $\chi(M_k) = 3 + k$ and $\tau(M_k) = 1 - k$. Hence if $k \geq 9$, then M_k does not admit an Einstein metric. As discussed in §3, M_k does admit an Einstein metric for $k \leq 8$.

Note that the characteristic numbers that enter the results above are all homotopy invariant, and so the results apply to all possible smooth structures on a given topological 4-manifold. Gompf and Mrowka show in [30] that there are infinitely many distinct smooth structures within the homeomorphism class of the K3 surface; hence it follows immediately from Theorem 4.2 that none of these exotic smooth structures on K3 admits an Einstein metric.

A strengthing of (4.4) holds in the case of complex surfaces with $c_1 < 0$. Namely, a standard result from complex surface theory gives $c_2 = \chi$ and $c_1^2 = 3\tau + 2\chi$, so that, for any compact complex surface,

$$\chi(M) - 3\tau(M) = 3c_2 - c_1^2.$$

Via Chern-Weil theory the terms c_2 and c_1^2 are given in terms of the curvature of Hermitian metrics on M and by inspection, it easy to see that if (M, J) admits a Kähler-Einstein metric, then

$$\chi(M) - 3\tau(M) = 3c_2 - c_1^2 \geq 0, \tag{4.5}$$

with equality if and only if (M, J) is biholomorphic to a complex hyperbolic space-form $\mathbb{CH}^2/\Gamma$. By Theorem 3.1, the Bogomolov-Miyaoka-Yau inequality (4.5) thus holds for any complex surface admitting a Kähler metric with $c_1 < 0$. (This equation is uninteresting when $c_1 = 0$ or $c_1 > 0$).

Yau's proof above of the rigidity of $\mathbb{CH}^2/\Gamma$ among complex surfaces and among Kähler-Einstein metrics does not extend to give a rigidity for (non-Kähler) Einstein metrics. In a series of papers, LeBrun has used the Seiberg-Witten theory to extend many of the rigidity and non-existence results of Kähler-Einstein metrics to general Einstein metrics. For instance:

Theorem 4.3. [51]. *Suppose (M, g) is Einstein, non-flat, and M admits an almost complex structure J. With respect to the orientation and spin^c structure induced by J, suppose the mod 2 Seiberg-Witten invariant $\eta_c(M) \neq 0$. Then*

$$\chi(M) \geq 3\tau(M), \tag{4.6}$$

with equality if and only if (M, g) is homothetic to a complex hyperbolic space-form $\mathbb{CH}^2/\Gamma$.

Corollary 4.4. [51]. *The locally symmetric metric g_0 on any compact space-form $M = \mathbb{CH}^2/\Gamma$ is the unique Einstein metric on M, up to rescaling and isometry.*

Proof: By a result of Witten and Kronheimer, the manifold M has non-zero Seiberg-Witten invariant. Theorem 4.3 then implies that any Einstein metric on M is homothetic to a locally symmetric metric, and the result then follows from Mostow rigidity. ■

Next we turn to a very different perspective on rigidity and non-existence results, developed by Besson-Courtois-Gallot [14]. This holds in all dimensions, so in the following we assume $M = M^n$ is a compact n-dimensional manifold.

For a given compact Riemannian manifold (M, g) let $\widetilde{M}$ denote the universal cover and define the volume entropy of (M, g) by

$$h(M, g) = \lim_{r\to\infty} \frac{\ln vol B_x(r)}{r}, \tag{4.7}$$

where $B_x(r)$ is the geodesic r-ball about $x \in \widetilde{M}$. It is easy to see this is independent of x. Note also that $h(M, g)$ scales inversely to the distance, so that $h^n(M, g) vol_g M$ is scale-invariant. If (M, g_0) is hyperbolic, then a simple computation gives $h(M, g_0) = n-1$. More generally, if (M, g) is Einstein, then by the Bishop-Gromov volume comparison theorem,

$$h(M,g) \leq (n-1)\sqrt{\frac{\lambda_-}{n-1}}, \tag{4.8}$$

where $\lambda_- = -\min(0, \lambda)$, with equality if and only if (M, g) is hyperbolic.

Theorem 4.5. [14]. *Let (X, g_0) be a compact oriented locally symmetric space of negative curvature, and let M be any compact manifold with $dim\, M = dim\, X$. Suppose $f : M \to X$ is smooth. Then for any metric g on M, one has*

$$h^n(M,g)vol_g(M) \geq |degf| h^n(X,g_0)vol_{g_0}(X), \tag{4.9}$$

with equality if and only if f is a covering map and g is locally homothetic to g_0.

Sketch of Proof: The main ideas are already present in the simplest case, where $X = M$ is hyperbolic, of curvature -1, and so we assume this in the following. Let $L^2(S^{n-1}(\infty), d\theta)$ be the Hilbert space of L^2 functions on the sphere at infinity $S^{n-1}(\infty)$ of the hyperbolic space $\mathbb{H}^n(-1) = \widetilde{X}$. Let S_+^∞ be the space of positive functions of norm 1 in $L^2(S^{n-1}(\infty), d\theta)$. The central objects of study are $\pi_1(M)$-equivariant Lipschitz maps

$$\Phi : \widetilde{M} \to S_+^\infty. \tag{4.10}$$

The canonical example here is the square root of the Poisson kernel,

$$\Phi_0 = \sqrt{p_0},$$

where $p_0(x, \theta) = e^{-(n-1)\beta_\theta(x)}$ and β_θ is the Busemann function associated with the base point $\theta \in S^{n-1}(\infty)$. Moreover, for any metric g on M, one has another natural class of examples, given by $\Phi_c(x, \theta) = \Psi_c(x, \theta)/|\Psi_c(x, \theta)|_{L^2(d\theta)}$, where

$$\Psi_c(x,\theta) = (\int_M e^{-cd(x,y)} p_0(y,\theta) dv_g(y))^{1/2}.$$

Here c is any number satisfying $c > h(M, g)$, so that $\Psi_c(x, \cdot)$ is well-defined in $L^2(S^{n-1}(\infty), d\theta)$.

In general, any such Φ induces a (possibly degenerate) metric g_Φ on M by pullback, i.e.

$$g_\Phi = \Phi^*(g_{can}),$$

where g_{can} is the canonical L^2 metric (product of L^2 functions) on S_+^∞. A simple computation shows that

$$g_{\sqrt{p_0}} = \frac{(n-1)^2}{4n} g_0,$$

so that the embedding by the Poisson kernel is a homothety. Further straightforward computation shows that

$$g_{\Phi_c} \leq \frac{c^2}{4} g.$$

Now define the spherical volume $vol_{Sph}(M)$ to be $\inf vol_{g_\Phi}(M)$, where the inf is taken over all Φ as in (4.10). A simple analysis using the functions Φ_c gives an upper bound:

$$vol_{Sph}(M) \leq (\frac{h(g)^2}{4n})^{n/2} vol(M,g), \tag{4.11}$$

for any metric g on M. In the case at hand, the result (4.9) then follows from the claim that

$$vol_{Sph}(M) = (\frac{h(g_0)^2}{4n})^{n/2} vol(M,g_0). \tag{4.12}$$

This is proved by exhibiting a calibration form for the Poisson kernel embedding. In slightly more detail, for any absolutely continuous measure $d\mu$ on $S^{n-1}(\infty)$, define the barycenter $B(\mu)$ to be the unique $x \in \widetilde{X}$ such that

$$\int_{S(\infty)} d\beta_{(x,\theta)}(v)d\mu(\theta) = 0,$$

for any $v \in T_x\widetilde{X}$. Standard convexity arguments in hyperbolic geometry show that x is uniquely defined. This defines a Γ-equivariant map

$$\pi : S_+^\infty \to \widetilde{X}, \quad \phi \to B(\phi^2(\theta)d\theta).$$

Now let ω_0 be the volume form of the constant curvature metric g_0 on $\widetilde{X}$. One then shows that the closed n-form $\pi^*\omega_0$ on S_+^∞ is a calibration for the Poisson kernel embedding $x \to \sqrt{p_0(x)}$ of comass $(\frac{4n}{h_0^2})^{n/2}$, which gives (4.12).

■

Theorem 4.5 gives easily the following uniqueness or rigidity result for Einstein metrics on hyperbolic 4-manifolds; note in particular that this result gives a new proof of the Mostow rigidity theorem for hyperbolic metrics.

Corollary 4.6. [14]. *Suppose N is a compact manifold homotopy equivalent to a hyperbolic 4-manifold (M, g_0). Then N admits an Einstein metric only if N is diffeomorphic to M, and moreover, g_0 is the unique Einstein metric on $N = M$, up to scaling and isometry.*

Proof: By (4.2) and (4.3), for (N, g) Einstein,

$$2\chi \pm 3\tau = \frac{1}{4\pi^2}\int_N 2|W_\pm|^2 + \frac{s^2}{24} - \frac{|z|^2}{2} \geq \frac{1}{6\pi^2}\lambda^2 vol(N, g).$$

For the hyperbolic metric, this gives $2\chi \pm 3\tau = \frac{3}{2\pi^2}vol(M, g_0)$, and hence

$$vol(M, g_0) \geq \frac{\lambda^2}{9}vol(N, g).$$

Combining this with Theorem 4.5 and (4.8) gives

$$3^4(\frac{\lambda_-}{3})^2 vol(N, g) \geq h^n(N, g)vol(N, g) \geq 3^4 vol(M, g_0) \geq 3^4(\frac{\lambda}{3})^2 vol(N, g).$$

It follows that $\lambda < 0$ and all the inequalities above are equalities. The rest of the proof follows from the rigidity statement in Theorem 4.5. ■

In contrast to the Ricci-flat and Ricci-negative results mentioned above, there are currently no rigidity or uniqueness results for Ricci-positive Einstein metrics, for instance for the standard metrics on S^4 or $\mathbb{CP}^2$. It is known, cf. [13], that the standard metrics on these spaces are locally rigid, i.e. the metrics are isolated points in the moduli space of Einstein metrics.

An extension of the reasoning in Corollary 4.6 gives the following non-existence result.

