# DEHN FILLING AND EINSTEIN METRICS IN HIGHER DIMENSIONS

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Abstract. We prove that many features of Thurston's Dehn surgery theory for hyperbolic 3-manifolds generalize to Einstein metrics in any dimension. In particular, this gives large, infinite families of new Einstein metrics on compact manifolds.

## 1. Introduction.

In this paper, we construct a large new class of Einstein metrics of negative scalar curvature on $n$-dimensional manifolds $M = M^n$, for any $n \geq 4$. Einstein metrics are Riemannian metrics $g$ of constant Ricci curvature, and we will assume the curvature is normalized as

$$Ric_g = -(n-1)g, \tag{1.1}$$

so that the scalar curvature $s = -n(n-1)$. The construction is a direct generalization of Thurston's theory of Dehn surgery or Dehn filling on hyperbolic 3-manifolds [31] to Einstein metrics in any dimension; in fact the proof gives a new, analytic approach to Thurston's cusp closing theorem [31], [32].

To describe the construction, start with any complete, non-compact hyperbolic $n$-manifold $N = N^n$ of finite volume, with metric $g_{-1}$ of constant curvature $-1$. The manifold $N$ has a finite number of cusp ends $\{E_j\}$, $1 \leq j \leq q$, with each end $E$ diffeomorphic to $F \times \mathbb{R}^+$, where $F$ is a compact flat manifold, with flat metric $g_0$ induced from $(N, g_{-1})$. For simplicity, assume that each $F$ is an $(n-1)$-torus $T^{n-1}$; this can always be achieved by passing to a finite covering space if necessary, cf. [6].

Now perform Dehn filling on any collection $\mathcal{C} = \{E_k\}$ of cusp ends of $N$, where $1 \leq k \leq p$, and $p \leq q$. Thus, fix a torus $T^{n-1} \subset E \in \mathcal{C}$ and let $\sigma$ be a simple closed geodesic $\sigma \subset (T^{n-1}, g_0)$. Attach a (generalized) solid torus $D^2 \times T^{n-2}$ onto $T^{n-1}$ by a diffeomorphism of $\partial D^2 \times T^{n-2} \simeq T^{n-1}$ sending $S^1 = \partial D^2$ onto $\sigma$. If $\sigma_k$ are such simple closed geodesics in tori $T_k^{n-1} \subset E_k$, let $\bar{\sigma} = (\sigma_1, ...\sigma_p)$ and let

$$M = M_{\bar{\sigma}} = M^n(\sigma_1, ..., \sigma_p) \tag{1.2}$$

be the resulting manifold obtained by Dehn filling the collection of ends $E_1, ..., E_p$ of $N$. The diffeomorphism type of $M$ depends on the homotopy class of each $\sigma_k$ in $\pi_1(T_k^{n-1}) \simeq \mathbb{Z}^{n-1}$ but is otherwise independent of the choice of attaching map.

If $p = q$ the manifold $M_{\bar{\sigma}}$ is compact, (without boundary); otherwise $M_{\bar{\sigma}}$ has $q-p$ remaining cusp ends. Define the Dehn filling $\bar{\sigma} = (\sigma_1, ..., \sigma_p)$ to be *sufficiently large* if, given $N$ and a fixed collection of tori $T_k^{n-1}$, the length $R_k$ of each geodesic $\sigma_k$, $1 \leq k \leq p$, is sufficiently large in $(T_k^{n-1}, g_0)$; this will be made more precise in §3.

The main result of the paper is then the following:

**Theorem 1.1.** *Let $(N, g_{-1})$ be a complete, non-compact hyperbolic $n$-manifold of finite volume, $n \geq 3$, with toral ends. Then any manifold $M_{\bar{\sigma}}$ obtained by a sufficiently large Dehn filling of the*

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*ends of $N$ admits a complete, finite volume Einstein metric $g$, of uniformly bounded curvature and satisfying (1.1).*

To place this result in some perspective, a well-known result of Wang [33] states that if $n \geq 4$, there are only finitely many complete hyperbolic $n$-manifolds with volume $\leq V$. On the other hand, let $\mathcal{H}(V)$ denote the number of (diffeomorphically) distinct complete non-compact hyperbolic $n$-manifolds of volume $\leq V$. Then $\mathcal{H}(V)$ grows super-exponentially in $V$; in fact, by a recent result in [13], there are constants $a$, and $b$, depending only on $n$, such that

$$e^{aV \ln V} \leq \mathcal{H}(V) \leq e^{bV \ln V}. \tag{1.3}$$

(The lower bound in (1.3) is stated in [13] only for compact hyperbolic manifolds, but using the work of Lubotzky in [24], this bound also holds for non-compact hyperbolic manifolds, [25]). For many such manifolds $N$, the number of cusp ends also grows linearly in $V$, cf. Remark 4.2.

With each such $N$, Theorem 1.1 associates infinitely many homeoeomorphism types of compact manifolds $M_{\bar{\sigma}}$, (as well as non-compact manifolds). Formally, the number of such compact manifolds is $\infty^q$, where $q$ is the number of cusps of $N$. The Einstein metrics all have volume close to $V = volN$. Further, although all hyperbolic manifolds are locally isometric, most of the Einstein metrics constructed are not locally isometric. Thus, the result gives a wealth of new examples of Einstein manifolds.

All of the manifolds $M_{\bar{\sigma}}$ are $K(\pi,1)$ manifolds, again for $\bar{\sigma}$ sufficiently large; in fact all admit metrics of non-positive sectional curvature. However, none of these manifolds admit metrics of negative sectional curvature. The curvature of the Einstein metrics $g$ on $M_{\bar{\sigma}}$ is not non-positive, (at least when $n > 4$), but one has the uniform bounds

$$-1 - \tfrac{1}{2}(n-3) - \varepsilon(\bar{\sigma}) \leq K \leq -1 + \tfrac{1}{2}(n-3)(n-2) + \varepsilon(\bar{\sigma}), \tag{1.4}$$

where $K$ denotes the sectional curvature of the metric, and $\varepsilon(\bar{\sigma})$ is small, with $\varepsilon(\bar{\sigma}) \to 0$ as $\bar{\sigma} \to \infty$ in the Dehn filling space attached to each cusp. When $n = 4$, note that (1.4) gives $K \leq \varepsilon(\bar{\sigma})$, so that the Einstein metrics are of almost non-positive curvature. When $n = 3$, the Einstein metrics are of course hyperbolic; the construction in Theorem 1.1 then gives an analytic proof of Thurston's cusp closing theorem.

The Einstein metrics $(M, g)$ given by Theorem 1.1 are all close to the initial hyperbolic manifold $(N, g_{-1})$ in the pointed Gromov-Hausdorff topology. This will be apparent in a precise sense from their construction, but can be formulated generally as follows. Note first that $N$ is embedded in any $M$ obtained by Dehn filling as the complement of a generalized link - the collection of the $(n-2)$ tori at the core of the solid tori $\{D_j^2 \times T_j^{n-2}\} \subset M$. Given $(N, g_{-1})$, let $g^k$ be a sequence of Einstein metrics on $M^k = M_{\bar{\sigma}^k}$, (constructed by the Theorem), such that the length of $\sigma_j^k$ diverges to infinity as $k \to \infty$, for each $\sigma_j^k \in \bar{\sigma}^k$. Then, given a fixed base point $y \in N \subset M^k$, the metrics $(M^k, g^k, y)$ converge to $(N, g_{-1})$ in the pointed Gromov-Hausdorff topology based at $y$.

The convergence is smooth on compact domains containing $y$, and the curvature tends to $-1$, uniformly on compact subsets. Thus, by the bounds (1.4) and the fact that the volume of $(M^k, g^k)$ is uniformly bounded, one finds that the metrics $(M^k, g^k)$ have uniformly small Weyl curvature in $L^p$, for any $p < \infty$:

$$\int_{M^k} |W|^p dV_{g^k} \leq \varepsilon(\bar{\sigma}^k, p), \tag{1.5}$$

where $\varepsilon$ depends only on $p$ and $\bar{\sigma}^k$; for any fixed $p$, $\varepsilon \to 0$ as the length of $\sigma_j^k$ diverges to infinity, for all $j$. This behavior does not hold w.r.t. the $L^\infty$ norm.

We also point out that each Einstein metric $g$ constructed on any $M = M_{\bar{\sigma}}$ is an isolated point in the moduli space of Einstein metrics on $M$, cf. Remark 3.8; thus such metrics are (locally) rigid.

Theorem 1.1 is an analogue of Thurston's cusp closing theorem [31]. The next result is an analogue of the Jorgensen-Thurston cusp opening theorem, cf. [31], [18]. Let $\mathcal{E}$ be the class of complete, finite volume Einstein metrics constructed via Theorem 1.1, together with the class of complete, non-compact hyperbolic $n$-manifolds $(N, g_{-1})$ of finite volume. Let $\mathcal{E}_V$ be the subset of $\mathcal{E}$ of metrics of volume $\leq V$.

**Theorem 1.2.** *The space $\mathcal{E}$ is closed with respect to the pointed Gromov-Hausdorff and $C^{\infty}$ topologies and the subspaces $\mathcal{E}_V$ are compact, for any $V < \infty$. Any limit point $(M_{\infty}, g_{\infty}) \in \mathcal{E}$ of a sequence $(M^k, g^k) \in \mathcal{E}$ satisfies*

$$C(M_{\infty}) > \max_k C(M^k),$$

*where $C(M)$ denotes the number of cusp ends of $M$.*

In fact, Theorem 1.1 is proved for compact manifolds, where one Dehn-fills all the cusp ends of a given hyperbolic manifold $N$. It is then shown that the closure of the class of resulting Einstein metrics in the pointed Gromov-Hausdorff topology consists of the Einstein manifolds satisfying the conclusions of Theorem 1.1. Given this, the main content of Theorem 1.2 is the compactness $\mathcal{E}_V$.

Taken together, these results are close analogues of Thurston's Dehn surgery theory for hyperbolic 3-manifolds. Note that the manifolds $M_{\bar{\sigma}}$ in Theorem 1.1 can be viewed as obtained by Dehn *surgery* on a fixed manifold $M = M_{\bar{\sigma}_0}$, where $\bar{\sigma}_0$ is any Dehn filling of all the ends of $N$. The original non-compact hyperbolic manifold $N$ is then given by $N = M(\infty, ..., \infty)$.

Several aspects of the Thurston-Jorgensen picture of the structure of the volumes of hyperbolic 3-manifolds also generalize to Einstein metrics in higher dimensions. We describe briefly here the picture in dimension 4; further details, and discussion of the volume behavior in higher dimensions, are given in §4.

The Chern-Gauss-Bonnet theorem shows that the volume of a complete, finite volume hyperbolic 4-manifold is given by

$$vol(N, g_{-1}) = \frac{4\pi^2}{3}\chi(N) \geq 0. \tag{1.6}$$

Further, it is known that given any $k \in \mathbb{Z}^+$, there are (many) complete, non-compact hyperbolic 4-manifolds $N^k$ of finite volume, with $\chi(N^k) = k$, cf. [29] for example. Let $(M, g)$ be any Einstein metric constructed via Theorem 1.1. Then the Chern-Gauss-Bonnet theorem gives

$$vol(M, g) = \frac{4\pi^2}{3}\chi(M) - \frac{1}{6}\int_M |W|^2. \tag{1.7}$$

By a standard Mayer-Vietoris argument, $\chi(M) = \chi(N)$ and thus by (1.6),

$$vol(M, g) = vol(N, g_{-1}) - \delta(\bar{\sigma}) < vol(N, g_{-1}); \tag{1.8}$$

here $\delta(\bar{\sigma})$ is small, and by (1.5) may be made arbitrarily small if the Dehn fillings in $\bar{\sigma} = (\sigma_1, ..., \sigma_p)$ are sufficiently large, depending on $\delta$. Thus, the volume decreases under Dehn filling.

Several features of the Thurston-Jorgensen theory of volumes of hyperbolic 3-manifolds thus generalize to Einstein metrics in higher dimensions. In particular, the set of volumes of metrics in $\mathcal{E}$ is a non-discrete, countable closed set in $\mathbb{R}$. However, it is not known if the set of volumes is well-ordered, (as a subset of $\mathbb{R}$, or finite-to-one, as in the Thurston-Jorgensen theory; again see §4 for further discussion.

The main idea of the proof is a glueing procedure now frequently used in constructing solutions to geometric PDE. Thus, one constructs an approximate Einstein metric on $M = M_{\bar{\sigma}}$, and shows this can be perturbed to an exact solution, i.e. an Einstein metric, by means of the inverse function theorem. Most of the technical work in the paper is concerned with the proof that the linearization of the Einstein operator (1.1) uniformly near the approximate solution is an isomorphism, (modulo diffeomorphisms).

Conceptually, the main issue is to construct the approximate solution. Since the hyperbolic manifold $N$ is already Einstein, one needs to find suitable complete Einstein metrics on $D^2 \times T^{n-2}$ which asymptotically approach a hyperbolic cusp metric. Now a model for such metrics was constructed long ago by physicists, see [23] for instance, and later by Berard-Bergery [7], cf. also [8,9.118]. More recently these model metrics have been frequently analysed in connection with the AdS/CFT correspondence, and are now commonly called toral AdS black hole metrics, cf. [12] and references therein for example. These metrics have the following simple explicit form:

$$g_{BH} = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{T^{n-2}}, \tag{1.9}$$

where $g_{T^{n-2}}$ is any flat metric on $T^{n-2}$ and $V = V_m(r)$ is the function

$$V = r^2 - \frac{2m}{r^{n-3}}, \tag{1.10}$$

If $n = 3$, this gives the usual hyperbolic metric on a tube about a single core geodesic. The parameter $r$ runs over the interval $[r_+, \infty)$, where $r_+ = (2m)^{1/n-1}$. In order to obtain a smooth metric, the circular parameter $\theta$ is required to run over the interval $[0, \beta]$, where

$$\beta = \frac{4\pi}{(n-1)r_+}. \tag{1.11}$$

The number $m$ is any positive number, and represents the mass of $g_{BH}$.

The metric $g_{BH}$ has infinite volume, and so is not asymptotic to a hyperbolic cusp in the usual sense. However, we will see that this can be remedied by suitably "twisting" these metrics. This has been previously described in [1] and is discussed further in §2 below. Briefly, all the metrics $g_{BH}$ in (1.9) are isometric in the universal cover $D^2 \times \mathbb{R}^{n-2}$. By taking suitable isometric actions of $\mathbb{Z}^{n-2}$ on the universal cover, the quotient has large regions closely approximating a given hyperbolic cusp metric. Thus, one may glue on a suitable quotient of the metric $g_{BH}$ onto a cusp of $N$ to obtain an approximate Einstein metric. This is exactly the same observation as Thurston's in the context of Dehn filling of hyperbolic 3-manifolds.

There is a large and growing literature on such glueing constructions for numerous geometric PDE. However, these have not been previously successful in constructing Einstein metrics; to our knowledge, the only exception is the work of Joyce on the construction of Einstein metrics of special holonomy in dimensions 7 and 8. More recently, Mazzeo and Pacard [26] have constructed new classes of conformally compact Einstein metrics on open manifolds, (of infinite volume), by a glueing technique on the boundary at conformal infinity.

The contents of the paper are briefly as follows. In §2, we discuss a number of background results and material needed for the proof of Theorem 1.1. The proof of Theorem 1.1 follows in §3. Several further results are then given after the proof. Thus, Proposition 3.9 proves that there are only finitely many Dehn fillings of a given $N$ which have the same homeomorphism type, while Corollary 3.11 discusses Dehn fillings on non-toral ends. In §4, we discuss a number of aspects of the geometry and topology of the manifolds $M_{\bar{\sigma}}$, as well as the convergence and volume behavior of the set of all Einstein metrics constructed by Dehn filling. Theorem 1.2 is proved at the end of §4.1.

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## 2. Background Material.

In this section, we assemble background results and material needed for the work in §3. We break the discussion into four subsections dealing with different topics.

**§2.1.** Let $(N, g_{-1})$ be a complete, non-compact hyperbolic manifold of finite volume. As mentioned in the Introduction, $N$ then has a finite number of ends $E_i$, each diffeomorphic to $F \times \mathbb{R}^+$, where $F$ is a flat manifold; the topological type of $F$ depends of course on the end $E$.

It is not difficult to show that there is a finite cover $\bar{N}$ of $N$ such that all ends of $\bar{N}$ are tori $T^{n-1}$, cf. [6, Cor. 2.4] for instance. For simplicity, from now on, we assume this is the case, and drop the bar from the notation; see Lemma 3.10 for discussion of non-toral ends.

The groups $\pi_1(T^{n-1}) \simeq \mathbb{Z}^{n-1}$ inject in $\pi_1(N)$ and are called the peripheral subgroups of $\pi_1(N)$. Any subgroup of $\pi_1(N)$ isomorphic to $\mathbb{Z}^{n-1}$ is conjugate to some peripheral subgroup; in fact any non-cyclic abelian subgroup is conjugate to a subgroup of some peripheral subgroup.

The hyperbolic metric $g_{-1}$ on any cusp end $E$ has the form

$$g_{-1} = dt^2 + e^{2t} g_0, \tag{2.1}$$

where $g_0$ is a flat metric on the $(n-1)$-torus $T^{n-1}$, and $t$ runs over the interval $(-\infty, 0]$. By the Margulis Lemma [18], [22], the flat metric $g_0$ may be chosen so that the injectivity radius $inj_{g_0}$ satisfies $inj_{g_0} T^{n-1} \geq \mu_0$, for a fixed constant $\mu_0$, depending only on $n$. For each end $E$ of $N$ on which Dehn filling is performed, we thus choose a fixed toral slice $T^{n-1} = \{0\} \times T^{n-1} \subset E$ satisfying this property. Given this, one may then write

$$(T^{n-1}, g_0) = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1},$$

where the lattice $\mathbb{Z}^{n-1}$ is generated by $(n-1)$ basis vectors $v_1, ..., v_{n-1} \in \mathbb{R}^{n-1}$. The vectors $v_i$ are naturally identified with simple closed geodesics in $(T^{n-1}, g_0)$ which intersect each other exactly once in a single base point. The choice of lattice vectors $(v_1, ..., v_{n-1})$ is of course not unique - it may be changed by any element in $SL(n-1, \mathbb{Z})$. However, we again fix such a basis of each $\pi_1(T^{n-1})$ once and for all.

Next, we describe the process of Dehn filling in higher dimensions; this is completely analogous to the situation in 3 dimensions.

Fix an end $E$ and $T^{n-1} \subset E$ as above. Elements $[\sigma]$ of $\pi_1(T^{n-1}) \simeq \mathbb{Z}^{n-1}$ are represented by closed geodesics in $(T^{n-1}, g_0)$. If $\sigma$ is then any simple closed geodesic in $(T^{n-1}, g_0)$, the class $[\sigma]$ may be represented in the form

$$[\sigma] = \sum \sigma^i [v_i],$$

where each $\sigma^i \in \mathbb{Z}$ and the collection $\sigma^I = (\sigma^1, ..., \sigma^{n-1})$ is primitive, in the sense that $\sigma^I$ is not a multiple of some $\sigma^{I'}$.

Now attach a (generalized) solid torus $D^2 \times T^{n-2}$ to $T^{n-1}$ by a diffeomorphism $\phi$ of the boundary $\partial(D^2 \times T^{n-2}) = S^1 \times T^{n-2}$ with $T^{n-1}$, which sends $S^1$ to the closed geodesic $\sigma$. This gives the Dehn filled manifold

$$M_\sigma = (D^2 \times T^{n-2}) \cup_\phi N. \tag{2.2}$$

By the Bieberbach rigidity theorem [9], any diffeomorphism of $T^{n-1}$ is isotopic to an element of $SL(n-1, \mathbb{Z})$, and so extends to a diffeomorphism of the solid torus $D^2 \times T^{n-2}$. Thus the topological type of $M_\sigma$ is well-defined by the homotopy class of $[\sigma] \in \pi_1(T^{n-1})$. In fact, the topological type of $M_\sigma$ depends only on the unoriented curve $\sigma$, i.e. the class $[\pm\sigma] \in \pi_1(T^{n-1})$, cf. [30]. The vector

$$\sigma = (\sigma^1, ..., \sigma^{n-1})$$

gives the filling coefficients associated to $\sigma$, (w.r.t. the basis $\{v_i\}$). The Dehn filling space associated to the end $E$ is the collection of primitive $(n-1)$-tuples $\{\sigma^i\}$, and thus a subset of $\mathbb{Z}^{n-1}/\{\pm 1\}$.

This process may be carried out separately on any collection of ends $E_j$, $1 \leq j \leq p \leq q$, of $N$ and gives the manifold $M = M_{\bar{\sigma}}$, $\bar{\sigma} = (\sigma_1, ..., \sigma_p)$, obtained by Dehn filling on the ends of $N$.

Next we make a number of remarks on the topology of the manifolds $M = M_{\bar{\sigma}}$. First, the hyperbolic manifold $N$ embeds in any $M$,

$$N \subset M \tag{2.3}$$

as the complement of the core tori $T^{n-2}$ of each Dehn filling. We recall a well-known result of Gromov-Thurston, the $2\pi$ theorem, cf. [19] or [11,Thm.7]; this states that when the length $L(\sigma)$ of $\sigma$ in the flat torus $(T^{n-1}, g_0)$ satisfies

$$L(\sigma) \geq 2\pi, \tag{2.4}$$

the resulting manifold $M_\sigma$ has a complete metric of non-positive sectional curvature and finite volume. Although proved in the context of 3-manifolds, the same result and proof holds in any dimension. (Briefly, one forms the Euclidean cone of length 1 on $\sigma$, and takes the constant skew product with the flat metric on $T^{n-2}$. This gives a singular flat metric on $D^2 \times T^{n-2}$, with cone angle $L(\sigma)$ along the core $T^{n-2}$. A natural smoothing of this cone singularity gives a metric of non-positive curvature on $M_\sigma$).

In particular, all the manifolds $M_{\bar{\sigma}}$ satisfying (2.4) for each geodesic $\sigma_j \in \bar{\sigma}$ are $K(\pi, 1)$ manifolds. Further, with respect to the metric of non-positive curvature on $M_{\bar{\sigma}}$, the core tori $T^{n-2}$ are totally geodesic. Since all closed geodesics in a manifold of non-positive curvature are essential in $\pi_1$, it follows that each core torus injects in $\pi_1$:

$$\pi_1(T^{n-2}) \hookrightarrow \pi_1(M_{\bar{\sigma}}). \tag{2.5}$$

In particular, by Preissman's theorem, one sees that $M_{\bar{\sigma}}$ does not admit a metric of negative sectional curvature when dim $M \geq 4$.

**§2.2.** In this subsection, we discuss some aspects of the geometry of the (standard) AdS toral black hole metrics (1.9):

$$g_{BH} = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{T^{n-2}}. \tag{2.6}$$

As in (1.10) and (1.11), $V = V(r) = r^2 - 2mr^{-(n-3)}$ and $\theta$ takes values in $[0, \beta]$, where $\beta = 4\pi/(n-1)r_+$, $r_+ = (2m)^{1/(n-1)}$ with $r \in [r_+, \infty)$. Although this metric appears to be singular at $r = r_+$, a simple change of coordinates, (analogous to the change from polar to Cartesian coordinates), shows that $g_{BH}$ is smooth everywhere. The metric is defined on the solid torus $D^2 \times T^{n-2}$ and $g_{T^{n-2}}$ is any flat metric on $T^{n-2}$.

From the physical point of view, the core $(n-2)$-torus $H = \{r = r_+\} \subset D^2 \times T^{n-2}$ represents the horizon of a black hole. Note that $H$ is the fixed point set of the isometric $S^1$ action given by rotation in $\theta$. Thus, $H$ is totally geodesic in $g_{BH}$; $H$ gives the usual core geodesic in a hyperbolic tube when $n = 3$.

The metric $g_{BH}$ is an Einstein metric, satisfying (1.1), which is asymptotically hyperbolic or conformally compact, cf. [2] or [10]. This is most easily seen by writing the complete hyperbolic cusp metric $g_{-1}$ on $\mathbb{R} \times T^{n-1}$ in the form

$$g_{-1} = r^{-2}dr^2 + r^2 g_{T^{n-1}}. \tag{2.7}$$

Here $r \in (0, \infty)$ is given by $r = e^t$ in terms of (2.1). The direction $r \to 0$ gives the contracting end of the cusp, while the direction $r \to \infty$ gives the expanding end.

As $r \to \infty$, the metrics $g_{BH}$ and $g_{-1}$ clearly approximate each other. In fact, the curvature tensor of $g_{BH}$ is easily calculated as follows: let $e_i$ be an orthonormal basis for $g_{BH}$ at a given point, with $e_1$ pointing in the $r$ direction, $e_2$ pointing in the $\theta$ direction, and $e_i$, $i \geq 3$ tangent to the toral factor. This basis diagonalizes the curvature tensor at every point, and the sectional curvatures $K$ in the corresponding 2-planes are given by

$$K_{12} = -1 + \frac{(n-3)(n-2)m}{r^{n-1}}, \quad K_{1i} = -1 - \frac{(n-3)m}{r^{n-1}}, \ i \geq 3, \tag{2.8}$$

$$K_{2i} = -1 - \frac{(n-3)m}{r^{n-1}}, \ i \geq 3, \quad K_{ij} = -1 + \frac{2m}{r^{n-1}}, \ i, j \geq 3.$$

Thus, the curvature decays to that of the hyperbolic metric at a rate of $r^{-(n-1)}$, as $r \to \infty$. Let $s$ denote the geodesic distance to the core torus $T^{n-2}$, so that $s = s(r)$ with $ds/dr = V^{-1/2}$. For $r$ large, $r \sim e^s$, and so the curvature decays to $-1$ as $O(e^{-(n-1)s})$. In particular, $|W| = O(e^{-(n-1)s})$, for the Weyl curvature $W$. Similarly, one easily computes that $|\nabla^k R| = O(e^{-(n-1)s}) = O(r^{-(n-1)})$ for the decay of the covariant derivatives of the curvature tensor.