Corollary 4.7. [64]. *For any given values (k, l) with $k - l \equiv 0$ $(mod\ 2)$, there are infinitely many non-homeomorphic closed 4-manifolds X_i which satisfy $(\chi(X_i), \tau(X_i)) = (k, l)$, and which admit no Einstein metric.*

Sketch of Proof: The idea is to take connected sums of a hyperbolic manifold M as above with copies of $\pm\mathbb{CP}^2$, $S^2 \times S^2$ or $S^2 \times T^2$ and use the degree theory part of Theorem 4.5. ■

There are a several further interesting obstructions to the existence of Einstein metrics on 4-manifolds, but for lack of space we will forgo a detailed discussion. First, there is an improvement of the Hitchin-Thorpe inequality due to Gromov [32], when the manifold M has non-zero simplicial volume, cf. in particular [44], [47]. Next, based on information from the Seiberg-Witten invariants developed by LeBrun in [52], Kotschick, LeBrun and many others have found a wide variety of simply connected 4-manifolds of a fixed homeomorphism type, which have an Einstein metric for one smooth structure, but without Einstein metrics for other smooth structures, analogous to the discussion following Theorem 4.2. Also, in the same context, there are Einstein metrics with $\lambda > 0$ for one smooth structure and with $\lambda < 0$ for a different smooth structure; we refer to [52, 53], [45, 46] and references therein for further details.

5. Moduli spaces I.

In this section, we discuss various aspects of the moduli space $\mathcal{E}$ of Einstein metrics on a given compact 4-manifold M. We begin with local results, (which hold in all dimensions), and then pass to more global issues on the structure of $\mathcal{E}$.

The Einstein equations (1.1) are invariant under scaling, and so throughout §5 and §6, we assume all metrics are normalized to have unit volume. Let $\mathbb{E} = \mathbb{E}(M)$ denote the space of all (unit volume) Einstein metrics on a given manifold M, viewed as a subset of the space $Met(M)$ of all unit volume Riemannian metrics on M. As noted in §2, Einstein metrics are C^∞ smooth, in fact real-analytic, in suitable local coordinate systems. The group $\mathcal{D}$ of C^∞ diffeomorphisms acts continuously on $\mathbb{E}$ and the quotient

$$\mathcal{E} = \mathcal{E}(M) \tag{5.1}$$

is the moduli space of Einstein metrics on M. It is standard that $\mathcal{E}$ is Hausdorff, with countably many components, cf. [13]. As noted in §2, Einstein metrics are critical points of the total scalar curvature (2.2). Hence s_g, or equivalently λ, is constant on each component of $\mathcal{E}$.

The Einstein equations (1.1) are not elliptic, due to their invariance under diffeomorphisms. In fact if $\hat{E}(g) = Ric_g - \lambda g$ denotes the Einstein operator, then the linearization $D\hat{E}$ is given by

$$2D\hat{E}_g(h) = D^*Dh - 2R(h) - 2\delta^*\beta(h),$$

where $R(h)$ is the action of the curvature tensor on the space $S^2(M)$ of symmetric bilinear forms on M, β is the Bianchi operator, $\beta(h) = \delta h + \frac{1}{2}dtrh$ and D^*D is the rough Laplacian. One has $D\hat{E}_g(\delta^*X) = 0$, for any vector field X on M, so that $Ker\, D\hat{E}_g$ is infinite dimensional.

As is usual in geometrically covariant problems, one needs to fix a gauge transverse to the action of $\mathcal{D}$ to obtain an elliptic system. To do this, first pass to the usual Einstein operator used in physics,

$$E(g) = Ric_g - \frac{s}{2}g + \Lambda g, \tag{5.2}$$

where $\Lambda = \frac{n-2}{2}\lambda$, $n = dimM$. Then $E^{-1}(0)$ consists of Eintein metrics (1.1). Choose a background metric $g_0 \in \mathbb{E}$ and consider the divergence-gauged Einstein operator

$$\Phi_{g_0}(g) = Ric_g - \frac{s}{2}g + \Lambda g + \delta_g^*\delta_{g_0}(g), \tag{5.3}$$

where δ_{g_0} is the divergence operator with respect to g_0. The linearization of Φ at $g = g_0$ is given by

$$2(D\Phi)_{g_0}(h) = L(h) = D^*Dh - 2R(h) - D^2trh - \delta\delta h\, g + \Delta trh\, g + \frac{s}{n}trh\, g. \tag{5.4}$$

The operator L is self-adjoint, and elliptic when $n \geq 3$. Thus L is Fredholm, and so has finite dimensional kernel and cokernel with closed range. Let $\mathbb{Z} = \Phi^{-1}(0)$ be the zero-set of Φ. We observe that for metrics g close to g_0 one has

$$\mathbb{Z} \subset \mathbb{E}. \tag{5.5}$$

To see this, apply δ_g to (5.3). By the Bianchi identity, $\delta_g E(g) = 0$, and hence, for $g \in \mathbb{Z}$ and $V = \delta_{g_0} g$, one has

$$\delta\delta^* V = 0.$$

Pairing this with V and integrating by parts gives $\delta^* V = 0$ which implies (5.5).

On the other hand, via the Ebin slice theorem, for any $g \in \mathbb{E}$ near g_0 there exists $\phi \in \mathcal{D}$ such that $\phi^* g$ is in divergence-free gauge, i.e.

$$\delta_{g_0}(\phi^* g) = 0.$$

Thus $\mathbb{Z}$ is a local slice for $\mathbb{E}$, transverse to the action of $\mathcal{D}$. This leads to the following result of Koiso.

Theorem 5.1. [41]. *Near any $g_0 \in \mathbb{E}$, the space $\mathbb{Z} \subset \mathbb{E}$ has the structure of a finite dimensional real-analytic subvariety of a smooth, finite dimensional manifold $\mathbb{W}$. Further the tangent space $T_{g_0}\mathbb{W}$ consists of the space of essential infinitesimal Einstein deformations of g_0.*

Proof: Let $\pi : S^2(M) \to Im\, L$ be the projection onto Im $L = Im\, D\Phi$, and recall that $Im\, L$ is of finite codimension in $S^2(M)$. The composition $F = \pi \circ \Phi : Met(M) \to Im\, L$ is then a submersion near g_0, i.e. the derivative DF is surjective, with splitting kernel. It follows from the implicit function theorem in Banach spaces that $\mathbb{W} \equiv F^{-1}(0)$ is smooth submanifold of $Met(M)$ near g_0 of dimension $dim\, Ker\, L = dim\, Coker\, L < \infty$.

The tangent space to $\mathbb{W}$ at $g = g_0$ equals $Ker\, L$. For $h \in Ker\, L$, the same arguments establishing (5.5) show that $\delta h = 0$. Using this, and taking the trace of (5.4) then gives

$$\Delta trh + \tfrac{s}{n} trh = 0.$$

If $s \leq 0$, then it is immediate that $trh = 0$. If $s > 0$, this conclusion also follows from the Lichnerowicz estimate on the first eigenvalue of the Laplacian: $\lambda_1 \geq \frac{s}{n-1}$. Hence, $Ker\, L$ consists of the forms h satisfying

$$D^* Dh - 2R(h) = 0. \tag{5.6}$$

This, together with the conditions $\delta h = trh = 0$ are the equations for essential infinitesimal Einstein deformations.

Finally, Φ is a real-analytic function on $\mathbb{W}$, so the zero set $\mathbb{Z}$ is a real-analytic subvariety of $\mathbb{W}$. ■

Remark 5.2. (i). In [41], Koiso finds examples where one has a strict inclusion

$$\mathbb{Z} \subset \mathbb{W},$$

and hence there are situations where infinitesimal Einstein deformations are not tangent to a curve of Einstein metrics in $\mathbb{E}$; the moduli space is non-integrable or obstructed. This occurs for instance on $\mathbb{CP}^1 \times \mathbb{CP}^{2k}$. However, there are no examples where this occurs in dimension 4.

In fact, there are still no examples where $\mathbb{Z}$ is not a finite dimensional manifold. Is the non-integrability related to the existence of Killing fields? For instance, if there are no Killing fields on (M, g), (for example $\lambda < 0$), is $\mathbb{Z} = \mathbb{W}$ near g?

(ii). Is there any method to compute the dimension of components of $\mathcal{E}$?

(iii). There are several well-known local rigidity results for Einstein metrics under various curvature conditions, which show that a given metric $g \in \mathcal{E}$ is an isolated point in the full moduli space. This is the case for instance for Einstein metrics of strictly negative sectional curvature, or for irreducible symmetric spaces, cf. [13] for further discussion. All of these results follow from an analysis (generally algebraic) on solutions of (5.6).

The first global results on the structure of moduli space $\mathcal{E}$ of unit volume Einstein metrics on a given 4-manifold M were obtained in [1], [9], [55]. These results bear some similarities to the results of Uhlenbeck [48] on the moduli space of self-dual Yang-Mills fields.

To describe the situation, we first need the following definition. An Einstein orbifold (V, g) associated to a 4-manifold M is a 4-dimensional orbifold, with a finite number of singular points q_k, each having a neighborhood homeomorphic to the cone $C(S^3/\Gamma)$, where $\Gamma \neq \{e\}$ is a finite subgroup of $SO(4)$. Let $V_0 = V \setminus \cup q_k$ be the regular, (smooth manifold), set of V. Then g is a smooth Einstein metric on V_0, which extends smoothly over $\{q_k\}$ in local finite covers. The manifold M is a resolution of V in the sense that there is a continuous surjection $\pi : M \to V$ such that $\pi|_{\pi^{-1}(V_0)} : \pi^{-1}(V_0) \to V_0$ is a diffeomorphism onto V_0. In particular, V is compact.

Theorem 5.3. [1], [9], [55]. *The completion $\bar{\mathcal{E}}_{GH}$ of $\mathcal{E}$ in the Gromov-Hausdorff topology consists of $\mathcal{E}$ together with unit volume Einstein orbifold metrics associated to M.*

Moreover, the completion is locally compact, in that any sequence $g_i \in \bar{\mathcal{E}}_{GH}$, bounded in the Gromov-Hausdorff topology, has a subsequence converging to an Einstein orbifold associated to M. There is a uniform bound on the number of orbifold singularities and the order of the local groups Γ in terms of $\chi(M)$.