The function $\rho = r^{-1}$ is a smooth, geodesic defining function for the boundary $S^1 \times T^{n-2} \simeq T^{n-1}$ of $D^2 \times T^{n-2}$ and hence the natural conformal compactification of $g_{BH}$ given by

$$\bar{g}_{BH} = \rho^2 g_{BH}, \tag{2.9}$$

extends smoothly to the boundary to give a metric $\gamma$ on the conformal infinity $T^{n-1}$. Clearly, the metric $\gamma$ is the flat product metric $d\theta^2 + g_{T^{n-2}}$, where the circle parametrized by $\theta$ has length $\beta$ given by (1.11). Note that the mass $m$ thus determines the length $\beta$ of the $S^1$ at conformal infinity. Further, it is important to note that (1.11) shows $\beta$ is strictly monotonically decreasing in $m$, $\beta'(m) < 0$.

**§2.3.** Next, we briefly discuss Einstein metrics and the linearization of the Einstein operator. Let $M$ be an arbitrary closed $n$-manifold, or the interior of a compact manifold with boundary. Let $\mathbb{M}^{m,\alpha}$ be the space of $C^{m,\alpha}$ complete Riemannian metrics on $M$, i.e. complete metrics which are $C^{m,\alpha}$ in a smooth atlas on $M$. A more precise description of the topology on $\mathbb{M}^{m,\alpha}$ is given later in §2.4. For convenience, we assume $m \geq 3$, $\alpha \in (0, 1)$. Similarly, let $\mathbb{S}_2^{m,\alpha}$ be the space of $C^{m,\alpha}$ symmetric bilinear forms on $M$.

The Einstein condition (1.1) is diffeomorphism invariant, and hence if $g$ is Einstein, so is $\phi^* g$, for any diffeomorphism $\phi$. In order to take this invariance into account, following Biquard [10], it is natural to consider the related operator

$$\Phi : \mathbb{M}^{m,\alpha} \longrightarrow \mathbb{S}_2^{m-2,\alpha}, \tag{2.10}$$

$$\Phi(g) = Ric_g + (n-1)g + (\delta_g)^* \left(\delta_{\bar{g}} g + \tfrac{1}{2} d(tr_{\bar{g}} g)\right). \tag{2.11}$$

Here $\bar{g}$ is any fixed, (background) metric in $\mathbb{M}^{m,\alpha}$, $\delta$ is the divergence operator, with respect to the given metric, and $\delta^*$ is its $L^2$ adjoint. Recall that $\beta_{\bar{g}} = \delta_{\bar{g}} + \frac{1}{2} dtr_{\bar{g}}$ is the Bianchi operator associated to $\bar{g}$. In the applications in this paper, $\bar{g}$ will be a constructed, approximate solution to the Einstein equations, (called $\widetilde{g}$ later), while $g$ will be a metric nearby to $\bar{g}$ in the $C^{m,\alpha}$ topology. The map $\Phi$ is clearly a $C^\infty$ smooth map.

There are two basic reasons for considering the operator $\Phi$. First:

**Lemma 2.1.** *Suppose $Ric_g - \lambda g \leq 0$, for some $\lambda < 0$ and $|\beta_{\bar{g}}(g)|$ is bounded. If $\Phi(g) = 0$, then $g$ is Einstein, and*

$$Ric_g = -(n-1)g.$$

**Proof:** This result is essentially proved in [10, Lemma I.1.4], in the context of asymptotically hyperbolic metrics. The proof in the case of complete manifolds with $Ric$ strictly negative is the same, but for completeness we give the proof. Applying the operator $\beta_g$ to both sides of (2.11), and using the Bianchi identity and a standard Weitzenbock formula, gives

$$(D^*D - Ric_g)(\beta_{\bar{g}}(g)) = 0.$$

Taking the inner product of this with $\beta_{\bar{g}}$ with respect to $g$ then gives $-\Delta|\beta_{\bar{g}}(g)|^2 + |D\beta_{\bar{g}}(g)|^2 - Ric_g(\beta_{\bar{g}}(g), \beta_{\bar{g}}(g)) = 0$. The last two terms are non-negative, with the last term positive wherever $|\beta_{\bar{g}}(g)| > 0$. The result then follows by a standard application of the maximum principle, or more precisely a maximum principle at infinity, cf. [35]. ∎

The map $\Phi$ is not equivariant with respect to the action of the diffeomorphism group, and so not every Einstein metric $h$ near $\bar{g}$ satisfies $\Phi(h) = 0$. On the other hand, the variety $\Phi^{-1}(0)$ gives a local slice for space of Einstein metrics near $\bar{g}$, transverse to the orbits of the diffeomorphism group, cf. [10].

The second reason is that the form of the linearization $D\Phi$ at $\bar{g}$ has an especially simple form, cf. [10, (1.9)]:

$$(D_{\bar{g}}\Phi)(h) = \frac{1}{2}[D^*Dh - 2R(h) + Ric \circ h + h \circ Ric + 2(n-1)h]. \tag{2.12}$$

Here all metric quantities on the right are with respect to $\bar{g}$ and $R(h)$ is the action of the curvature tensor of $\bar{g}$ on symmetric bilinear forms, cf. [8, 1.131]. In particular, the operator $D_{\bar{g}}\Phi$ is elliptic. For metrics $\bar{g}$ of constant curvature $-1$, one easily computes that

$$R(h) = h - (trh)\bar{g}. \tag{2.13}$$

For later use, we record here the Weitzenbock formula on symmetric bilinear forms, cf. [8, 12.69]

$$D^*Dh = (\delta d + d\delta)h + R(h) - h \circ Ric, \tag{2.14}$$

where $d = d^{\nabla}$ is the exterior derivative induced by the metric connection $\nabla$, and $\delta$ is the adjoint of $d$. Hence, (2.12) may be rewritten in the form

$$2(D_{\bar{g}}\Phi)(h) = L(h) = (\delta d + d\delta)h - R(h) + Ric \circ h + 2(n-1)h. \tag{2.15}$$

For Einstein metrics, this becomes

$$L(h) = (\delta d + d\delta)h - R(h) + (n-1)h. \tag{2.16}$$

The kernel $K = \mathrm{Ker}L$ is the space of (essential) infinitesimal Einstein deformations.

**§2.4.** We conclude with a discussion of topologies on the space of metrics that will be used below. As above $\mathbb{M}$ denotes the space of complete Riemannian metrics on a given manifold $M$. The tangent space to $\mathbb{M}$ at any point is $\mathbb{S}_2$ - the space of symmetric bilinear forms on $M$. Let $\mathbb{M}^m$ be the space of $C^m$ complete Riemannian metrics on $M$ - i.e. there exist (smooth) local coordinates in which the metric is $C^m$. The space $\mathbb{M}^m$ may be defined intrinsically, (without use of local coordinates) by means of a $C^m$ norm on the tangent spaces $T_g\mathbb{M}$. Thus, given $h \in T_g\mathbb{M}$, define

$$||h||_{C^m(g)} = \sup_{x \in M}[|h|(x) + |Dh|(x) + ... + |D^m h|(x)],$$

where $D^j$ is the $j^{\text{th}}$ covariant derivative; both the covariant derivative and (pointwise) norm are taken with respect to $g$. One may then define $\mathbb{M}^m$ to be the completion of the space of $C^\infty$ complete metrics with respect to this norm. It is standard that these two definitions of $\mathbb{M}^m$ agree.

However, the spaces $C^m$ are not suitable for estimates for elliptic equations, (as in (2.12)) - which will be needed in the proof. For this, one must use the Hölder spaces $C^{m,\alpha}$, $\alpha \in (0,1)$. We are not aware of any intrinsic definition of such Hölder spaces of metrics and so local coordinates are needed to define them.

For a given metric $g$ on an $n$-manifold $M$, the coordinates giving the optimal regularity properties for the metric are harmonic coordinates. Let $\rho^{m,\alpha}(x)$ be the $C^{m,\alpha}$ harmonic radius at $x \in M$, cf. [3]. This is the largest radius such that, for any $r < \rho^{m,\alpha}(x)$, the geodesic ball $B_x(r)$ has harmonic coordinates in which the metric components $g_{ij}$ satisfy

$$Q^{-1}(\delta_{ij}) \leq (g_{ij}) \leq Q(\delta_{ij}), \tag{2.17}$$

$$\sum_{1\leq|\beta|\leq m} r^{|\beta|} \sup_y |\partial^\beta g_{ij}(y)| + \sum_{|\beta|=m} r^{m+\alpha} \sup_{y_1,y_2} \frac{|\partial^\beta g_{ij}(y_1) - \partial^\beta g_{ij}(y_2)|}{|y_1 - y_2|^\alpha} \leq Q - 1. \tag{2.18}$$

Here $Q > 1$ is a constant, fixed once and for all, (close to 1).

It is proved in [3] that there is a lower bound on $\rho^{m,\alpha}$, $\rho^{m,\alpha} \geq \rho_0 > 0$, on any Riemannian manifold, where $\rho_0$ depends only on an upper bound for $||\nabla^{m-1}Ric||_{L^\infty}$ and a lower bound for the injectivity radius $inj$:

$$||\nabla^{m-1}Ric||_{L^\infty} \leq \Lambda < \infty, \text{ and } inj \geq i_0 > 0. \tag{2.19}$$

Given a Riemannian manifold $(M, g)$ satisfying (2.19), choose a covering $\mathcal{U}_\lambda$ of $(M, g)$ by a collection of $\rho_0/2$ balls such that the $\rho_0/4$ balls are disjoint. The bounds (2.17) imply a uniform upper bound on the multiplicity of such a covering. Now let $g'$ be another metric on $M$ and set $g' - g = h$, so that $h \in \mathbb{S}_2(M)$. As in (2.18), define then

$$||g'||_{C^{m,\alpha}(g)} \equiv ||h||_{C^{m,\alpha}} = \sup_\lambda\{ \sum_{1\leq|\beta|\leq m} \rho_0^{|\beta|} \sup_y |\partial^\beta h^\lambda_{ij}(y)| + \sum_{|\beta|=m} \rho_0^{m+\alpha} \sup_{y_1,y_2} \frac{|\partial^\beta h^\lambda_{ij}(y_1) - \partial^\beta h^\lambda_{ij}(y_2)|}{|y_1 - y_2|^\alpha}\}, \tag{2.20}$$

where the components $h^\lambda_{ij}$ are taken in local $g$-harmonic coordinates $u^\lambda_i$ satisfying (2.17)-(2.18), and the supremum (2.20) is taken over all such local coordinate systems in $\mathcal{U}_\lambda$.

This defines the $C^{m,\alpha}$ topology on $\mathbb{M}$, denoted as $\mathbb{M}^{m,\alpha}$, in a neighborhood of a given metric $g$ on which one has bounds on the Ricci curvature and injectivity radius as above.

In the course of the arguments to follow, we will have good control on the Ricci curvature, to all orders. However, for the classes of metrics to be considered, there will not be a uniform lower bound on the injectivity radius; this will cause the norm (2.20) to degenerate.

In general, when the injectivity radius is very small, the geometry of small balls may be very complicated; (this involves the structure of collapsed manifolds in the sense of Cheeger-Gromov, with bounds on Ricci curvature). Fortunately, we need only deal with situations where the metrics have *bounded local covering geometry*, in the following sense.

**Definition 2.2.** *Let $i_0 > 0$ be given. Then $(M, g)$ has bounded local covering geometry, (with respect to $i_0$), if for any $x$ where $inj(x) \leq i_0$, there is a finite covering space $\bar{B}_{x,i_0}$ of the geodesic ball $B_x(i_0)$ with $diam_g \bar{B}_{x,i_0} \leq 1$ and*

$$inj_g(\bar{x}) \geq i_0.$$

*Here $\bar{x}$ is a lift of $x$ to $\bar{B}_{x,i_0}$, and $g$ is lifted to $\bar{B}_{x,i_0}$ so that the projection is a local isometry.*

Thus, by passing to a finite covering space locally, one can unwrap to obtain a metric of bounded geometry, and thus good local harmonic coordinates as in (2.17)-(2.18), given suitable control on the Ricci curvature. The degree of the covering of course depends on the injectivity radius at $x$; the smaller the injectivity radius, the larger the degree of the covering. This definition depends on a choice of $i_0$. For our purposes, $i_0$ will be a fixed small number, depending only on dimension, throughout the paper. One may take for instance $i_0$ to be a fixed small multiple of the Margulis constant in dimension $n$, cf. [18], [31].

Let $(M, g)$ be any complete Riemannian manifold satisfying the bound

$$||\nabla^{m-1}Ric||_{L^\infty} \leq \Lambda < \infty, \tag{2.21}$$

and which has bounded local covering geometry with respect to $i_0$. One may then define a "modified" $C^{m,\alpha}$ norm $\widetilde{C}^{m,\alpha}$ of a metric $g'$ by setting $h = g' - g$, and defining

$$||g'||_{\widetilde{C}^{m,\alpha}(g)} \equiv ||h||_{\widetilde{C}^{m,\alpha}} \tag{2.22}$$

exactly as in (2.20) where the charts are defined in finite covering spaces as above in regions where the injectivity radius is $\leq i_0$.

## 3. Proof of Theorem 1.1.

This section is mainly concerned with the proof of Theorem 1.1. Following the proof, Proposition 3.9 proves that the homeomorphism type of $M_{\bar{\sigma}}$ is determined up to finite ambiguity by the curves in $\bar{\sigma}$. Corollary 3.11 is a version of Theorem 1.1 on non-toral ends.

We break the proof of Theorem 1.1 into two main steps.

**Step I. Construction of the Approximate Solution.**

One begins with a complete non-compact hyperbolic $n$-manifold $(N, g_{-1})$ of finite volume, and its collection of toral ends $T^{n-1} \times \mathbb{R}^+$. Fix any such end $E$, and a flat torus $T^{n-1} \subset E$, normalized as in §2.1. Given a simple closed geodesic $\sigma$ in $(T^{n-1}, g_0)$, the discussion in §2.1 describes Dehn filling topologically on the end $E$. In this step, we construct this filling metrically.

Consider the standard toral AdS black hole metric (2.6) on $D^2 \times T^{n-2}$:

$$g_{BH} = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{T^{n-2}}. \tag{3.1}$$

On the universal cover $D^2 \times \mathbb{R}^{n-2}$, the metric $g_{BH}$ lifts to a metric $\widetilde{g}_{BH}$ of the form (3.1), with flat metric on $T^{n-2}$ lifted to $\mathbb{R}^{n-2}$, i.e.

$$\widetilde{g}_{BH} = V^{-1}dr^2 + Vd\theta^2 + r^2(ds_1^2 + ... + ds_{n-2}^2). \tag{3.2}$$

The change of variable $r \to r_m = m^{1/(n-3)}r$ shows that the metrics $\widetilde{g}_{BH} = \widetilde{g}_{BH}(m)$ are all isometric. Thus, for convenience, we fix $m$ once and for all, by setting, (for example), $m = \frac{1}{2}$, so that $r_+ = 1$.

Let $D(R) = \{r \leq R\}$ in $(D^2 \times \mathbb{R}^{n-2}, \widetilde{g}_{BH})$ and let $S(R) = \partial D(R) = \{r = R\}$. The induced metric on the boundary $S(R)$ is then a flat metric

$$V(R)d\theta^2 + (dt_1^2 + ... + dt_{n-2}^2) \tag{3.3}$$

on $S^1 \times \mathbb{R}^{n-2}$, where $t_i = Rs_i$ are coordinates on $\mathbb{R}^{n-2}$. Choose $R$ so that

$$V(R)^{1/2} \cdot \beta = L(\sigma). \tag{3.4}$$

Thus, the length of $S^1 \times \{pt\} \subset S(R)$ equals $L(\sigma)$. Recall that $V = V_m$ and $\beta = \beta(m)$ are determined since $m = \frac{1}{2}$.

Given the flat structure $g_0$ on the torus $T^{n-1}$, observe that there is a unique (up to conjugacy) free isometric $\mathbb{Z}^{n-2}$ action on the flat product $\partial S(R) = S^1 \times \mathbb{R}^{n-2}$ such that the projection map to the orbit space

$$\pi : S^1 \times \mathbb{R}^{n-2} \to T^{n-1} \tag{3.5}$$

satisfies $\pi(S^1) = \sigma$, and the flat structure on $T^{n-1}$ induced by $\pi$ is the given $g_0$. In fact the map $\pi$ is just the covering space of $(T^{n-1}, g_0)$ corresponding to the subgroup $\langle\sigma\rangle \subset \pi_1(T^{n-1})$. In more detail, $\sigma = \sum \sigma^i v_i$ may be viewed as a vector in $\mathbb{R}^{n-1}$. This may be completed to an integral basis $(\sigma, b_2, ..., b_{n-1})$ of $\mathbb{R}^{n-1}$ in such a way that the lattice generated by $(\sigma, b_2, ..., b_{n-1})$ equals the lattice generated by $(v_1, ..., v_{n-1})$, i.e. there is a matrix in $SL(n-1, \mathbb{Z})$ taking $(v_1, ..., v_{n-1})$ to $(\sigma, b_2, ..., b_{n-1})$. Without loss of generality, we may assume that the length of the projection of each

$b_i$ onto $\sigma$ has length at most $|\sigma|$, i.e. $|\langle b_i, \sigma\rangle| < |\sigma|^2$. Then $S(R)$ may be identified with $\mathbb{R}^{n-1}/\langle\sigma\rangle$, where $\langle\sigma\rangle \simeq \mathbb{Z}$ is the group generated by $\sigma$. The vectors $b_2, ..., b_{n-1}$ generate a $\mathbb{Z}^{n-2}$ action on $\mathbb{R}^{n-1}$ commuting with $\langle\sigma\rangle$, and hence generate a $\mathbb{Z}^{n-2}$ action on $S(R)$. The map $\pi$ is then the map to the orbit space of this action.

This $\mathbb{Z}^{n-2}$ action extends radially to an isometric action on the domain $D(R)$ contained in the universal cover $D^2 \times \mathbb{R}^{n-2}$. To see this, the isometry group of $\widetilde{g}_{BH}$ is Isom$(S^1)\times$ Isom$(\mathbb{R}^{n-2})$, corresponding to rotations in the $\theta$-circle and Euclidean isometries on $\mathbb{R}^{n-2}$. Any isometry of the boundary $\partial D(R) = S(R)$ thus extends uniquely to an isometry of $D(R)$. It is clear that the resulting action on $D(R)$ or the full universal cover $D^2 \times \mathbb{R}^{n-2}$ is smooth and free.

The quotient space $(D^2 \times \mathbb{R}^{n-2})/\mathbb{Z}^{n-2} \simeq D^2 \times T^{n-2}$ gives the (twisted) toral AdS black hole metric

$$g_{BH} = [V^{-1}dr^2 + Vd\theta^2 + r^2 g_{\mathbb{R}^{n-2}}]/\mathbb{Z}^{n-2}. \tag{3.6}$$

If now $D(R)$ denotes the domain $\{r \leq R\}$ in the quotient space, the boundary $S(R) = \partial D(R)$ is isometric to the initially given flat torus $(T^{n-1}, g_0)$.

As $r$ varies over $(r^+, R]$, the tori $S(r)$ with metric induced from $g_{BH}$ give a curve of flat metrics on $T^{n-1}$. To describe this curve, let $\lambda(r) = \beta V^{1/2}(r)/|\sigma| = (V(r)/V(R))^{1/2}$, so that $\lambda(r) \in (0, 1]$. Then the torus $S(r)$ is generated by $(\sigma(r), b_2(r), ..., b_{n-1}(r))$, where

$$\sigma(r) = \lambda(r)\cdot\sigma, \text{ and } b_i(r) = b_i + (\lambda(r) - 1)(\langle b_i, \sigma\rangle/|\sigma|^2)\sigma. \tag{3.7}$$

Note that $L(\sigma(r)) \to 0$, as $r \to r_+$, and at $\{r = r^+\}$, the generators $b_i(r^+)$ of the core $(n-2)$-torus $T^{n-2}$ are orthogonal to $\sigma$.

Observe also that for $R$ large, equivalently $L(\sigma)$ large, the core totally geodesic $T^{n-2}$ at $r = r_+$ shrinks to 0 size; in fact

$$diam T^{n-2} \sim R^{-1}.$$

In particular, the injectivity radius of $g_{BH}$ at and near $T^{n-2}$ is $O(R^{-1})$. On the other hand, the metrics $g_{BH}$ clearly have uniformly locally bounded covering geometry, independent of $R$, cf. §2.4. When $n = 3$, the metric $g_{BH}$ is hyperbolic, and is a complete hyperbolic tube metric about a closed geodesic of length $\sim R^{-1}$, cf. [18].

Since the boundaries $(\partial D(R), g_{BH}) = S(R)$ and $(T^{n-1}, g_0) \subset (E, g_{-1})$ are isometric, they may be identified; this gives the Dehn filling $M_\sigma$ of the end $E$ along the curve $\sigma$.

Although the intrinsic flat metrics on the boundaries agree, the union of the two ambient metrics $g_{BH}$ and $g_{-1}$ forms a corner at the seam $\partial S(R)$. To estimate the difference of the metrics, it is convenient to write the hyperbolic cusp metric $g_{-1}$ from (2.7) in the form

$$g_{-1} = r^{-2}dr^2 + r^2 g_{\frac{1}{R}T^{n-1}}, \tag{3.8}$$

so that $R^2 g_{R^{-1}T^{n-1}} = g_0$. This just amounts to replacing $r$ by $r/R$ in (2.7) and has the effect that the glueing seam is located at $\{r = R\}$ for both metrics. Thus, comparing (3.6) and (3.8), one sees that $g_{BH}$ and $g_{-1}$ differ on the order of $O(R^{1-n})$ near the seam. A simple computation also shows that the 2nd fundamental forms $A_{-1}$ and $A_{BH}$ of the boundary with respect to $g_{-1}$ and $g_{BH}$ are

$$A_{-1} = g_{-1}|_{T^{n-1}},$$

$$A_{BH} \sim (1 + O(R^{1-n}))g_{BH}|_{T^{n-1}}.$$

Thus, the 2nd fundamental forms differ on the order of $O(R^{1-n})$. Similarly, from (2.8), the curvatures of the two metrics also differ on the order of $O(R^{1-n})$.

One may then smooth the corner at the toral seam $S(R)$ by setting

$$\widetilde{g} = [\widetilde{V}^{-1}dr^2 + \widetilde{V}d\theta^2 + r^2 g_{\mathbb{R}^{n-2}}]/\mathbb{Z}^{n-2}, \tag{3.9}$$

where, recalling $m = \frac{1}{2}$,

$$\widetilde{V} = r^2 - \frac{\chi \circ r}{r^{n-3}}.$$

Here $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\chi(r) = 1$, for $1 \leq r \leq R/2$, $\chi(r) = 0$, for $r \geq 2R$ and $|\partial^k \chi| = O(R^{-k})$. Note here also that the geodesic distance between the $r$-levels $R/2$ and $2R$ is on the order of 1.

The smooth metric $\widetilde{g}$ extends to a globally defined metric on $M_\sigma$, by letting $\widetilde{g}$ be the hyperbolic metric on $N$. This process may be carried out on any collection of toral ends $E_j$, $1 \leq j \leq p$ of $N$ and gives a smooth metric $\widetilde{g}$ on $M_{\bar{\sigma}} = M(\sigma_1, ..., \sigma_p)$. This gives a collection of numbers $\bar{R} = (R_1, ..., R_p)$ corresponding to $\{\sigma_j\}$ via (3.4). Let

$$R_{min} = \min_j R_j.$$

We also set $R_{max} = \max_j R_j$, but note that $R_{max} = \infty$ if $p < q$, i.e. if there is an end of $N$ which is not capped off by Dehn filling.

These metrics will be called approximate solutions of the Einstein equation (1.1).

The discussion above proves the following result:

**Proposition 3.1.** *The approximate solutions $\widetilde{g}$ constructed above on $M_{\bar{\sigma}}$ are complete, and of uniformly bounded local covering geometry. Outside a tubular neighborhood $U_j$ of radius 1 about each fixed torus $T_j^{n-1}$, $1 \leq j \leq p$, $\widetilde{g}$ is the hyperbolic metric $g_{-1}$ on $N$ or the black hole metric (3.6) on $D^2 \times T^{n-2}$. The curvature of $\widetilde{g}$ is uniformly bounded by that of $g_{BH}$, in that its sectional curvature is bounded by the values in (2.8) with $r = 2m = 1$; if $n = 3$, then the curvature of $\widetilde{g}$ is $-1 + O(R_{min}^{-2})$.*

*The metric $\widetilde{g}$ satisfies the Einstein equation*

$$Ric_{\widetilde{g}} + (n-1)\widetilde{g} = 0, \tag{3.10}$$

*outside $U = \cup U_j$, while inside each $U_j$,*

$$Ric_{\widetilde{g}} + (n-1)\widetilde{g} = O(R_j^{1-n}), \text{ and } |\nabla^k Ric_{\widetilde{g}}| = O(R_j^{1-n}), \text{ for any } k < \infty. \tag{3.11}$$

■

**Step II. Analysis of the Linearization.**

The strategy now is to use the inverse function theorem to perturb the approximate solution $\widetilde{g}$ constructed on $M = M_{\bar{\sigma}}$ into an exact solution of the Einstein equation (1.1). To do this, one needs to study the linearization of the Einstein operator (2.11) at $\widetilde{g}$. Thus, set

$$L = 2D_{\widetilde{g}}\Phi,$$

so that, from (2.12),

$$L(h) = D^*Dh - 2R(h) + Ric \circ h + h \circ Ric + 2(n-1)h. \tag{3.12}$$

where the metric quantities on the right are with respect to $\widetilde{g}$. For reasons that will soon be apparent, we assume throughout Step II that

$$M = M_{\bar{\sigma}} \text{ is compact,} \tag{3.13}$$

so that $p = q$ and all ends of $N$ are Dehn filled. This assumption will be removed later, cf. . Under the assumption (3.13), we will show that $L$ is invertible on suitable function spaces, and obtain a bound on the inverse $L^{-1}$, for all sufficiently large Dehn fillings $\bar{\sigma}$. In addition, these statements hold for metrics sufficiently close to $\widetilde{g}$.