Sketch of Proof: By (4.2), unit volume Einstein metrics on M have a lower bound on their scalar curvature and hence a uniform lower bound on their Ricci curvature. The completion in the Gromov-Hausdorff topology is then equivalent to the completion with respect to a diameter bound, so that any Cauchy sequence $\{g_i\} \in \mathbb{E}$ satisfies

$$vol_{g_i} M = 1,\ s_{g_i} \geq s_0 > 0,\ diam_{g_i} M \leq D, \tag{5.7}$$

for some $D = D(g_i) < \infty$. Gromov's weak compactness theorem [31] implies that the Cauchy sequence $\{g_i\}$ converges in the Gromov-Hausdorff topology to a complete length space (X, d_∞). One needs then to understand the structure of the limit (X, d_∞).

Given $r > 0$, let $\{x_k\}$ be a maximal $r/2$ separated set, (depending on i), in (M, g_i). Thus, the geodesic balls $B_{x_k}(\frac{r}{2})$ are disjoint while the balls $B_{x_k}(r)$ cover M. Choose a fixed $\delta_0 > 0$ small, and let

$$G_i^r = \cup\{B_{x_k}(r) : \int_{B_{x_k}(2r)} |R|^2 dV < \delta_0\}, \tag{5.8}$$

and similarly, let

$$B_i^r = \cup\{B_{x_k}(r) : \int_{B_{x_k}(2r)} |R|^2 dV \geq \delta_0\}. \tag{5.9}$$

All quantities here are with respect to (M_i, g_i). For each i, one has $M_i = G_i^r \cup B_i^r$. Observe via (4.2) that there is uniform bound $K = 8\pi^2\chi(M)$ on the number Q_i^r of r-balls in B_i^r:

$$Q_i^r \leq \frac{K}{\delta_0}. \tag{5.10}$$

Einstein metrics satisfy the inequality

$$\Delta|R| + c|R|^2 \geq 0,$$

where c is a constant, depending only on dimension. Using this together with the deGiorgi-Nash-Moser method in elliptic PDE, cf. [29], one shows that for δ_0 small, depending only on D in (5.7), one has the L^∞ estimate

$$|R| \leq C\delta_0 r^{-2} \ \text{ on } \ G_i^r. \tag{5.11}$$

In fact (5.11) holds on the $r/2$ thickening of G_i^r. It then follows from the smooth Gromov compactness theorem that for any given $r > 0$, a subsequence of G_i^r converges in the $C^{1,\alpha}$ topology to a limit manifold G_∞^r with limit $C^{1,\alpha}$ metric g_∞^r. In particular, G_∞^r and G_i^r are diffeomorphic, for i large, and there exist smooth embeddings $F_i^r : G_\infty^r \to G_i^r \subset M_i$ such that $(F_i^r)^*(g_i)$ converges in $C^{1,\alpha}$ to g_∞^r. Via regularity of the Einstein equation as in (2.2), both the limit metric g_∞^r and the convergence are in fact C^∞ smooth.

Now choose a sequence $r_j \to 0$, with $r_{j+1} = \frac{1}{2}r_j$, and perform the above construction for each j. Let $G_i(r_m) = \{x \in (M_i, g_i) : x \in G_i^j, \text{for some } j \leq m\}$, so that one has inclusions

$$G_i(r_1) \subset G_i(r_2) \subset ... \subset M_i$$

By the argument above, each $G_i(r_m) \subset (M_i, g_i)$, for m fixed, has a subsequence converging smoothly to a limit $G_\infty(r_m)$. Clearly $G_\infty(r_m) \subset G_\infty(r_{m+1})$ and we set

$$G = \cup_1^\infty G(r_m),$$

with the induced metric g_∞. Thus, (G_∞, g_∞) is C^∞ smooth and for any m, there are smooth embeddings $F_i^m : G(r_m) \to M_i$, for i sufficiently large, such that $(F_i^m)^*(g_i)$ converges smoothly to the metric g_∞.

Let $\bar{G}$ be the metric completion of G with respect to g_∞. Then there is a finite set of points q_k, $k = 1, \cdots, Q$, such that

$$\bar{G} = G \cup \{q_k\}.$$

This follows since there is a uniform upper bound (5.10) on the cardinality of B_i^r, for all r small, and all i, independent of r, i. It is then easy to see that a subsequence of (M_i, g_i) converges to the length space $(\bar{G}, g_\infty)$ in the Gromov-Hausdorff topology, so that $X = \bar{G}$.

It remains to prove that $\bar{G}$ is an orbifold, with orbifold singular points $\{q_k\}$, i.e. $\bar{G} = V$. This follows by an analysis of the tangent cone of the limit metric g_∞ near each q_k, i.e. by a blow-up analysis. The curvature of G is locally bounded in L^p, for any $p < \infty$. Further, by lower semi-continuity of the norm under weak convergence, the L^2 norm of the curvature on (G, g_∞) is globally bounded:

$$\int_G |R|^2 dV_{g_\infty} \leq 8\pi^2 \chi(M). \tag{5.12}$$

In particular, for any $q = q_k \in \bar{G}$,

$$\int_{A_q(\frac{1}{2}r, 2r)} |R|^2 dV \to 0, \quad \text{as} \quad r \to 0, \tag{5.13}$$

and hence, as in (5.11), near any singular point q one has

$$|R| \leq \varepsilon(r) r^{-2}, \tag{5.14}$$

where $r(x) = dist(x, q)$ and $\varepsilon(r) \to 0$ as $r \to 0$.

Let $s_j = 2^{-j}$, for j large, and rescale the metric g_∞ on G near q by s_j^{-2}, i.e. consider the metrics $\bar{g}_j = s_j^{-2} \cdot g_\infty$. The bounds (5.7), together with the Bishop-Gromov volume comparison theorem and the curvature bound (5.14) imply that a subsequence of $(G \backslash q, \bar{g}_j, q)$ converges, modulo diffeomorphisms, smoothly to a flat limit $(T^\infty, \bar{g}^\infty)$. Next one shows that T^∞ has a bounded number of components and the metric completion $\bar{T}^\infty$ of each component of T^∞ has a single isolated singularity $\{0\}$. Thus, $\bar{T}^\infty$ is a finite collection of complete flat manifolds joined at a single isolated singularity $\{0\}$. From this, it follows easily that $\bar{T}^\infty$ is isometric to a union of flat cones $C(S^{n-1}/\Gamma_j)$. By the smooth convergence, this (unique) structure on the limit is equivalent to the structure of (G, g_∞) on small scales near the singular point q. Via the Cheeger-Gromoll splitting theorem, one proves that the regular set V_0 is locally connected and hence each singular point is an orbifold singularity. It is proved in [9] that in the local finite cover resolving a singularity q, the lifted metric g_∞ extends smoothly across the origin.

The remaining parts of Theorem 1.1 are now easily established, via (5.7), (4.2) and the volume comparison theorem. For the proof of Theorem 5.3 in the case that g_i are Einstein orbifold metrics associated to M, see [3]. ■

A main point of the Uhlenbeck completion for self-dual connections is that the completion is *compact*. In the current context of Einstein metrics, consider the components $\mathcal{E}_{\lambda_0}$ of $\mathcal{M}$ for which

$$\lambda \geq (n-1)\lambda_0 > 0. \tag{5.15}$$

Myers' theorem, (for manifolds of positive Ricci curvature), then implies that

$$diam_g M \leq \pi/\sqrt{\lambda_0},$$

so that (5.7) holds automatically. Hence, the completion of $\mathcal{E}_{\lambda_0}$ in the Gromov-Hausdorff topology is compact. However, this is certainly not the case when $\lambda \leq 0$. For example, the moduli space $\mathcal{E}$ or $\bar{\mathcal{E}}_{GH}$ on a torus T^4, or on a product $\Sigma_{g1} \times \Sigma_{g2}$ of surfaces of genus at least 2, is certainly not compact. Thus, for a better understanding one needs to consider what happens when the Gromov-Hausdorff distance goes to infinity. This will be discussed further in §6.

There is another strong difference compared with the Uhlenbeck completion. Namely, the frontier $\partial_o\mathcal{E} = \bar{\mathcal{E}}_{GH} \setminus \mathcal{E}$ should not really be thought of as a boundary, but instead as a filling in of "missing pieces" in $\mathcal{E}$. For example, as discussed in §6, in the case of K3 surfaces this frontier consists of subvarieties of codimension 3 in $\bar{\mathcal{E}}$ and so does not form a boundary in the sense of a wall at which the moduli space comes to an end.

Although there is currently very little evidence, we venture the following (optimistic) conjecture, which would confirm this picture in general:

Conjecture. The space $\partial_o\mathcal{E} \subset \bar{\mathcal{E}}_{GH}$ of Einstein orbifold metrics associated to M is of codimension at least 2 in $\bar{\mathcal{E}}_{GH}$.

The orbifold limits (V, g_∞) arise from the "bubbling off" of so-called gravitational instantons, (again in analogy to the case of Yang-Mills fields). These spaces, called EALE spaces here, are complete Ricci-flat metrics (N, g) which have curvature in L^2,

$$\int_N |R|^2 dV_g < \infty, \tag{5.16}$$

and which are ALE, (asymptoticaly locally Euclidean), in that the metric g at infinity is asymptotic to a flat cone $C(S^3/\Gamma)$, where $\{e\} \neq \Gamma \subset SO(4)$ is a finite subgroup. Thus, outside a compact set $K \subset N$, there is a finite cover of $N \setminus K$ which is diffeomorphic to $\mathbb{R}^4 \setminus B$, and a chart in which the (lifted) metric g has the form

$$|g_{ij} - \delta_{ij}| = \varepsilon(r), \quad |R| = \varepsilon(r)r^{-2},$$

where $\varepsilon(r) \to 0$ as $r \to \infty$. It is proved in [9] that in fact one has $\varepsilon(r) = r^{-2}$. A similar definition holds for EALE orbifolds.

To describe this bubbling process, let q be a singular point of the limit V. If $x_i \in (M, g_i)$ is any sequence of points such that $x_i \to q$ in the Gromov-Hausdorff topology, then the curvature of (M, g_i) blows up near x_i, i.e. diverges to infinity, as $i \to \infty$. If one rescales the metrics g_i so that the curvature remains bounded near x_i, then a subsequence converges to a complete EALE space (N, g). Blowing this limit (N, g) down, i.e. rescaling g by factors converging to 0, gives a spherical cone at a single vertex $\{0\}$.