To begin, as function spaces, we will use the modified Hölder spaces and norms, discussed in §2.4; these are well-adapted to the approximate solutions $\widetilde{g}$, since by (3.11), the metrics $\widetilde{g}$ have uniformly bounded Ricci curvature, (in fact uniformly bounded curvature), to all orders, for all $\bar{\sigma}$. Further, the metrics $\widetilde{g}$ have uniformly bounded local covering geometry, again independent of $\bar{\sigma}$.

Thus, fix any $m \geq 3$, $\alpha \in (0,1)$. The map $\Phi$ is a smooth map

$$\Phi : \mathbb{M}^{m,\alpha} \to \mathbb{S}_2^{m-2,\alpha},$$

with derivative at $\widetilde{g}$, (modulo the factor of 2), a smooth linear map

$$L : \mathbb{S}_2^{m,\alpha} \to \mathbb{S}_2^{m-2,\alpha},$$

$$L(h) = f. \tag{3.14}$$

Recall that $R_{max} = \max_j R_j$.

**Proposition 3.2.** *For $M = M_{\bar{\sigma}}$ as in (3.13) with $\bar{\sigma}$ sufficiently large, there is a constant $\Lambda$, independent of $\bar{\sigma}$, such that*

$$||h||_{\widetilde{C}^{m,\alpha}} \leq \Lambda(\log R_{max})||L(h)||_{\widetilde{C}^{m-2,\alpha}}. \tag{3.15}$$

*It follows that $L$ is invertible and the norm of $L^{-1} : \mathbb{S}_2^{m-2,\alpha} \to \mathbb{S}_2^{m,\alpha}$ is uniformly bounded by $\Lambda \log R_{max}$.*

**Proof:** Note first that the estimate (3.15) is local, in the sense that the norms are taken with respect to controlled local harmonic coordinate charts (2.17)-(2.18), in suitable covers where the injectivity radius is small.

The operator $L$ is an elliptic operator on $h$, and by an examination of the form of $L$ in (3.12), one has uniform control on all the coefficients of $L$ in local harmonic coordinates. More precisely, the leading order term $D^*D$ has (uniformly bounded) $C^{m,\alpha}$ coefficients, while the 0-order terms involving curvature give (uniformly bounded) $C^{m-2,\alpha}$ coefficients. Hence, the Schauder estimates for elliptic systems, cf. [17], [27], give the estimate

$$||h||_{\widetilde{C}^{m,\alpha}} \leq \Lambda\{||L(h)||_{\widetilde{C}^{m-2,\alpha}} + ||h||_{L^\infty}\}, \tag{3.16}$$

where $\Lambda$ is independent of the Dehn filling. Note that the $L^\infty$ norm is invariant under passing to (local) covering spaces. Setting $f = L(h)$ as above, it then suffices to prove that there exists $\Lambda < \infty$ such that

$$||h||_{L^\infty} \leq \Lambda \log R_{max}||f||_{\widetilde{C}^{m-2,\alpha}}. \tag{3.17}$$

The claim is that the estimate (3.17) holds provided all Dehn fillings $\sigma_j \in \bar{\sigma}$ are sufficiently large with $\Lambda$ independent of $\bar{\sigma}$. We prove this by contradiction; some comments on the possibility of a more effective proof are given in Remark 3.5 below.

Thus, suppose (3.17) is false. Then there is a sequence of Dehn-filled manifolds $M_i = M_{\bar{\sigma}_i}$, with $(\sigma_j)_i \to \infty$ for each $(\sigma_j)_i \in \bar{\sigma}_i$, together with approximate solutions $\widetilde{g}_i$ on $M_i$, and symmetric forms $h_i \in \mathbb{S}_2^{m,\alpha}(M_i)$, such that

$$||h_i||_{L^\infty} = 1, \quad \text{but} \quad \log(R_{max})_i||f_i||_{\widetilde{C}^{m-2,\alpha}} \to 0, \tag{3.18}$$

where $f_i = L_i(h_i)$. Observe that the estimate (3.16) now implies that

$$||h_i||_{\widetilde{C}^{m,\alpha}} \leq \Lambda, \tag{3.19}$$

where $\Lambda$ is fixed, (independent of $i$).

The idea of the proof then is to pass to limits, and produce a non-trivial limit form $h$ in Ker $L$. Roughly speaking, the manifold $(M_i, \widetilde{g}_i)$ divides into three regions - the hyperbolic region $N$, the cusp regions and the black hole regions. The cusp regions arise as a transition between the hyperbolic and black hole geometries. A well-known argument, essentially due to Calabi [14], implies that $L$ has no kernel on $N$. We will prove that the cusp and black hole regions also have no kernel. Taken together, these facts will give a contradiction to the behavior (3.18). We now supply the details of this description.

First, we prove an elementary Lemma, (which will be needed however only in Appendix A).

**Lemma 3.3.** *Under the assumptions (3.18), one has*

$$||trh_i||_{L^\infty} \to 0 \quad \text{as } i \to \infty. \tag{3.20}$$

**Proof:** Taking the trace of (3.16), using (3.12) and the fact that $trR(h) = \langle Ric, h\rangle$, gives, (dropping the $i$ from the notation),

$$-\Delta trh - \frac{2s}{n} trh = trf + \langle z, h\rangle,$$

where $z$ is the trace-free Ricci curvature. The metric $\widetilde{g}$ is almost Einstein; $|z| \leq O([R_{min}]_i^{-(n-1)})$, cf. (3.11). Since $|h|$ is uniformly bounded, one has $|\langle z, h\rangle| \to 0$, as $i \to \infty$. Since also $|f| \to 0$, the right side of the equation above tends to 0 in $L^\infty$ as $i \to \infty$. The left side is a strictly positive operator, since $s \sim -n(n-1)$. Hence, the result follows by evaluating the equation above at points realizing the maximum and minimum of $trh$. ■

We now continue with the proof of Proposition 3.2 itself. Let $T = \cup T_j^{n-1}$ be the collection of tori $T^{n-1}$ in $N$ to which the solid tori are attached by Dehn filling, and let $N_T$ be the hyperbolic manifold obtained by removing these cusp ends $T_j^{n-1} \times \mathbb{R}^+$ from $N$. The manifold $M_i = M_{\bar{\sigma}_i}$ is a union of black hole and hyperbolic regions:

$$M_i = \{\cup_j D(R_i^j)\} \cup N_T,$$

where for each $j$, $D(R_i^j)$ is the black hole region defined as following (3.6); thus $\partial D(R_i^j)$ is attached to $T_j^{n-1}$. Observe that for any fixed $j$, $R_i^j \to \infty$, as $i \to \infty$. In the following, we will often work with each component of $D(R_i^j)$ separately, and thus usually drop $j$ from the notation.

Let $x_i$ be a sequence of base points in $(M_i, \widetilde{g}_i)$. By passing to a subsequence if necessary, we may assume that $\{x_i\}$ has exactly one of the following behaviors:

(i). (Hyperbolic) One has

$$dist_{\widetilde{g}_i}(x_i, y_0) < \infty, \tag{3.21}$$

for some fixed point $y_0 \in N$. In this case, the pointed sequence $(M_i, \widetilde{g}_i, x_i)$ converges in the pointed Gromov-Hausdorff topology, and smoothly and uniformly on compact sets, to the limit $(N, g_{-1}, x)$, $x = \lim x_i$; $(N, g_{-1})$ is the original hyperbolic manifold.

(ii). (Cusps) For all $j$,

$$dist_{\widetilde{g}_i}(x_i, (T_j^{n-2})_i) \to \infty, \text{ and } dist_{\widetilde{g}_i}(x_i, y_0) \to \infty, \tag{3.22}$$

where $T_j^{n-2}$ is the core torus of the Dehn filling on $E_j$, $1 \leq j \leq q$. In this case, the pointed sequence $(M_i, \widetilde{g}_i, x_i)$ collapses. However, as discussed below, one may unwrap the collapse and obtain a complete limit which is a complete hyperbolic cusp as in (2.7).

(iii). (Black hole) For some $j$,

$$dist_{\widetilde{g}_i}(x_i, (T_j^{n-2})_i) < \infty. \tag{3.23}$$

Again the pointed sequence $(M_i, \widetilde{g}_i, x_i)$ collapses, but by passing to a subsequence, the collapse may be unwrapped and one obtains convergence to a complete black hole metric (2.6).

We deal with each of these cases in turn.

**Case (i).** The forms $h_i$ satisfying (3.18) converge smoothly (in a subsequence) to a limit form $h$ on the complete manifold $N$ satisfying

$$L(h) = 0, \tag{3.24}$$

i.e. $h$ is an infinitesimal Einstein deformation of the hyperbolic metric on $N$.

Now we use the form (2.15) for the linearization $L = 2D\Phi$: recall this is

$$L(h) = (\delta d + d\delta)h - R(h) + Ric \circ h + 2(n-1)h.$$

Since $N$ is hyperbolic, $R(h) = h - (trh)g$, and so

$$L(h) = (\delta d + d\delta)h - R(h) + (n-1)h.$$

Pick any $r_0$ large and pair (3.24) with $h$. Integrating by parts over the domain $N_{r_0} = \{r \geq r_0 > 0\}$, where $r$ is the parameter for any of the cusp ends of $N$ as in (2.7), one thus obtains

$$\int_{N_{r_0}} |dh|^2 + |\delta h|^2 + (n-2)|h|^2 + (trh)^2 = \int_{\partial N_{r_0}} Q(h, \partial h),$$

where the boundary term involves only $h$ and its first derivative. By (3.18) and (3.19), $Q$ is thus uniformly bounded, while the volume form of $\partial N_{r_0}$ is $O(e^{-(n-1)r_0})$. Letting $r_0 \to 0$, it follows that

$$h \equiv 0 \text{ on } N.$$

By the smooth convergence of $h_i$ to the limit form $h$, it follows that $h_i(x_i) \to 0$ for $x_i$ satisfying (3.21). This shows that in fact there is an exhaustion $K_j \subset N$, with $K_j \subset M_i$ for $i = i(j)$ sufficiently large, a sequence $\varepsilon_j \to 0$, and a subsequence $\{h_{i_j}\}$ of $\{h_i\}$ such that

$$|h_{i_j}(x)| \leq \varepsilon_j \quad \forall x \in K_j. \tag{3.25}$$

In the following, we work only with this subsequence, and relabel $\{h_{i_j}\}$ to $\{h_i\}$. This shows that the support of $h_i$ must either wander down the cusp-like regions of $(M_i, \widetilde{g}_i)$, or meet the black hole region of $(M_i, \widetilde{g}_i)$.

**Case (ii).**

In this case, $x_i$ becomes further and further distant from any given point in $N$, as well as any of the black hole regions. Without loss of generality, assume that $\{x_i\}$ is contained in a fixed end $E$ of $N$. Then (3.22) is equivalent to the statements that $(r/R_i)(x_i) \to 0$, and $r(x_i) \to \infty$ as $i \to \infty$.

By construction, the manifolds $(M_i, \widetilde{g}_i, x_i)$ are collapsing in domains of uniformly bounded diameter about $x_i$. However, this collapse may be unwrapped, (cf. §2.4 and Proposition 3.1), in larger and larger finite covering spaces to obtain a complete limit manifold $(C, g_{-1}, x)$. The limit is clearly the complete hyperbolic cusp metric (2.7) on $\mathbb{R} \times T^{n-1}$, with parameter $r$ normalized so that $r(x) = 1$. Similarly, the forms $h_i$, when lifted to forms $\widetilde{h}_i$ on the covering spaces, are uniformly bounded in $\widetilde{C}^{m,\alpha}$. Hence, a subsequence converges in the $\widetilde{C}^{m,\alpha'}$ topology, for any $\alpha' < \alpha$, to a limit form $\widetilde{h}$ satisfying, by (3.18),

$$L(\widetilde{h}) = D^*D\widetilde{h} - 2R(\widetilde{h}) = 0, \tag{3.26}$$

on $(C, g_{-1})$, i.e. $\widetilde{h}$ is an infinitesimal Einstein deformation. Since the forms $\widetilde{h}_i$ have been lifted to covering spaces, they are invariant under the corresponding group of covering transformations. These groups restrict to cyclic groups $\mathbb{Z}_{k_i}$ acting on each circle $S^1$ in $T^{n-1} = S^1 \times S^1 \cdots \times S^1$, with $k_i \to \infty$ as $i \to \infty$. As $i \to \infty$, these covering groups converge to the isometric $T^{n-1}$ action on $(C, g_{-1})$. Hence, by the smooth convergence, the limit form $\widetilde{h}$ is also $T^{n-1}$ invariant. This implies that $\widetilde{h}$ has the form

$$\widetilde{h} = \sum h_{ab}(r)\theta^a \cdot \theta^b, \tag{3.27}$$

where $h_{ab}$ is a function of $r$ only, and $\theta^a$ is the natural orthonormal coframing of the cusp metric (2.7), with $\theta^1 = r^{-1}dr$. It is also clear that $\widetilde{h}$ is bounded on the complete cusp $C$, since the bound (3.18) on $h$ passes continuously to the limit by (3.19).

It is shown in Appendix A that $\widetilde{h}$ then necessarily satisfies $h_{1a} = 0$ for any $a$, (see (A.11) and (A.13)), while for any $a, b \geq 2$, the coefficient functions $h_{ab}$ satisfy

$$\Delta h_{ab} = r^2 h''_{ab} + nrh_{ab} = 0, \tag{3.28}$$

see (A.8). Here $r \in (0, \infty)$ and again $r(x) = 1$. (The proof of these statements is deferred to Appendix A, since it is purely computational, and unrelated to the issues at hand). The general solution of (3.28) is given by $c_1 r^{-(n-1)} + c_2$, cf. (A.9). Since $\widetilde{h}$ is bounded on $C$, it follows that

$$h_{ab} = const = c_{ab}. \tag{3.29}$$

Geometrically, this means that all bounded $T^{n-1}$-invariant infinitesimal Einstein deformations of the cusp metric arise from deformations of the flat structure on $T^{n-1}$.

However, the constants $c_{ab}$ in (3.29) may apriori vary with different choices of the base point sequence $\{x_i\}$. (For instance, consider the function $q(r) = \sin(\log r)$; any sequence $r_i \to \infty$ has a subsequence such that $q(r)$ converges to a constant on $[-k+r_i, k+r_i]$, for any given $k$. Nevertheless, the constants vary with different choices of sequence $r_i$).

We claim that all constants $c_{ab}$ in (3.29) satisfy

$$c_{ab} = 0, \tag{3.30}$$

for all $x_i$ satisfying (3.22). The proof of (3.30) requires the assumption (3.15), not just the weaker the assumption that $||f_i||_{\widetilde{C}^{m-2,\alpha}} \to 0$.

To prove (3.30), return to the black hole metric (3.6), viewed as part of the approximate solution $\widetilde{g} = \widetilde{g}_i$. The injectivity radius and diameter of the tori $T^{n-1}(r)$ then satisfy $inj(T^{n-1}(r)) \sim O(r/R)$ and $diam(T^{n-1}(r)) \sim O(r/R)$; recall here that $R = R_i^j \to \infty$, as $i \to \infty$, for any given $j$. To see this, as discussed in §2.2, the parameter $r$ and the geodesic distance $s$ from the black hole horizon are related by $r \sim e^s$, for $r$ large. Let $R = e^S$. Then the diameter and injectivity radius of the torus at the locus $r$ are approximately $e^{s-S} \sim r/R$, as claimed.

As above, we then unwrap in large covering spaces so that $inj(T^{n-2}) \sim 1$, and $diam(T^{n-2}) \sim 1$. The lifted forms $h = h_i$ are then invariant under the corresponding covering transformations; here and in the following, we drop the tilde from the notation. Given any fixed, large $i$ and with $h = h_i$, let

$$h_{ab}(r) = \frac{1}{volT^{n-1}(r)} \int_{T^{n-1}(r)} h_{ab}(r, \theta) d\theta$$

be the average of $h_{ab}$ over $T^{n-1}(r)$. The same definition applies to $f_{ab}(r)$, so that $h(r)$, $f(r)$ are $T^{n-1}$-invariant forms, as in (3.27). Abusing notation slighly, let $U(r) = \{x \in E : r(x) \in [\frac{1}{2}r, 2r]\}$, so that $U(r)$ is a tubular neighborhood about $T^{n-1}(r)$ of geodesic size on the order of 1, independent of $r$. Using (3.18)-(3.19), we note that one has

$$||h - h(r)||_{C^2(U(r))} = O(\frac{r}{R}) \text{ and } ||f - f(r)||_{C^0(U(r))} = O(\frac{r}{R}),$$

independent of $i$. This is because the coefficients of the lifted forms $h = h_i$ and $f = f_i$ are uniformly bounded in $C^{m,\alpha}$ and $C^{m-2,\alpha}$ respectively, and invariant under rotations by an angle of order $r/R << 1$ on each circle of $T^{n-1}(r)$; here $r_i/R_i \to 0$ as $i \to \infty$. A function on a circle which is bounded in $C^k$ norm by 1, and which is periodic of period $\delta << 1$ is $\varepsilon$-close to its average value in $C^{k-1}$, where $\varepsilon$ depends linearly on $\delta$ for $\delta$ sufficiently small.

Moreover, in the region where $r(x) \sim r$, the black hole metric $g_{BH}$ differs from the cusp metric $g_C$ on the order of $O(r^{-(n-1)})$, cf. (3.8ff). It then follows that the equation (3.14), i.e. $(L(h))_{ab} = f_{ab}$, $a, b \geq 2$, may be written in the form

$$r^2(h_{ab}(r))'' + nr(h_{ab}(r))' = f_{ab}(r) + e_{ab}(r), \tag{3.31}$$

where $e_{ab}(r) = O(r/R) + O(r^{-(n-1)})$, and as above the index $i$ has been supressed, (compare with (3.28)).

By (3.25), we already know that there exist $r_i \to \infty$ such that $r_i/R_i \to 0$ and $|h_i|(x) \to 0$ whenever $r(x) \geq r_i$. Hence view (3.31) for $r$ in the interval $[C_0, r_i]$, where $C_0$ is a fixed but arbitrarily large constant. The equation (3.31) may be integrated explicitly to give

$$h'_{ab}(r) = \frac{1}{r^n}[\int_{C_0}^{r} r^{n-2}(f_{ab}(r) + e_{ab})dr + c_1], \tag{3.32}$$

where we recall $h = h_i$, $f = f_i$. Let $\alpha_i = \sup f_i(r) \log R_i$ on the interval $[C_0, r_i]$, so that by (3.18), $\alpha_i \to 0$ as $i \to \infty$. Then (3.32) gives

$$|h'_{ab}(r)| \leq C[\frac{\alpha_i}{\log R_i}\frac{1}{r} + \frac{1}{R_i} + r^{-n}\log r],$$

on $[C_0, r_i]$. Integrating further from $r$ to $r_i$ then gives

$$\begin{aligned} |h_{ab}(r)| &\leq C'[\alpha_i \frac{\log r_i}{\log R_i} + \frac{r_i}{R_i} + r_i^{-(n-1)}\log r_i + r^{-(n-1)}\log r] + |h_{ab}(r_i)| \\ &\leq C'\delta_i + C' r^{-(n-1)}\log r + |h_{ab}(r_i)|, \end{aligned} \tag{3.33}$$

uniformly on $[C_0, r_i]$, where $\delta_i \to 0$ as $i \to \infty$. By (3.25), $|h_i(r_i)| \to 0$ as $i \to \infty$.

This proves the claim (3.30), and as in Case (i), it follows that $h_i(x_i) \to 0$ as $i \to \infty$, for any $x_i$ satisfying (3.22).

**Case (iii).**

For $x_i$ satisfying (3.23), the metrics $(M_i, \widetilde{g}_i, x_i)$ are also highly collapsed in regions of arbitrary but uniformly bounded diameter about $x_i$. However, just as above in Case (ii), the collapse may be unwrapped by passing to sufficiently large finite covering spaces and one may then pass to a limit. The limit is a complete black hole metric $g_{BH}$ on $D^2 \times T^{n-2}$ as in (3.6). Similarly, as above, the forms $h_i$, (and $f_i$), lift to forms $\widetilde{h}_i$ on the covering spaces and converge, (in a subsequence), in the $\widetilde{C}^{m,\alpha'}$ topology, to a limit $T^{n-1}$-invariant form $\widetilde{h}$ satisfying the kernel equation (3.26) on $(D^2 \times T^{n-2}, g_{BH})$. The assumption (3.18), together with the results above in Cases (i) and (ii) and the smooth convergence to the limit imply that one must have

$$||\widetilde{h}||_{L^\infty} = 1. \tag{3.34}$$

In particular, $\widetilde{h} \neq 0$. Further, by (3.33) the limit form $\widetilde{h}$ satisfies

$$|\widetilde{h}| \leq C' r^{-(n-1)}\log r \tag{3.35}$$

as $r \to \infty$ in $(D^2 \times T^{n-2}, g_{BH})$.

The following Lemma now shows this situation is impossible.

**Lemma 3.4.** *Any bounded $T^{n-1}$-invariant Einstein deformation $h$ of a black hole metric $(D^2 \times T^{n-2}, g_{BH})$ in (3.6) satisfies*

$$|h|(y) \to c_0 \geq 0, \quad \text{as} \quad y \to \infty, \tag{3.36}$$

*for some constant $c_0$. Further, $c_0 = 0$ if and only if $h \equiv 0$. In particular, the operator $L$ has trivial $L^2$ kernel, i.e. there are no non-trivial solutions $h$ of (3.26) with $h \in L^2$.*

**Proof:** It is possible to prove Lemma 3.4 by a direct, although rather lengthy computation, by solving the system of ODE's for the coefficients of $h$ as in (3.28) above. Thus, the main point is to prove that $L$ has no $L^2$ kernel, i.e. the black hole metric is non-degenerate, cf. [26]. Since $g_{BH}$ has regions where the sectional curvature is positive when $n > 4$, this is not so easy to prove computationally. Thus, instead of going through the extensive computational details, we give a more conceptual proof at the non-linear level.

Thus, we first note that any complete Einstein metric (1.1) on $D^2 \times T^{n-2}$ with an isometric $T^{n-1}$ action, with codimension 1 principal orbits, is a black hole metric $g_{BH}$ as in (3.6). This is proved in [5] when $n = 4$ and the same proof holds in all dimensions. A black hole metric is uniquely determined, up to isometry, by the flat structure induced on $T^{n-2}$, the mass parameter $m$, giving the length of the remaining $S^1$, (parametrized by $\theta$), and the homotopy class of $\sigma$. In particular, the only *small* deformations of $g_{BH}$ are those induced by variation of the flat structure on $T^{n-2}$ and variation of the mass $m$, cf. (2.6).

Next we claim that the infinitesimal deformation $h$ is tangent to the moduli space of $C^2$ conformally compact (or asymptotically hyperbolic) Einstein metrics on the given manifold. To see this, since $h$ is invariant with respect to the standard $T^{n-1}$-action on $g_{BH}$, it may be written in the form (3.27), i.e.

$$h = \sum h_{ab}(r)\theta^a \cdot \theta^b,$$

where $\theta^a$ is the natural co-framing of $g_{BH}$, dual to $e_a$ as in (2.8). As noted in (2.9), the function $\rho = r^{-1}$ is a smooth defining function, and gives a smooth compactification $\bar{g}_{BH} = \rho^2 g_{BH}$ of $g_{BH}$. The associated compactification $\bar{h} = \rho^2 h$ of $h$ satisfies $|\bar{h}|_{\bar{g}_{BH}} = |h|_{g_{BH}}$. Further, the equation (3.24) for an infinitesimal Einstein deformation may be reexpressed in terms of the compactified metric $\bar{g}_{BH}$ and $\bar{h}$, where it gives a system of ODE's for the functions $\bar{h}_{ab}(\rho)$. Since $g_{BH}$ is asymptotic to the hyperbolic cusp metric, it is easy to see that to leading order, the system (3.24) has the same form as that for the hyperbolic cusp metric, given in (A.8), (A.10) and (A.12). Hence a straightforward calculation for conformal changes of metric shows the coefficients $\bar{h}_{ab}(\rho)$ satisfy

$$\bar{h}''_{ab} - \frac{n-2}{\rho}\bar{h}'_{ab} = o(1),$$

when $a, b \geq 2$. A similar expression holds for the coefficients $\bar{h}_{1a}$. It follows by elementary integration that $\bar{h}$ extends $C^2$ up to the boundary at $\rho = 0$. This means that $h$ defines a tangent vector to the space of conformally compact Einstein metrics, as required. (A similar but much more elementary argument holds when $n = 3$, using the fact that infinitesimal Einstein deformations are infinitesimal hyperbolic deformations; we will not carry out the details).

Now the space of such $C^2$ conformally compact Einstein metrics is a smooth Banach manifold, and any tangent vector $h$ is tangent to a curve of conformally compact Einstein metrics, cf. [2], [4]. Since $h$ is $T^{n-1}$ invariant, it follows by the classification above that $h$ is tangent to the space of black hole metrics on $D^2 \times T^{n-2}$. Thus, $h$ corresponds to an infinitesimal deformation of the flat structure on $T^{n-2}$ and the mass $m$.