However, the curvature of (M, g_i) may diverge to infinity at a number of different scales near any singular point q, giving rise to a collection of such EALE spaces associated with each scale. This gives rise to a so-called "bubble-tree" of EALE spaces and scales. The structure of the limit orbifold (V, g_∞) near any singular point q is recaptured by the structure at infinity of the complete EALE orbifold corresponding to the *smallest* rate at which the curvature of (M_i, g_i) diverges to infinity near q; this corresponds to the *largest* distance scale, since the curvature scale corresponds to the inverse square of the distance scale.

In more detail, for $x_i \to q$ as above, there is sequence of scales $r_i = r_i^1 \to 0$ such that the rescalings $(M, r_i^{-2} g_i, x_i)$ with $x_i \to q$, converge, (in a subsequence), in the pointed Gromov-Hausdorff topology to a complete, EALE orbifold (V^1, g^1) with a finite number of singular points, and

$$\int_{V^1} |R|^2 \geq \delta_0, \tag{5.17}$$

for a fixed $\delta_0 > 0$, (as in (5.9)). If (V^1, g^1) is not a smooth manifold, then there are second level scales $\{r_i^2\}$ associated with each singular point of V^1; (the scales r_i^2 depend on the choice of singular point in V^1). Rescaling $r_i^{-2} g_i$ by such factors at base points converging to the singular points of V^1 gives a collection of 2^{nd} level EALE orbifolds $\{(V^2, g^2)\}$ associated with each singular point of (V^1, g^1). Each iteration of this process satisfies (5.17), and hence, via (4.2), (as in (5.10)), terminates after a finite number of iterations. At the last stage, corresponding to the smallest scales, the resulting blow-up limits are non-flat smooth manifolds, cf. [10], [7] for full details.

The topology of the original manifold M may then be reconstituted from that of V_0, and the scale of orbifolds associated with each singular point $q \in V$. In particular, the homology groups of M are determined, (by the Mayer-Vietoris sequence), by the homology of V and the homology of the collection of EALE orbifold spaces $\{V^j\}$. Thus the orbifold singularities correspond to a generalized connected sum decomposition of M, in that M is the union of the regular set V_0 with a finite collection of EALE spaces:

$$M = V_0 \cup \{N_m\}, \tag{5.18}$$

where the union is along non-trivial spherical space forms. The collection $\{N_m\}$ itself could well consist of orbifolds, in which case it also splits inductively as a union along spherical space forms.

Clearly then it is important to understand the geometry and topology of these EALE spaces in detail. In the special case where (N, g) is simply connected and Kähler, (and so hyperkähler), one has a complete description and classification due to Kronheimer. To describe this, let $\Gamma \subset SU(2)$ be a finite group. Then Γ belongs to one of the following five classes.

- A_k: Γ is the cyclic group $\mathbb{Z}_{k+1}$.
- D_k: Γ is the binary dihedral group of order $4(k-2)$.
- E_6, E_7, E_8: Γ is the binary tetrahedral, binary octahedral group or binary icosahedral group respectively.

The quotient $\mathbb{C}^2/\Gamma$ is called a rational double point. Let $\pi : N_\Gamma \to \mathbb{C}^2/\Gamma$ be a minimal resolution of $\mathbb{C}^2/\Gamma$. The exceptional divisor $E = \pi^{-1}(0)$ is a union of $\mathbb{CP}^1$'s, $E = \Sigma_1 + \cdots + \Sigma_l$, and so in particular $H_2(N_\Gamma, \mathbb{Z}) = \oplus_1^l \mathbb{Z}$ is generated by $\{\Sigma_i\}$. Moreover, the intersection form $(\Sigma_i \cdot \Sigma_j)$ is given by the negative of the Cartan matrix associated to the root system of the Lie group associated to Γ.

Theorem 5.4. [49, 50]. *For any such N_Γ, there is a hyperkähler EALE metric (N_Γ, g), which is uniquely determined by three cohomology classes α_1, α_2, $\alpha_3 \in H^2(N_\Gamma, \mathbb{R})$ which satisfy the following nondegeneracy condition: for each $\Sigma \in H_2(N_\Gamma, \mathbb{Z})$ with $\Sigma \cdot \Sigma = -2$, there exists i such that $\alpha_i(\Sigma) \neq 0$.*

Conversely, any ALE hyperkähler 4-manifold is diffeomorphic to some N_Γ, and any such metric is uniquely determined by the cohomology classes of the Kähler forms α_i.

Kronheimer's result above arose from earlier work of Eguchi-Hanson [24], Gibbons-Hawking [27] and Hitchin [38], (among others), and it is useful to describe in detail the Gibbons-Hawking metrics. These are the hyperkähler EALE spaces for which $\Gamma = A_k$.

Thus, choose $(k+1)$ points $\{p_i\}$ in $\mathbb{R}^3$ and let $U = \mathbb{R}^3 \setminus \{p_i\}$. Let $\pi_0 : N_0 \to U$ be the principal S^1 bundle whose first Chern class is -1 when restricted to a small sphere about any p_i. Since $H_2(U, \mathbb{Z}) = \oplus_1^k \mathbb{Z}$, this uniquely determines the principal S^1 bundle π_0. Moreover, for r small, the domain $\pi_0^{-1}(B_{p_j}(r)) \subset N_0$ is diffeomorphic to a punctured 4-ball and so adding a point to each such neighborhood gives a closed, non-compact manifold N and a smooth map $\pi : N \to \mathbb{R}^3$ extending the map π_0. If l_i is a line segment joining p_i to p_{i+1}, then $\pi^{-1}(l_i)$ is a 2-sphere $S^2 \subset N$, with self-intersection -2. The collection of such 2-spheres generates the homology group $H_2(N, \mathbb{Z}) = \oplus_1^k \mathbb{Z}$. Alternately, the topology of N is given by plumbing a collection of k spheres S^2 according to the Cartan matrix of A_k.

To describe the Gibbons-Hawking metric, let $V : U \to \mathbb{R}^+$ be given by

$$V = \frac{1}{2} \sum_1^k \frac{1}{|p - p_j|}, \tag{5.19}$$

and note that V is harmonic on (U, g_0), where g_0 is the Euclidean metric on $\mathbb{R}^3$. It is easily verified that the cohomology class of the closed 2-form $\frac{1}{2\pi} * dV$ represents the Chern class $c_1(\pi_0)$ of the S^1 bundle π_0 in deRham cohomology. Thus, there is a connection form $\omega \in \Omega^1(M_0)$ whose curvature is given by $*dV$, so that

$$\pi_0^*(*dV) = d\omega.$$

The Gibbons-Hawking metric has the simple explicit form

$$g_{GH} = V^{-1} w \cdot \omega + V \pi_0^*(g_0). \tag{5.20}$$

It is easy to see that g_{GH} is smooth, and self-dual, and hence (Kähler) and Ricci-flat.

An important special case, corresponding to $k = 2$, is the Eguchi-Hanson metric on TS^2:

$$g_{EH} = V^{-1} dr^2 + V \sigma_1^2 + r^2(\sigma_2^2 + \sigma_3^2), \tag{5.21}$$

where $V(r) = (1 - (\frac{a}{r})^4)$ and $\{\sigma_i\}$ is the standard coframing of $SO(3) \simeq \mathbb{RP}^3$. Note that this metric has a free, isometric $\mathbb{Z}_2$ action, and hence the metric desends to a metric on the $\mathbb{Z}_2$ quotient, (where $b_2 = 0$).

The following is a well-known open problem: are there any non-Kähler EALE spaces (N, g), (at least if N is simply connected)? In [56], Nakajima proved that if N is spin, and the group Γ is a subgroup of $SU(2) \subset SO(4)$, and acting suitably on $\mathbb{C}^2$, then N is necessarily hyperkähler. On the other hand, since there are many Yang-Mills connections which are not self-dual, one might expect that the answer to this problem is negative.

For later purposes, we note that if one changes the potential function V in (5.18) to

$$V = 1 + \frac{1}{2} \sum_1^k \frac{1}{|p - p_j|}, \tag{5.22}$$

then the metric (5.19) remains Ricci-flat and Kähler, and gives the so-called multi-Taub-NUT metrics, cf. [28]. These are no longer ALE, but are ALF, (asymptotically locally flat), with asymptotics of the form S^3/Γ, where the S^1 fiber of the Hopf fibration is shrunk to bounded length near infinity. When $k = 1$, the potential (5.22) gives the (Riemannian) Taub-Nut metric.

Returning now to the general topological issues as in (5.18), it is clearly useful to understand the topology of general non-flat EALE spaces N. First, by the Cheeger-Gromoll splitting theorem ∂N is connected and so the inclusion map $i : \partial N = S^3/\Gamma \to N$ induces a surjection

$$\pi_1(S^3/\Gamma) \to \pi_1(N) \to 0. \tag{5.23}$$

Next, (by [1] for instance), one has

$$|\pi_1(N)| < |\Gamma|. \tag{5.24}$$

In particular, the universal cover of an EALE space is still EALE. The following result essentially appears in [3].

Proposition 5.5. *If N is simply connected, then N has the homotopy type of a bouquet of 2-spheres. In particular,*

$$b_2(N) > 0. \tag{5.25}$$

Proof: A result of Wu [74] applies in this setting, and implies that there exists a smooth Morse-Smale exhaustion function ρ on N, which has non-degenerate critical points of index at most 2; ρ is obtained by a smoothing of the distance function from any given point in N. All flow lines of $\nabla\rho$ either connect critical points of ρ, or diverge to infinity in N, and hence N can be retracted onto the space of flow lines connecting critical points. In particular, general N, (not necessarily simply connected), have the homotopy type of a 2-dimensional CW complex.

The Morse-Smale condition means that if x, y are any distinct critical points of ρ, with $\rho(y) > \rho(x)$, then the space $M_{x,y}$ of flow lines starting at x and ending at y is a smooth submanifold of N, (given by the intersection of the unstable manifold of x and the stable manifold of y). Further,

$$dim M_{x,y} = ind_y - ind_x,$$

where ind_x is the index of the critical point x.