Because $h$ is $T^{n-1}$ invariant near infinity, it is now clear that $|h| \to c_0$ at infinity, for some constant $c_0$. This gives (3.36). To prove the second statement, suppose $h$ is non-trivial, i.e. $h \neq 0$. If $h$ induces a non-trivial deformation of the $T^{n-2}$ factor, then it is clear from the form of $h$ above that $c_0 \neq 0$. If instead the variation of the $T^{n-2}$ factor is trivial, consider the deformation of the mass $m$. This induces a variation of the length $\beta$ of the $S^1$ factor parametrized by $\theta$. Since $h \neq 0$, the variation of $m$ is non-trivial. Now as noted following (2.9), $\beta$ is strictly monotone decreasing in $m$, and from (1.11), $\beta'(m) < 0$. Hence, the variation of the $S^1$ factor is non-trivial. This implies that $c_0 \neq 0$, which completes the proof of Lemma 3.4. ■

Combining the results obtained in Cases (i)-(iii) above, this now also completes the proof of (3.15). To prove the last statement in Proposition 3.2, (3.15) implies that Ker$L = 0$ on $\mathbb{S}_2^{m,\alpha}$ in the $\widetilde{C}^{m,\alpha}$ norm. Since $L$ is essentially self-adjoint, and $M$ is assumed compact, standard Fredholm theory implies that that $L$ is surjective onto $\mathbb{S}_2^{m-2,\alpha}$ with the $\widetilde{C}^{m-2,\alpha}$ norm. Moreover, (3.15) then gives a bound $\Lambda$ on the norm of the inverse mapping $L^{-1}$ on these spaces. ■

**Remark 3.5.** With some further work, it should be possible to give a direct, effective proof of Proposition 3.2, avoiding the use of a contradiction. However, this requires understanding of the possible limit behaviors discussed above anyway, and carrying along effective estimates at each stage of the proof. We do not know of any proof that holds without addressing the structure of the possible limits.

A more explicit estimate of the constant $\Lambda$ would give more precise information on the set of Dehn fillings which carry Einstein metrics.

Next, we observe that the proof of Proposition 3.2 also shows that the conclusion (3.15) holds for all smooth metrics sufficiently close to the approximate solution $\widetilde{g}$. More precisely, let $B_{\widetilde{g}}(\varepsilon)$ be the $\varepsilon$-ball about $\widetilde{g}$ in the $\widetilde{C}^{m,\alpha}$ topology on $\mathbb{M}$, cf. (2.22).

**Corollary 3.6.** *There exists $\varepsilon_0 > 0$ such that (3.15) holds, for all metrics $g' \in B_{\widetilde{g}}(\varepsilon_0)$, with again $\Lambda$ independent of $\bar{\sigma}$, (provided $\bar{\sigma}$ is sufficiently large).*

**Proof:** The proof is exactly the same as that of Proposition 3.2. Briefly, if not, then there exists a sequence $(M_i, \widetilde{g}_i)$, together with symmetric forms $h_i$ such that (3.18) holds, for some sequence of metrics $g'_i \in B_{\widetilde{g}_i}(\varepsilon_i)$, with $\varepsilon_i \to 0$. However, the proof of Proposition 3.2 applies just the same to this sequence, (as with the sequence $\widetilde{g}_i$ before), and gives the same contradiction.

■

**Step III. (Solution of the Nonlinear Problem).**

We are now in position to prove Theorem 1.1. This is done first in the case (3.13) where all the ends of $N$ are capped by Dehn filling,

**Proof of Theorem 1.1. ($M_{\bar{\sigma}}$ compact)**

Let $M = M_{\bar{\sigma}} = M(\sigma_1, ..., \sigma_q)$ be obtained from $N$ by Dehn filling all the toral ends of $N$. Let $\widetilde{g}$ be the approximate Einstein metric on $M$ constructed in Step I. By (3.10)-(3.11), $\Phi(\widetilde{g}) = 0$ outside the glueing region $U = \cup U_j$. Write $M \setminus U = B \cup N_T$, where $B$ is the union of the black hole regions and $N_T \subset N$.

Let

$$\mathcal{W} = \{f \in \mathbb{S}_2^{m-2,\alpha} : f(x) = 0, \forall x \text{ s.t. } dist_{\widetilde{g}}(x, B) \geq 2\}. \tag{3.37}$$

Note that $\mathcal{W}$ is closed in $\mathbb{S}_2^{m-2,\alpha}$ and so is a Banach subspace of $\mathbb{S}_2^{m-2,\alpha}$. Set $f_0 = \Phi_{\widetilde{g}}(\widetilde{g})$, and note that $f_0 \in \mathcal{W}$. We let $\mathcal{W}_\varepsilon = \mathcal{W} \cap B_{f_0}(\varepsilon)$, where $B_{f_0}(\varepsilon)$ is the $\varepsilon$-ball about $f_0$ in $\mathbb{S}_2^{m-2,\alpha}$, and set

$$\mathcal{U}_\varepsilon = \Phi^{-1}(\mathcal{W}_\varepsilon),$$

so that

$$\Phi_0 = \Phi|_{\mathcal{U}_\varepsilon} : \mathcal{U}_\varepsilon \to \mathcal{W}_\varepsilon. \tag{3.38}$$

By Proposition 3.2 and Corollary 3.6, for $\varepsilon_0$ sufficiently small, every point in $\mathcal{W}_{\varepsilon_0}$ is a regular value of $\Phi$ and so of $\Phi_0$. Hence by the inverse function theorem, $\mathcal{U}_{\varepsilon_0}$ is a Banach submanifold of $\mathbb{M}^{m,\alpha}$, (of infinite codimension), and $\Phi_0$ is a local diffeomorphism onto $\mathcal{W}_{\varepsilon_0}$. Of course the use here of Proposition 3.2 and Corollary 3.6 means that $\varepsilon_0$ might depend on $\bar{\sigma}$, via $R_{max}$. Further, by construction,

$$\widetilde{g} \in \mathcal{U}_{\varepsilon_0}.$$

We now consider the mapping $\Phi_0$ in place of $\Phi$. Being the restriction of a smooth map to a submanifold, $\Phi_0$ is of course still smooth. The linearization $L = D\Phi$ restricted to the tangent spaces $T_{g'}\mathcal{U}_{\varepsilon_0}$ of $\mathcal{U}_{\varepsilon_0}$, gives a linear mapping

$$L_0(h) = f, \tag{3.39}$$

from $h \in T_{g'}\mathcal{U}_{\varepsilon_0}$ to $f \in T_{\Phi_0(g')}\mathcal{W}_{\varepsilon_0}$. Observe that $f$ has restricted support on $M$; supp $f \subset$ supp $\eta$. Of course in general $h$ does not have this form; one may well have supp $h = M$.

We now claim that Proposition 3.2, (and Corollary 3.6), can be improved when $\Phi$ is restricted to $\Phi_0$.

**Proposition 3.7.** *Let $M = M_{\bar{\sigma}}$ be compact as in (3.13). Then there exist $\varepsilon_0 > 0$ and $\Lambda < \infty$, independent of $\bar{\sigma}$, such that for any $g' \in \mathcal{U}_{\varepsilon_0}$ and $\bar{\sigma}$ sufficiently large, one has*

$$||h||_{\widetilde{C}^{m,\alpha}} \leq \Lambda ||f||_{\widetilde{C}^{m-2,\alpha}}, \tag{3.40}$$

*for $h$ and $f$ as in (3.39). Thus, $L_0$ is invertible on $\mathcal{U}_{\varepsilon_0}$, and one has a uniform bound $\Lambda$ for the norm of $L_0^{-1}$, independent of $\bar{\sigma}$.*

**Proof:** Given the work above, this is now essentially an immediate consequence of the proof of Proposition 3.2. Thus, suppose first that $g' = \widetilde{g}$. The proof that (3.40) holds at $\widetilde{g}$ then follows exactly the proof of Proposition 3.2, with $f \in T_{f_0}\mathcal{W}$ and $h \in T_{\widetilde{g}}\mathcal{U}$ in place of the general $f$ and $h$ from before. The $\log R_{\max}$ term in the estimate (3.15) arises only because of the behavior in (3.32)-(3.33) in Case (ii). For $f \in T_{f_0}\mathcal{W}$, one has $f \equiv 0$ in this region and hence the same analysis following (3.32) shows that (3.33) holds. It follows that (3.30) and (3.35) both hold also. The proof of Cases (i) and (iii) then also holds without any changes. This establishes the estimate (3.40) at $\widetilde{g}$. The proof that (3.40) is also valid for $g' \in \mathcal{U}_{\varepsilon_0}$, with $\varepsilon_0$ independent of $\bar{\sigma}$ for $\bar{\sigma}$ sufficiently large, is then exactly the same as Corollary 3.6, with $\Phi_0$ in place of $\Phi$. The last statement also follows as before, since $L_0$ is still essentially self-adjoint as a mapping $T(\mathcal{U}_{\varepsilon_0}) \to T(\mathcal{W}_{\varepsilon_0})$. ■

The proof of Theorem 1.1 when $M$ is compact is now quite straightforward. First, the estimate (3.11) implies that

$$||\Phi_0(\widetilde{g})||_{\widetilde{C}^{m-2,\alpha}} \leq (R_{min})^{-(n-1)}, \tag{3.41}$$

where via (3.4), $R_{min}$ is the shortest length of the collection of geodesics $\{\sigma_j\}$ in $\bar{\sigma}$, up to a fixed constant.

Next, let $h$ be any symmetric bilinear form in $T_{\widetilde{g}}\mathcal{U}$ satisfying $||h||_{\widetilde{C}^{m,\alpha}} \leq 1$. Then (3.11) and (3.12) show that

$$||D_{\widetilde{g}}\Phi_0(h)||_{\widetilde{C}^{m-2,\alpha}} \leq K.$$

The constant $K$ depends only on the local geometry of $\widetilde{g}$, (in covering spaces in sufficiently collapsed regions), and hence is independent of $\bar{\sigma}$. For the same reasons, choosing $\varepsilon_0 > 0$ smaller if necessary, one has

$$||D_{g'}\Phi_0(h)||_{\widetilde{C}^{m-2,\alpha}} \leq 2K, \tag{3.42}$$

for all $g' \in \mathcal{U}_{\varepsilon_0}$, and $h$ as above, where $K$ is independent of $\bar{\sigma}$. Next, Proposition 3.7 shows that

$$||(D_{g'}\Phi_0)^{-1}(f)||_{\widetilde{C}^{m,\alpha}} \leq \Lambda, \tag{3.43}$$

for all $g' \in \mathcal{U}_{\varepsilon_0}$ and $f \in T\mathcal{W}_{\varepsilon_0}$ with $||f||_{\widetilde{C}^{m-2,\alpha}} \leq 1$. The bounds (3.42)-(3.43) prove that $\Phi_0$ is a bi-Lipschitz map, with Lipschitz constant $2K$ for $\Phi_0$ and $\Lambda$ for $\Phi_0^{-1}$.

The inverse function theorem applied to the mapping $\Phi_0$ between the Banach manifolds $\mathcal{U}_{\varepsilon_0}$ and $\mathcal{W}_{\varepsilon_0}$ then implies that there is a domain $\Omega \subset U_{\varepsilon_0}$ and $\varepsilon_1 > 0$ such that

$$\Phi_0 : \Omega \to \mathcal{W}_{\varepsilon_1}, \tag{3.44}$$

is a diffeomorphism onto $\mathcal{W}_{\varepsilon_1}$. The constant $\varepsilon_1$ is of the form $\varepsilon_1 = (4K/\Lambda)\varepsilon_0$. By (3.41), one may now choose $R_{min}$ sufficiently large, i.e. $\bar{\sigma}$ sufficiently large, so that $0 \in \mathcal{W}_{\varepsilon_1}$. Via (3.44), this implies that there exists a metric $g \in \mathcal{U}_{\varepsilon_0}$, such that

$$\Phi(g) = \Phi_0(g) = 0.$$

By Lemma 2.1, $g$ is then an Einstein metric on $M$, smoothly close to $\widetilde{g}$. ■

**Remark 3.8.** Since $\Phi_0$ in (3.44) is a diffeomorphism on $\Omega$, the metric $g$ is the unique Einstein metric, (up to isometry), with the normalization (1.1) in $\Omega$. Moreover, since $\Phi$ is a local diffeomorphism near $g$, it follows that the metrics $g$ constructed above are isolated points in the moduli space of Einstein metrics on $M$.

Next, we complete the proof of Theorem 1.1 by discussing the case where not all the cusps of $M$ are capped by Dehn filling.

**Proof of Theorem 1.1. ($M_{\bar{\sigma}}$ non compact)**

Let $\mathcal{E}_c = \mathcal{E}_c(N)$ be the collection of Einstein metrics constructed on the compact manifolds $M_{\bar{\sigma}}$ above associated to a given $N$. This is an infinite collection of metrics, parametrized by $\bar{\sigma}$. Now let $M = M_{\bar{\sigma}} = M(\sigma_{j_1}, ..., \sigma_{j_p})$ be any manifold obtained by Dehn filling a collection of $p$ toral ends $E_{j_k}$ of $N$, with each $\sigma_{j_k}$ sufficiently large. By relabeling, assume $1 \leq j_k \leq p$, so that the ends $E_j$, $p+1 \leq j \leq q$ are cusp ends of $M$. Further, we assume $p < q$, so that $M$ is non-compact. The manifold $M$ may be written in the form $M = M(\sigma_1, ..., \sigma_p, \infty, ..., \infty)$.