Thus, for any $x \neq y$, one has dim $M_{x,y} = 1$ or 2. There are a finite number of 1-dimensional components $M_{x,y}$, given by arcs from x to y. The closure $\bar{M}_{x,y}$ of a 2-dimensional component is either a 2-sphere S^2, or a disc D^2 with edge identifications along $S^1 = \partial D^2$, cf. [73] for instance. The components of the latter type have non-trivial π_1, and the result then follows from the Seifert-van Kampen theorem for the fundamental group. ■

Proposition 5.6. *If (N, g) is an EALE space, then*

$$H_2(N, \mathbb{F}) \neq 0, \tag{5.26}$$

for some field F.

Proof: From the results discussed above, one has $b_1(N) = b_3(N) = 0$, and so

$$\chi(N) = 1 + b_2(N) \geq 1.$$

If $b_2(N) \neq 0$, then of course (5.26) holds with $\mathbb{F} = \mathbb{R}$, so suppose $b_2(N) = 0$, so that $\chi(N) = 1$. The Euler characteristic can be computed with homology with coefficients in any field $\mathbb{F}$, so that

$$\chi(N) = 1 - H_1(N, \mathbb{F}) + H_2(N, \mathbb{F}),$$

since again $H_3(N, \mathbb{F}) = H_4(N, \mathbb{F}) = 0$. If (5.26) does not hold, then it follows that also $H_1(N, \mathbb{F}) = 0$, for all fields $\mathbb{F}$. By the universal coefficient theorem, this implies that $H_1(N, \mathbb{Z}) = H_2(N, \mathbb{Z}) = 0$.

This means that N is an integral homology ball, with finite π_1. A standard argument using the exact sequence of the pair $(N, \partial N)$ and Poincaré-Lefschetz duality shows that $\partial N \simeq S^3/\Gamma$ is a homology 3-sphere, and hence is the Poincaré homology sphere, with Γ equal to the binary icosahedral group of order 120. One can now use a simple argument based on the η-invariant of S^3/Γ to obtain a contradiction, cf. [2] for further details. ■

Proposition 5.6 shows that EALE spaces N have non-trivial 2-dimensional topology. By construction, such spaces are naturally embedded in M, via (5.18) for instance.

Proposition 5.7. *For any EALE space $N \subset M$, the inclusion $\iota : N \hookrightarrow M$ induces an injection*

$$0 \to H_2(N, \mathbb{F}) \to H_2(M, \mathbb{F}). \tag{5.27}$$

Proof: The Mayer-Vietoris sequence for a thickening of the pair $(N, M \setminus N)$ gives

$$H_2(S^3/\Gamma, \mathbb{F}) \to H_2(N, \mathbb{F}) \oplus H_2(M \setminus N, \mathbb{F}) \to H_2(M, \mathbb{F}). \tag{5.28}$$

It suffices then to show that the inclusion map $i : \partial N = S^3/\Gamma \to N$ induces 0 on homology, i.e. the map

$$H_2(S^3/\Gamma, \mathbb{F}) \to H_2(N, \mathbb{F})$$

is the zero map. If N is simply connected, then this is clear since $H_2(S^3/\Gamma, \mathbb{F})$ is torsion while $H_2(N, \mathbb{F})$ is torsion-free, by Proposition 5.5 and the universal coefficient theorem. In general, suppose Σ is an essential 2-cycle in S^3/Γ; one needs to prove it bounds a 3-chain in N. Any 2-cycle in S^3/Γ with coefficients in $\mathbb{F}$ may be represented by a collection of maps $f : S^2 \to S^3/\Gamma$. Let $\pi : \widetilde{N} \to N$ be the universal cover of N, so that π is a finite cover. Also, let $\partial\widetilde{N} = S^3/\widetilde{\Gamma}$, so that π induces a map $\pi_\partial : S^3/\widetilde{\Gamma} \to S^3/\Gamma = \partial N$. The map f lifts to $\widetilde{f} : S^2 \to S^3/\widetilde{\Gamma}$. As noted above, $\widetilde{f}$ bounds a 3-chain in $\widetilde{N}$. Composing this with the projection map π shows that f bounds a 3-chain in N, as required. ■

Via Proposition 5.6, one sees that orbifold limits in $\bar{\mathcal{E}}$ necessarily crush essential 2-cycles in N to points. This leads easily to the following:

Corollary 5.8. *Suppose*

$$H_2(M, \mathbb{F}) = 0, \tag{5.29}$$

for any field $\mathbb{F}$. Then $\bar{\mathcal{E}}_{GH} = \mathcal{E}$, i.e. the moduli space $\mathcal{E}$ is complete and locally compact in the Gromov-Hausdorff, (or finite diameter), topology.

Proof: If N is any EALE space associated to an orbifold limit in $\bar{\mathcal{E}}_{GH}$, then Proposition 5.7 and (5.29) imply that $H_2(N, \mathbb{F}) = 0$, for any $\mathbb{F}$, which contradicts (5.25). Hence, there are no such EALE spaces, and the result follows. ■

The condition (5.29) holds for instance for S^4, with any (potentially exotic) differentiable structure or more generally any integral homology sphere with torsion-free π_1; there are many such 4-manifolds M, cf. §7.

It also follows from Propositions 5.6 and 5.7 that when (5.29) holds, there are only finitely many components to the part

$$\mathcal{E}_{\lambda_0} \subset \mathcal{E}$$

of the moduli space for which $\lambda \geq \lambda_0 > 0$, and further each such component is compact in the C^∞ topology. While this seems interesting and useful, we know of no applications this result. For instance, can it be useful in deciding whether S^4 has any Einstein metrics of positive scalar curvature besides the round metric?

Without the homology condition (5.29), it is an open question whether $\bar{\mathcal{E}}_{\lambda_0}$ for $\lambda_0 > 0$ is compact or not. For example, it may apriori be possible that there exists a sequence g_i in distinct components of $\mathcal{E}_{\lambda_0}$ which converges to an Einstein orbifold metric in a limiting component of $\bar{\mathcal{E}}_{GH}$, (so that $\mathcal{E}_{\lambda_0}$ has infinitely many components). This is related to the following question.

Question. Let (M, g_i) be a sequence of Einstein metrics on M converging to an Einstein orbifold metric (V, g) associated to M. Does there exist a (continuous) curve $\gamma(t)$, $t \in (0, 1]$, of Einstein metrics on M such that, for i sufficiently large, $\gamma(t_i) = g_i$, for some sequence $t_i \to 0$?

If the answer to the question is yes, then it follows that $\mathcal{E}_{\lambda_0}$ has finitely many components for $\lambda_0 > 0$. Another approach toward the compactness of $\bar{\mathcal{E}}_{\lambda_0}$ is the following:

Question. Is the completion $\bar{\mathcal{E}}_{GH}$ a real-analytic variety?

In other words, can one extend Theorem 5.1 to include orbifold formation? Note that in a real-analytic variety $\mathcal{V}$, an infinite sequence of points which converges to a limit point in $\mathcal{V}$ can be connected by a real-analytic curve in $\mathcal{V}$.

In this context, it is worth bearing in mind however that currently, except in the context of Kähler-Einstein metrics, one has no examples where the positive moduli space $\mathcal{E}_{\lambda_0}$, $\lambda_0 > 0$, is larger than a single point.

6. Moduli Spaces II.

A more complete theory of the global behavior of the components of the moduli space $\mathcal{E}$ was developed in [3]. From a broad perspective, the overall picture has a strong resemblance with the moduli space of constant curvature metrics on surfaces, which we survey very briefly.

Recall that the moduli space of unit volume constant curvature metrics on an oriented surface Σ has the following structure:

- $\Sigma = S^2$. Then $\mathcal{E} = pt$.
- $\Sigma = T^2$. Then $\mathcal{E} = \mathbb{H}^2/SL(2,\mathbb{Z})$, the modular quotient of the upper half plane. Any divergent sequence $\{g_i\} \in \mathcal{E}$ collapses, in that the injectivity radius $inj_{g_i}(x)$ of g_i at x satisfies

$$inj_{g_i}(x) \to 0,$$

for all $x \in \Sigma$. Any pointed Gromov-Hausdorff limit of $\{g_i\}$ is a line $\mathbb{R}$.

- $\Sigma = \Sigma_g$, $g \geq 2$. Then the Riemann moduli space $\mathcal{E}$ is a connected orbifold, of dimension $6g-6$. An element in $\partial\mathcal{E}$ is given by a finite collection of cusps, i.e. complete hyperbolic metrics on a collection of punctured surfaces of total genus g.

One may study the completion of $\mathcal{E}$ (or $\bar{\mathcal{E}}_{GH}$) in the pointed Gromov-Hausdorff topology, which allows the diameter of sequences of metrics in $\mathcal{E}$ to diverge to infinity. However, as in the case of surfaces, it is more useful to study the boundary $\partial\mathcal{E}$ with respect to other, more natural, metric topologies. In dimension 2, one of the most natural and well-studied metrics is the Weil-Petersson metric. This metric is just the restriction of the usual L^2 metric on the space $Met(\Sigma)$ of all metrics on Σ to the moduli space. The L^2 metric on $Met(M)$ is given by

$$\langle h_1, h_2\rangle_g = \int_M \langle h_1(x), h_2(x)\rangle_g dV_g,$$

where $h_1, h_2 \in T_gMet(M)$. Thus, we consider the completion $\bar{\mathcal{E}}$ of $\mathcal{E}$ with respect to the L^2 metric, which provides more information than the completion with respect to the pointed Gromov-Hausdorff topology.

Before stating the main result, one needs the following definition. A domain Ω (i.e. an open 4-manifold) weakly embeds in M, $\Omega \subset\subset M$, if for any compact subdomain $K \subset \Omega$, there is a smooth embedding $F = F_K : K \to M$. The same definition applies if Ω is an orbifold.

Theorem 6.1. [3]. *The completion $\bar{\mathcal{E}}$ of $\mathcal{E}$ with respect to the L^2 metric on $\mathcal{E}$ is a complete Hausdorff metric space, whose frontier $\partial\mathcal{E}$ consists of two parts: the orbifold part $\partial_o\mathcal{E}$ and the cusp part $\partial_c\mathcal{E}$.*

I. *$\partial_o\mathcal{E}$ consists of Einstein orbifolds associated to M, of unit volume and finite diameter. The partial completion $\mathcal{E} \cup \partial_o\mathcal{E}$ is locally compact and all the results of §5 hold as before.*

II. *An element in the cusp boundary $\partial_c\mathcal{E}$ is given by a pair (Ω, g), where Ω is a non-empty maximal orbifold domain weakly embedded in M, consisting of a finite number of components Ω_k called cusps, each with a bounded number (possibly zero) of orbifold singularities. The metric g is a complete Einstein metric on Ω, with*

$$vol_g\Omega = 1, \tag{6.1}$$

and outside a compact set K, Ω carries an F-structure along which g collapses with locally bounded curvature as one goes to infinity in Ω; thus as $x \to \infty$ in Ω,

$$inj(x) \to 0 \text{ and } (|R|inj^2)(x) \to 0. \tag{6.2}$$

To describe the behavior of the region $M \setminus K$, let g_i be a sequence in $\mathcal{E}$ with $g_i \to g \in \partial_c\mathcal{E}$ in the L^2 metric. Then $M \setminus K$ also carries an F-structure on the complement of a finite number of arbitrarily small balls. Thus, there exists a finite collection of points $z_k \in M$, with $dist_{g_i}(z_k, K) \to \infty$ as $i \to \infty$, and a sequence $\varepsilon_i \to 0$ such that outside $B_{z_k}(\varepsilon_i)$, $M \setminus K$ has an F-structure. If K_i is an exhaustion of Ω, then $M \setminus K_i$ collapses everywhere as $i \to \infty$, and collapses with locally bounded curvature as in (6.2) *away from the singular points $\{z_k\}$.*

Further, Case II, *i.e. cusps, can form only on the components of $\mathcal{E}$ for which there is a constant λ_0 such that*

$$\lambda \le \lambda_0 < 0. \tag{6.3}$$

Note that the product in (6.2) is scale invariant, so that (6.2) shows that the metric becomes flat on the scale of the injectivity radius.

The convergence in Case I is also in the Gromov-Hausdorff topology, while that in Case II is also in the pointed Gromov-Hausdorff topology. One sees of course some similarities here with the structure of $\mathcal{E}$ in the case of surfaces. Note for instance that the compactness of $\bar{\mathcal{E}}_{GH}$ or $\bar{\mathcal{E}}$ on the components of positive scalar curvature resembles the compactness of Einstein metrics on S^2.

As in the case of surfaces, (e.g. the moduli of flat metrics on T^2), the L^2 completion $\bar{\mathcal{E}}$ is not compact in general. An important example is the case of Einstein metrics on the K3 surface, which for illustration is worth discussing in some detail.

By Theorem 3.2, any Einstein metric on K3 is Ricci-flat and Kähler, and so hyperkähler. To any $g \in \mathcal{E}$ is associated a 3-dimensional subspace $P_g \subset H^2(K3, \mathbb{R}) \simeq \mathbb{R}^{22}$, given by the span of the Kähler forms associated to g. The intersection form is of type $(3, 19)$ on H^2, and restricts to a positive definite form on P_g. This assignment gives the period map

$$P : \mathcal{E} \to \Gamma / G^+_{3.19},$$

where $G^+_{3,19} \simeq SO(3, 19)/(SO(3) \times SO(19))$ is the Grassmannian of positive 3-planes in H^2 and Γ is the integral lattice $SO(3, 19 : \mathbb{Z}) \simeq Aut(H^2, I)$. The space $\Gamma/G^+_{3.19}$ is a 57-dimensional orbifold and the local Torelli theorem for K3 surfaces implies P is a local diffeomorphism.

However, the period map P is not surjective. Namely, let $\Delta \subset H^{1,1}(K3, \mathbb{R}) \cap H^2(K3, \mathbb{Z})$ be the set of roots, i.e. the class of effective divisors d of self-intersection -2, $(d, d) = -2$. Then ImP is contained in the open subset of $\Gamma/G^+_{3.19}$ for which, for any $d \in \Delta$, $\omega(d) \neq 0$, for some Kähler form $\omega \in P_g$.

One has $\bar{\mathcal{E}}_{K3} = \mathcal{E}_{K3} \cup \partial_o\mathcal{E}_{K3}$, and the set of Einstein orbifold metrics $\partial_o\mathcal{E}_{K3}$ is a countable collection of codimension 3 varieties in $\bar{\mathcal{E}}_{K3}$, corresponding to the locus where $\omega(d) = 0$, for some Kähler form ω and $d \in \Delta$. It is proved in [3] that the period map P extends continuously to $\bar{\mathcal{E}}_{K3}$ and that the extended period map

$$\bar{P} : \bar{\mathcal{E}}_{K3} \to \Gamma / G^+_{3.19}, \tag{6.4}$$

is an isometry between the L^2 metric on $\bar{\mathcal{E}}_{K3}$ and the complete, non-compact, finite volume locally symmetric metric on $\Gamma/G^+_{3.19}$.

Returning to the discussion in general, the behavior of $\bar{\mathcal{E}}$ at infinity is described as follows:

Theorem 6.2. [3]. *Suppose g_i is a divergent sequence in $\bar{\mathcal{E}}$ such that*

$$\lambda_{g_i} \to 0, \quad \text{as} \quad i \to \infty. \tag{6.5}$$

Then $\{g_i\}$ collapses everywhere, i.e. $inj_{g_i}(x) \to 0$, $\forall x \in M$. The collapse is along a sequence of F-structures $\mathcal{F}_i$ and with locally bounded curvature (6.2) *metrically on the complement of finitely many singular points $\{z_k\}$.*

Suppose instead that g_i is a divergent sequence in $\bar{\mathcal{E}}$ such that

$$\lambda_{g_i} \leq \lambda_0 < 0, \quad \text{as} \quad i \to \infty. \tag{6.6}$$

Then $\{g_i\}$ either has the same behavior as above in (6.5)*, or as in Case* II *(cusps) of Theorem 6.1, where Ω may instead have possibly infinitely many components, of total volume at most 1.*

Note that in the case of (6.5), the collapsing singularities $\{z_k\}$ must exist. Namely, if $\{z_k\} = \emptyset$, then it follows from Theorem 6.2 that the manifold M admits an F-structure. However, it is easy to see that any manifold M with an F-structure has vanishing Euler characteristic, $\chi(M) = 0$, cf. [17]. By Theorem 4.1, this can happen only if (M, g) is flat. Thus, one expects that small neighborhoods of $\{z_k\}$ account for all of $\chi(M)$.

Recently, building on the work in [3], Cheeger-Tian have improved Theorem 6.2, and proved the following, (answering a conjecture in [3]):

Theorem 6.3. [18]. *Suppose that g_i is a divergent sequence in $\bar{\mathcal{E}}$ such that*

$$\lambda_{g_i} \leq \lambda_0 < 0, \quad \text{as} \quad i \to \infty. \tag{6.7}$$

Then $\{g_i\}$ cannot collapse everywhere, and has the same behavior as described in Case II *of Theorem 6.1. Moreover, the collapse in* (6.2) *is with uniformly bounded curvature away from a finite number of singular points.*

We briefly describe the basic idea in the proof of this non-collapse result. The proof is by contradiction, and so suppose $\{g_i\}$ is a divergent sequence in $\bar{\mathcal{E}}$ (of unit volume) which collapses everywhere, i.e. $inj_{g_i}(x) \to 0$, for all x. By Theorem 6.1, there is a finite number of singularities $\{z_k\}$ outside of which the metrics g_i are collapsing M everywhere along a sequence of F-structures with locally bounded curvature. Let $U_i = (M \setminus \cup B_{z_k}(\varepsilon), g_i)$, where $\varepsilon > 0$ is (very) small. As above, $\chi(U_i) = 0$. By the Chern-Gauss-Bonnet theorem for manifolds with boundary, one then has

$$\frac{1}{8\pi^2}\int_{U_i} |R|^2 + \int_{\partial U_i} Q = \chi(U_i) = 0. \tag{6.8}$$

Here the boundary term Q consists of two terms; one of the form A^3, where A^3 is cubic in the eigenvalues of the 2^{nd} fundamental form A of $\partial U_i \subset U_i$ and a second, $R(A)$, which is linear in the curvature of g_i and A. Now suppose one can prove that there is a constant $K < \infty$ such that

$$|Q| \leq K. \tag{6.9}$$

Since one can easily arrange that $vol \partial U_i \to 0$, it then follows from (6.8) and (6.9) that

$$\frac{1}{8\pi^2}\int_{U_i} |R|^2 \to 0,$$

as $i \to \infty$. However, $vol U_i \to 1$ while by the assumption (6.7), $|R| \geq c_0 > 0$. This gives a contradiction, showing that it is not possible that $\{g_i\}$ collapses everywhere.

Given this idea, the main point is then to prove that the collapse away from $\{z_k\}$ is necessarily with uniformly bounded curvature, (in place of locally uniformly bounded curvature as in (6.2)), from which (6.9) then follows easily. ∎

To complete the analogy with the case of surfaces, it is natural to conjecture, (cf. [3]), that there are *no* divergent sequences in $\bar{\mathcal{E}}$ when (6.7) holds, i.e. (6.7) implies that $\bar{\mathcal{E}}$ is compact. This remains currently an open problem. We recall here that the completion of the moduli space of hyperbolic metrics on a surface with respect to the Weil-Petersson metric is compact, and agrees with the Deligne-Mumford compactification.

The results described above lead of course to many new questions, and we discuss some of these next.

First, while the process leading to the formation of orbifold singularities in $\bar{\mathcal{E}}_{GH}$, as well as the structure of the Einstein orbifold metrics themselves, is comparatively well understood, almost nothing is known about the structure of the singular points $\{z_k\}$ arising in collapsing regions. One would expect that such singularities can be modelled on spaces such as the multi-Taub-NUT metrics (5.22) and other ALF spaces, as in [20] for instance; however, with the exception of the work of Gross-Wilson [34] discussed in §7, there are no studies along these lines.

For instance, the fact that the F-structure does not extend through the singularities $\{z_k\}$ suggests the following question, which is of independent interest:

Question. Suppose (N, g) is a complete, non-compact Ricci-flat 4-manifold, with a free isometric S^1 action. Is (N, g) necessarily flat?

This is of course false if the S^1 action is allowed to have a non-trivial fixed point set; for example, the Gibbons-Hawking metrics (5.20) have an isometric S^1 action. This result is also easy to prove if (N, g) is ALE for instance. Moreover, there is a positive answer to the question in the setting of Ricci-flat Lorentz metrics with a non-vanishing time-like Killing field, cf. [5]. However, the method of proof used there does not carry over to the Riemannian case.

Similarly, very little is known in detail about the possible formation of Einstein cusp manifolds as described in Theorem 6.1, beyond the obvious examples of this behavior on products of surfaces of genus > 2, (and the Kähler-Einstein case on surfaces of general type); see however §7 for some further results.

For instance, it is not known if the cusp ends must be of finite topological type. A typical cross-section of a cusp end is a closed 3-manifold which collapses with bounded curvature. These are graph manifolds, i.e. unions of Seifert fibered 3-manifolds along tori. Can general graph manifolds arise in the cusp ends? One has the standard examples of 3-tori T^3 and the 3-dimensional Nil manifolds, arising from cusps of hyperbolic and complex hyperbolic space-forms, as well as collapse along circles in the case of products of hyperbolic surfaces; however, beyond this, no further examples are known.

As discussed in §5, orbifold singularities can only arise when essential 2-cycles in M are crushed to points, cf. Propositions 5.5-5.8. No analog of this result is known for cusps, and it would be very interesting to show cusps or their complements are topologically essential in M in some way.

Another very interesting question arises again by comparison with the case of surfaces. For surfaces, the Deligne-Mumford or Weil-Petersson compactification of the Riemann moduli space $\bar{\mathcal{E}}$ is a *closed* complex analytic variety, for which the boundary $\partial\mathcal{E} = \bar{\mathcal{E}} \backslash \mathcal{E}$ is a complex subvariety of real codimension 2. Thus, the boundary is not "topological", in the sense that $\bar{\mathcal{E}}$ is a (singular) manifold with boundary, (with boundary of codimension 1). Is the same true for the L^2 compactification $\bar{\mathcal{E}}$ in dimension 4, for the components with $\lambda < 0$? In other words, can one give the completion $\bar{\mathcal{E}}$ the structure of a cycle?

7. Constructions of Einstein metrics, II.

In this section, we discuss a different approach to constructing Einstein metrics, namely by "glueing" or resolving singular Einstein metrics. Beginning with the work of Taubes [68], this singular perturbation method has been very successful in constructing interesting solutions of a wide variety of geometric equations and it is of interest to study this issue in the context of Einstein metrics.

We have seen in prior sections that one can describe in some detail the singular Einstein structures that form at the boundary or at infinity of the moduli space $\mathcal{E}$. The singular behaviors are of three types.

- orbifold singularities.
- collapsing singularities or limits.

• cusp limits.

The collapsing singularities are point-type singularities whose blow-ups are modeled on complete Ricci-flat spaces which are not ALE; for example the ALF Taub-NUT or multi-Taub-NUT metrics discussed in §5. The collapsed or cusp limits describe global limit behavior, as opposed to local singularities.

For each of these types of singular behavior, the natural glueing question is: when can this process be reversed, i.e. given in general such a singular Einstein metric or configuration, when can one "resolve the singularity" by finding a sequence or curve of Einstein metrics, on a given compact manifold M or sequence of compact manifolds M_i, which tend to the given singular space or configuration in the limit.

As with all glueing procedures, the construction proceeds in two steps, one more conceptual and one more technical. First, one constructs an (essentially explicit) good approximate solution to the Einstein equations on M, (or M_i), by patching together exact Einstein metrics on the domains which together cover M. Next one needs to prove that such an approximate solution can be perturbed to a nearby exact solution of the Einstein equations. This perturbation to an exact solution is carried out by means of the inverse or implicit function theorem, and often requires a considerable amount of technical work to establish.

I. Orbifold singularities.

It has long been an open question whether Einstein orbifold metrics (V, g_∞) can be resolved to smooth Einstein metrics (M, g) close to (V, g_∞) in the Gromov-Hausdorff topology. The idea here would be to reverse the process of formation of orbifold singularities described in §5. Here there is often no difficulty in carrying out the first step in the construction. For instance, if a singularity q of V is of the type $C(S^3/\Gamma)$ with $\Gamma \subset SU(2)$, then it is easy to construct good approximate solutions by gluing the punctured Einstein orbifold $V \setminus \{q\}$ with a truncation and rescaling of a hyperkähler EALE space given by Theorem 5.4. The main issue is then the perturbation to an exact Einstein metric.

There are certainly some situations where one can resolve orbifold singularities. Notably, this is the case on K3 surfaces. As a concrete example, cf. [59], consider the classical Kummer construction of K3 surfaces. Thus, let $T^4 = \mathbb{C}^2/\Lambda$ be a complex torus, and consider the involution $A : T^4 \to T^4$, $A(x) = -x$. This map has 16 fixed points, (the points of order 2), and the quotient $V = T^4/\mathbb{Z}_2$ is an orbifold with 16 singular points, each of the form $C(\mathbb{RP}^3)$. The flat metric g_0 is an orbifold singular Einstein metric on V. One may then take 16 copies of the Eguchi-Hanson metric (5.21), truncated and scaled down, and glue this onto the regular set (V_0, g_0). This gives an approximate Ricci-flat metric on K3.

It follows from the discussion on the K3 surface in §6 that there are smooth Einstein metrics on the K3 surface very close to such appoximate solutions, cf. also [40]. In fact, by (6.4), all the orbifold Einstein metrics associated with K3 can be resolved. However, this proof is indirect, relying on the structure of $\bar{\mathcal{E}}_{K3}$, and does not lead to the actual construction of new Einstein metrics.

Moreover, the resolution of orbifold singularities in this way does not work in general for Kähler-Einstein orbifolds. This follows from fact that there are many more orbifold singular Kähler-Einstein metrics than smooth Kähler-Einstein metrics, cf. [42, 43] for instance. For example, consider log del Pezzo surfaces; these are surfaces with $c_1 > 0$ with quotient singularities, i.e. singularities of the form $\mathbb{C}^2/\Gamma$, $\Gamma \subset U(2, \mathbb{C})$. It is proved in [42] that there are log del Pezzo surfaces V having orbifold Kähler-Einstein metrics, with $\lambda > 0$, which have arbitrarily large b_2. However, although all smooth del Pezzo surfaces have Kähler-Einstein metrics with $\lambda > 0$, (by Theorem 3.3), all such satisfy $b_2 \leq 9$.

If one could carry out a glueing construction in the Kähler-Einstein context, resolving the singularities of V, then a smooth Kähler-Einstein metric would also have $\lambda > 0$, and hence is defined on a smooth del Pezzo surface M, as in Theorem 3.3. The same argument establishing (5.28) also proves $H_2(M \setminus N, \mathbb{R})$ injects in $H_2(M, \mathbb{R})$, which gives a contradiction.

While it is still conceivable that such orbifold metrics could be resolved by (real) Einstein metrics, this seems unlikely. It would of course be very interesting to establish this one way or the other.

II. Collapsing configurations.

An interesting glueing result was proved by Gross-Wilson [34], describing the collapse behavior of (some) sequences of Einstein metrics tending to infinity in the moduli space $\mathcal{E}_{K3}$ of Einstein metrics on the K3 surface.

To describe the result, a large family of K3 surfaces are given as elliptic (i.e. torus) fibrations; there is a holomorphic map $K3 \to S^2$, with 24 singular fibers, each of type I_1, (a pinched torus). The approximate solution is given by glueing together two types of Ricci-flat Kähler metrics. First, away from the singular fibers, one takes a "semi-flat" metric, defined by a Riemannian submersion to a base given by a disc $D^2 \subset S^2$ with flat metric induced on the fibers. The fibers are scaled to be highly collapsed, i.e. of small diameter. In a neigbhorhood of each singular fibers, one takes a suitable truncation and rescaling of the Ooguri-Vafa metric [57], i.e. a periodic Taub-NUT metric. Thus, choose the potential V as in (5.22), with infinitely many points p_i of distance ε apart along the z-axis in $\mathbb{R}^3$. By adding a suitable constant, the potential may be renormalized to sum to a smooth, periodic harmonic function. The resulting metric as in (5.20) is then also periodic, and taking the $\mathbb{Z}$ quotient gives a metric on a manifold topologically equivalent to a neighborhood of a singular fiber, and which, after rescaling, is close to a semi-flat metric. One then perturbs this approximate solution to an exact Ricci-flat Kähler metric by obtaining uniform estimates for the complex Monge-Ampere equation (3.3).

Although this glueing result does not actually construct any new Einstein metrics, (they are already given by Yau's Theorem 3.2 on the K3 surface), the construction is important in understanding various aspects of mirror symmetry on K3.

Next, we describe a much simpler glueing method, which, if successful, would lead to many interesting, new Einstein metrics. We will describe the method only in a special case, since this illustrates the main idea. Thus, let (M, g) be a Ricci-flat manifold of the form $M = S^1 \times N$, with g a product metric. Assume that the metric g is highly collapsed, so that the length of S^1 is small. Let $\hat{M}$ be the manifold obtained by performing surgery to kill the S^1 factor in $\pi_1(M)$. Thus, one removes $S^1 \times B^{n-1} \subset M$ and glues in $D^2 \times S^{n-2}$.

In [4], a good approximate Einstein metric was constructed on $\hat{M}$ in that, for any given $\varepsilon > 0$, there are metrics g_ε on $\hat{M}$ such that

$$|Ric_{g_\varepsilon}| \leq \varepsilon. \tag{7.1}$$

To see this, consider first the family of Schwarzschild metrics g_{Sch}:

$$g_{Sch} = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{S^{n-2}}, \tag{7.2}$$

where $V(r) = 1 - \frac{2m}{r^{n-3}}$ and $\theta \in [0, \beta]$; here $\beta = \beta(m)$ is chosen so that the metric is smooth at the horizon where $V = 0$. This metric lives on the manifold $D^2 \times S^{n-2}$, and is asymptotic to the product metric on $S^1 \times \mathbb{R}^{n-1}$. Truncating and rescaling the metric to a small size, one may then glue this into $\hat{M} \setminus (S^1 \times B^{n-1})$ and construct g_ε satisfying (7.1).

Question. Are there situations where the approximate solutions g_ε on $\hat{M}$ can be perturbed to a nearby exact Einstein metric.

If so, one would expect to be able to repeat this surgery arbitrarily many times.

III. Cusp configurations.

Here we describe a construction of Einstein metrics obtained by "resolving" cusp configurations. The results are closely analogous to the Thurston theory [69] of Dehn surgery on hyperbolic 3-manifolds. The construction holds in all dimensions $n \geq 3$ but for simplicity we work with $n = 4$.

One starts with any complete, non-compact hyperbolic 4-manifold $N = N^4$ of finite volume, with metric g_{-1}. The manifold N has a finite number of cusp ends $\{E_j\}$, $1 \leq j \leq k$, each diffeomorphic to $\mathbb{R} \times F$, where F is a compact flat 3-manifold. By passing to covering spaces if necessary, it may and will be assumed that each F is a 3-torus T^3.

Now perform Dehn filling on each of the cusp ends. Thus, fix a flat torus T^3 in a given end E and let σ be any simple closed geodesic in T^3. Attach a 4-dimensional solid torus $D^2 \times T^2$ onto T^3 by a diffeomorphism $\partial(D^2 \times T^2) = T^3$, sending the $S^1 = \partial D^2$ onto σ. Carrying out this Dehn filling for each of the cusp ends of N gives a compact manifold M_σ, $\sigma = (\sigma_1, \cdots, \sigma_k)$. We will say that σ is sufficiently large if the length $l(\sigma_j)$ of each closed geodesic $\sigma_j \subset T_j^3$ is sufficiently long.

Theorem 7.1. [6]. *For any choice of σ sufficiently large, the manifold M_σ admits an Einstein metric with negative scalar curvature.*

To give an idea of how many Einstein metrics are constructed in this way, it is known from results of [15] that the number $\mathcal{H}(V)$ of complete non-compact hyperbolic 4-manifolds of finite volume with volume $\leq V$ grows super-exponentially: in fact

$$e^{aV \ln V} \leq \mathcal{H}(V) \leq e^{bV \ln V}.$$

With each such N, Theorem 7.1 associates infinitely many homeomorphism types of compact manifolds M_σ. Formally, the number of such compact manifolds is ∞^q, where q is the number of cusp ends. (The number of cusp ends also grows linearly with V). Most of these Einstein metrics are not locally isometric (while all hyperbolic manifolds are locally isometric).

The Euler characteristic and signature of M_σ are given by

$$\chi(M_\sigma) = \chi(N), \quad \tau(M_\sigma) = 0.$$

Each M_σ is aspherical, i.e. a $K(\pi, 1)$, (for σ sufficiently large), and in fact admits metrics of non-positive, (but not negative), sectional curvature. A surprising result of [63] shows that there exists N and infinitely many choices of σ for which M_σ is an integral homology 4-sphere.

It is worth describing in some detail how the approximate Einstein metric is constructed on M_σ. First, one has the given hyperbolic, and so Einstein, metric on N. One needs then an Einstein metric on the solid torus $D^2 \times T^2$ which closely matches the hyperbolic metric on a cusp end of N. A model for such metrics was constructed long ago by physicists: this is the family of toral AdS black hole metrics

$$g_{BH} = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{T^2},$$

where g_{T^2} is any flat metric on the torus and $V = V(r)$ is given by

$$V = r^2 - \frac{2m}{r},$$

compare with the Schwarzschild metric in (7.2). The set $r = r_+ = (2m)^{1/3}$ where the potential V vanishes is called the horizon H. Note that H is a totally geodesic and flat torus T^2 in g_{BH}. For the metric g_{BH} to be smooth at H, one requires that the circular parameter θ runs over the interval $\theta \in [0, \beta]$, where $\beta = 4\pi/3r_+$.

This metric is asymptotically hyperbolic, in that the curvature tends to -1 at infinity, but the metric has infinite volume, and so is not at all close to a hyperbolic cusp metric on N. However, one can take quotients of g_{BH} to obtain metrics which are almost cusp-like in large regions.

To see this more clearly, consider first the universal cover $D^2 \times \mathbb{R}^2$ of the solid torus $D^2 \times T^2$ with lifted metric

$$\widetilde{g}_{BH} = V^{-1}dr^2 + Vd\theta^2 + r^2(dx_1^2 + dx_2^2).$$

The metrics g_{BH} depend on the mass parameter m, but all metrics $\widetilde{g}_{BH}$ are isometric, as one easily seems by the change of variable $r \to r_m = mr$; thus for convenience, set $m = \frac{1}{2}$, so that $r_+ = 1$. Let $D(R) = \{r \leq R\} \subset (D^2 \times \mathbb{R}^2, \widetilde{g}_{BH})$ and let $S(R) = \partial D(R) = \{r = R\}$. The induced metric on the boundary $S(R)$ is then a flat metric

$$V(R)d\theta^2 + (dz_1^2 + dz_2^2),$$

on $S^1 \times \mathbb{R}^2$, where $z_i = Rx_i$. Choose R so that

$$\sqrt{V(R)}\beta = l(\sigma),$$

so that the length of $S^1 \times \{pt\} \subset S(R)$ equals $l(\sigma)$. Now given the flat stucture g_0 on $T^3 \subset N$, observe that up to conjugacy there is a unique free isometric $\mathbb{Z}^2$ action on the flat product $S(R) = S^1 \times \mathbb{R}^2$ such that the projection map to the orbit space

$$\pi : S^1 \times \mathbb{R}^2 \to T^3$$

satisfies $\pi(S^1) = \sigma$, and for which the flat structure on T^3 induced by π is the given structure g_0. In fact, the map π is just the covering space of (T^3, g_0) corresponding to the subgroup $\langle\sigma\rangle \subset \pi_1(T^3)$.

One may easily verify that this $\mathbb{Z}^2$ action extends radially to a smooth and free isometric action on the domain $D(R) \subset D^2 \times \mathbb{R}^2$. The quotient space $(D^2 \times \mathbb{R}^2)/\mathbb{Z}^2 \simeq D^2 \times T^2$ gives the twisted toral AdS black hole metric

$$\hat{g}_{BH} = [V^{-1}dr^2 + Vd\theta^2 + r^2 g_{\mathbb{R}^2}]/\mathbb{Z}^2.$$

Note that diameter of the core T^2 at $r = r_+$ has size about R^{-1}, and so is very small. The locus $\{r = R\}$ with the metric induced from g_{BH} is now isometric to (T^3, g_0). Since this is also the metric at the boundary of the torus-truncated hyperbolic manifold, the boundaries may be identified, giving a C^0 metric on M_σ. The 2nd fundamental forms also are very close (for R large), as are the curvatures. Smoothing the seam at the identification locus gives the approximate solution of the Einstein equations.

One then shows via the inverse function theorem that the approximate metric may be perturbed to a nearby exact Einstein metric on M_σ. As a consequence, it follows that the constructed Einstein metric g_σ on M_σ is locally rigid, i.e. is an isolated point in the full moduli space $\mathcal{E}$ on M_σ. This local rigidity is essentially a feature of negative curvature. Note that it could not hold in the context of the conjectured Ricci-flat Schwarzschild glueings in (7.1)-(7.2), which would give rise to non-trivial curves in the moduli space. It is unknown if g_σ is the unique Einstein metric on M_σ.

The homeomorphism type of M_σ is essentially uniquely determined by the choice of $[\sigma_j] \in \pi_1(T_j^3)$, $1 \leq j \leq k$. The Einstein metrics (M_σ, g_σ) are all close to the original hyperbolic manifold (N, g_{-1}) in the pointed Gromov-Hausdorff topology, and are smoothly close away from the horizon region. On any sequence where $l(\sigma_j) \to \infty$ for all j, one has $(M_\sigma, g_\sigma) \to (N, g_{-1})$ in the pointed Gromov-Hausdorff topology, i.e. such infinite sequences tend toward the original "singular" cusp configuration. Note that the original tori $T_j^3 \subset N$, which are also embedded in each M_σ, are now no longer incompressible in M_σ; only the core tori T_j^2 are topologically essential in M_σ.

It is well-known that $\chi(N)$ can take on arbitrary values in $\mathbb{Z}^+$, although $\chi(N)$ is even if and only if N is orientable. Recall that for hyperbolic manifolds of finite volume, one has the relation, (via (4.2)),

$$volN = \frac{4\pi^2}{3}\chi(N).$$

It then follows again from (4.2) that if the Einstein metric g_σ on M_σ is scaled so that $\lambda = -3$, then

$$volM_\sigma < volN,$$

for any σ, (large). Thus, all manifolds obtained by performing Dehn filling on the ends of N have volume strictly less than $volN$, as in the Thurston theory of Dehn surgery.

This cusp formation on limits of sequences does not take place within the moduli space $\mathcal{E}$ on a fixed manifold. In fact, with the exception of the case Kähler-Einstein metrics with $c_1 < 0$, there are no examples of *curves* of Einstein metrics in a fixed component of $\mathcal{E}$ which limit on a cusp configuration. It would be very interesting to find examples of such curves. Similarly, it would be interesting to understand if such "cusp resolution" can be carried out on other types of cusps, for instance on complex hyperbolic cusps, or cusps arising from products of surfaces.

It is also worth pointing out that this process of Dehn surgery is an important method of constructing exotic smooth structures on a 4-manifold of a given homeomorphism type, cf. [25]. It remains completely open whether the construction above can be carried out in this context, to give a construction of Einstein metrics on such exotic smooth structures.

8. Concluding Remarks

The survey above shows that we are far from any theory describing the structure of Einstein metrics on 4-manifolds. One has instead an interesting collection of different methods, ideas and results which, at the moment, do not assemble to give any coherent picture or structure.

A basic issue is to what extent the Thurston picture of 3-manifolds, proved by Perelman, carries over to dimension 4; thus to what extent does a general 4-manifold decompose into a collection of domains, each of which carries a complete Einstein, or "Einstein-like" metric, or collapses along an F-structure. From the natural viewpoint of the Ricci flow, one should allow Ricci solitons as natural generalizations of Einstein metrics. As a first step for instance, it would be very interesting to know if there is an analog of Hamilton's structure theorem [36] for non-singular solutions of the Ricci flow in dimension 4.

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