Let $M_i = M(\sigma_1, ..., \sigma_p, \sigma_{p+1}^i, \cdots, \sigma_q^i)$, where $\sigma_k^i$, $p+1 \leq k \leq q$, is any sequence such that $\sigma_k^i \to \infty$ as $i \to \infty$, for each fixed $k$. Let $\widetilde{g}_i$ be the approximate Einstein metrics constructed on $M_i$ and let $g_i \in \mathcal{E}_c$ be the associated Einstein metrics on $M_i$ given by Theorem 1.1, (in the compact case). If $y_0$ is any fixed point in $N$, it is clear that the pointed sequence $(M_i, \widetilde{g}_i, y_0)$ has a subsequence converging smoothly and uniformly on compact sets to the limit manifold $(M, \widetilde{g}, y_0)$, where $\widetilde{g}$ is the approximate Einstein metric constructed on $M$ in Step I. Since the Einstein metrics $g_i$ are smoothly close to the approximate metrics $\widetilde{g}_i$, $\{g_i\}$ also converges, again smoothly and uniformly on compact sets, to a limit Einstein metric $g$ on $M$. The limit $g$ is complete, and of uniformly bounded curvature. This completes the proof of Theorem 1.1. $\blacksquare$

Having completed the proof of Theorem 1.1, we next show that the homeomorphism type of the Dehn-filled manifolds $M_{\bar{\sigma}}$ is determined up to finite ambiguity by the data $\bar{\sigma} = (\sigma_1, ..., \sigma_p)$. Let $\mathrm{Out}(\pi_1(N))$ be the group of outer automorphisms of $\pi_1(N)$. By Mostow-Prasad rigidity, this is a finite group, isomorphic to the isometry group $\mathrm{Isom}(N, g_{-1})$ of $N$.

**Proposition 3.9.** *Let $n \geq 4$. The number of manifolds $M_{\bar{\sigma}}$ homeomorphic to a given manifold $M_{\bar{\sigma}_0}$ is finite, and bounded by the cardinality of Out($\pi_1(N)$).*

**Proof:** If $M = M_{\bar{\sigma}}$ is obtained from $N$ by Dehn filling a collection of cusp ends $\{E_j\}$ of $N$, then by the Seifert-Van Kampen theorem, the fundamental group $\pi_1(M)$ is given by

$$\pi_1(M) = \pi_1(N)/\langle \cup R_j \rangle, \tag{3.45}$$

where $R_j \simeq \mathbb{Z}$ is the subgroup generated by the closed geodesic $\sigma_j \in \bar{\sigma}$, (i.e. the meridian circle is annihilated). As noted in §2.1, if the Dehn filling is sufficiently large, then $M$ is a $K(\pi, 1)$ and each core torus injects in $\pi_1$:

$$\pi_1(T_j^{n-2}) \hookrightarrow \pi_1(M).$$

Thus to each peripheral subgroup $\mathbb{Z}^{n-1} \simeq \pi_1(E_j) \subset \pi_1(N)$ is associated a subgroup $\mathbb{Z}^{n-2} \subset \pi_1(M)$, obtained by dividing $\mathbb{Z}^{n-1}$ by $\mathbb{Z}$. This gives a distinguished collection of (conjugacy classes of) subgroups isomorphic to $\mathbb{Z}^{n-2}$ and $\mathbb{Z}^{n-1}$, corresponding to the filled and unfilled ends of $N$; call these the peripheral subgroups of $\pi_1(M)$. As before with $N$, any non-cyclic abelian subgroup of $\pi_1(M)$ is conjugate to a subgroup of a peripheral subgroup. This is because $M$ admits a complete metric of non-positive sectional curvature naturally associated to the Dehn filling, cf. §2.1. With respect to such a metric, any non-cyclic abelian subgroup is carried by an essential torus embedded in $M$. However, up to isotopy, all such tori are contained in the core tori $T^{n-2}$ of $M$ or the end tori $T^{n-1}$ of $M$.

Now suppose $M_i$, $i = 1, 2$, are two $n$-manifolds obtained by Dehn fillings of a given hyperbolic $N$. If $M_1$ is homeomorphic to $M_2$, then $\pi_1(M_1) \simeq \pi_1(M_2)$, and we may choose a fixed isomorphism identifying both with the (abstract) group $\pi_1(M)$. A homeomorphism $F : M_1 \to M_2$ then defines an automorphism

$$F_* : \pi_1(M) \to \pi_1(M). \tag{3.46}$$

By the uniqueness mentioned above, it follows that $F_*$ permutes the collection of peripheral subgroups onto themselves, inducing an isomorphism of each $\mathbb{Z}_i^{n-2}$ to some $\mathbb{Z}_j^{n-2}$, and $\mathbb{Z}_k^{n-1}$ to some $\mathbb{Z}_l^{n-1}$ up to conjugacy; of course one may have $i = j$ or $k = l$. Each such subgroup is carried by an embedded, essential torus $T^{n-2}$ or $T^{n-1}$ in $M$. Let $\hat{T}_i^{n-2} = F(T_i^{n-2})$ and set $T = \cup T_i^{n-2}$, $\hat{T} = \cup \hat{T}_i^{n-2}$. Then $F$ gives a homeomorphism of $N = M \setminus T$ onto $\hat{N} = M \setminus \hat{T}$. Equivalently, $F$ induces a homeomorphism of the original hyperbolic manifold $N$,

$$F : N \to N, \tag{3.47}$$

permuting the cusp ends of $N$. Further, if $F$ maps the end $E_i$ to $E_j$ then by (3.46), $F_*\langle\sigma_i\rangle = \langle\sigma_j\rangle$, up to conjugacy, in $\pi_1(N)$; here $\langle\sigma\rangle$ is the subgroup generated by $[\sigma]$.

If $F$ is homotopic to the identity on $N$, then the filling data of $M_1$ and $M_2$ are the same, up to sign, and so $M_1$ and $M_2$ are diffeomorphic, cf. §2.1. If not, then $F$ induces a non-trivial automorphism $F_*$ of $\pi_1(N)$, so that $F_*$ is an element of the outer automorphism group $\mathrm{Out}(\pi_1(N))$. Since this group is finite, it follows that only a finite number of filling data can give rise to homeomorphic manifolds $M_{\bar{\sigma}}$. One obtains a bound on this number by a bound on the order of $\mathrm{Isom}(N)$, or more precisely a bound on the order of the corresponding effective group acting on the corresponding Dehn filling spaces $\mathbb{Z}^{n-1}$. ■

We complete this section with a discussion of Dehn filling on non-toral ends. Thus, let $(N, g_{-1})$ be a complete hyperbolic $n$-manifold of finite volume, with an end $E$ of the form $F \times \mathbb{R}^+$, where $F$ is a flat manifold with induced metric $g_0$. By the Bieberbach theorem, cf. [34],

$$F = T^{n-1}/\Gamma, \tag{3.48}$$

where $\Gamma$ is a finite group of Euclidean isometries acting freely on $T^{n-1}$. Let $\bar{E}$ be the covering space of $E$ with covering group $\Gamma$, so that $\bar{E}$ is of the form $T^{n-1} \times \mathbb{R}^+$, with hyperbolic metric $g_{-1}$. For $\sigma$ a simple closed geodesic in $(T^{n-1}, g_0)$, let $\phi_\sigma$ be a diffeomorphism of $\partial(D^2 \times T^{n-2})$ to $T^{n-1}$ sending $S^1 = \partial D^2$ to $\sigma \subset T^{n-1}$, so that $\phi_\sigma$ attaches a solid torus to $T^{n-1}$ along $\sigma$. Now suppose that the action of $\Gamma$ on $T^{n-1}$ extends to a free action of $\Gamma$ on $D^2 \times T^{n-2}$ and that $\Gamma$ commutes with the diffeomorphism $\phi_\sigma$ on the boundary $T^{n-1}$. Then the quotient manifold

$$M_\sigma = (D^2 \times T^{n-2})/\Gamma \cup_{\phi_\sigma} N$$

is well-defined, and is the manifold obtained by performing Dehn filling the end $E$ along the geodesic $\pi(\sigma) \subset F$, where $\pi : T^{n-1} \to F$ is the covering projection.

The following result gives a necessary and sufficient condition for the existence of such Dehn fillings of an end $E$.

**Lemma 3.10.** *For $F$ and $\sigma$ as above, the quotient $M_\sigma$ is well-defined, and carries a corresponding quotient of the AdS black hole metric $g_{BH}$ in (3.6) if and only if, for any $\gamma \in \Gamma$ acting on the universal cover $\mathbb{R}^{n-1}$, one has*

$$\langle\gamma(\sigma)\rangle \parallel \langle\sigma\rangle, \tag{3.49}$$

*where $\langle\tau\rangle$ is the line through $\tau$.*

**Proof:** In the process of Dehn filling a toral end, the initial flat structure on $T^{n-1}$ is deformed along a curve of flat structures, by smoothly changing the length of the meridian curve $\sigma$ from its initial length to length 0. This is described explicitly in (3.7). Thus, one has to check if the deformation (3.7) is invariant under a corresponding deformation of the action of $\Gamma$.

As discussed following (3.5), let $(\sigma, b_2, ..., b_{n-1})$ be a basis for the lattice giving $T^{n-1}$, and set $\sigma = b_1$. Let $b_i^r = b_i + (\lambda(r) - 1)(\langle b_i, \sigma\rangle/|\sigma|^2)\sigma$ be as in (3.7), and let $t^r_{b_i}$ denote the generators for the lattice $(\mathbb{Z}^{n-1})(r)$ defining $T^{n-1}(r)$; thus $t^r_{b_i}$ is translation by the vector $b_i(r)$ on $\mathbb{R}^{n-1}$.

By the Bieberbach theorem (3.48), the group $\pi_1(F)$ is a semi-direct product of $\mathbb{Z}^{n-1}$ with $\Gamma$. The group $\Gamma$ acts by affine transformations on $\mathbb{R}^{n-1}$; each $\gamma \in \Gamma$ acts by $(A_\gamma, t_\gamma)$, where $A_\gamma \in O(n-1)$ and $t_\gamma$ is a translation on $\mathbb{R}^{n-1}$ by the vector $t_\gamma$. Thus $\gamma(v) = A_\gamma(v) + t_\gamma$ and

$$(A_{\gamma_1}, t_{\gamma_1})(A_{\gamma_2}, t_{\gamma_2})(v) = A_{\gamma_1}A_{\gamma_2}(v) + A_{\gamma_1}(t_{\gamma_2}) + t_{\gamma_1} = (A_{\gamma_1}A_{\gamma_2}, t_{A_{\gamma_1}(t_{\gamma_2})+t_{\gamma_1}})(v). \tag{3.50}$$

Define then a deformation of the action of $\Gamma$ by setting

$$A^r_\gamma = A_\gamma, \text{ and } t^r_\gamma = t_\gamma + (\lambda(r) - 1)\frac{\langle t_\gamma, \sigma\rangle}{|\sigma|^2}\sigma = t^\perp_\gamma + \lambda(r)\frac{\langle t_\gamma, \sigma\rangle}{|\sigma|^2}\sigma, \tag{3.51}$$

where $t^\perp_\gamma$ is the component of $t_\gamma$ orthogonal to $\langle\sigma\rangle$. Thus, the orthogonal part $A_\gamma$ of $\gamma$ remains unchanged, while the translation part $t^r_\gamma$ varies along $\sigma$, and is orthogonal to $\sigma$ at $r = r_+$, where $\lambda(r_+) = 0$. Observe that the deformation $t^r_\gamma$ has exactly the same form as $t^r_{b_i}$.

To verify that this gives a well-defined action of $\pi_1(F)$ on $\mathbb{R}^{n-1}$ one needs to check that the relations of $\pi_1(F)$ are preserved. This is clear for the orthogonal (or $A$) part of the action by (3.50)-(3.51), and so one only needs to consider the translation or vector part of the action.

Each relation $R$ is a word in some generators $A_\gamma$, $t_\gamma$, $t_{b_i}$. Thus, as a vector, $R(A_\gamma, t_\gamma, t_{b_i}) = 0$, where each $t$ acts by translation, (i.e. addition), and each $A_\gamma$ acts by an orthogonal matrix on some $t$ vector. To verify that $R^r = R(A_\gamma, t^r_\gamma, t^r_{b_i}) = 0$, suppose first that $R$ involves no rotational part, i.e. $R = R(t_\gamma, t_{b_i}) = 0$. The components of $R$ parallel and orthogonal to $\sigma$ then also both vanish. Since the deformations $t^r_\gamma$ and $t^r_{b_i}$ have exactly the same form along these components, and orthogonal projection commutes with translation, it follows that $R^r = R(t^r_\gamma, t^r_{b_i}) = 0$.

Next, consider the action of any $A = A_\gamma$ on some translation $t = t_\gamma$ or $t_{b_i}$. The condition (3.49) implies that $A$ leaves the subspaces $\langle\sigma\rangle$ and $\langle\sigma\rangle^\perp$ invariant, i.e. $A(\sigma) = \pm\sigma$. As above, the components of the vector $R = R(A_\gamma, t_\gamma, t_{b_i})$ along $\langle\sigma\rangle$ and $\langle\sigma\rangle^\perp$ vanish. Since any $A$ commutes with translation by $\sigma$, it follows that $R_\sigma = R((A_\gamma)_\sigma, (t_\gamma)_\sigma, (t_{b_i})_\sigma) = 0$, where $t_\sigma$ is the $\sigma$ component of $t$ and $A_\sigma = A|_{\langle\sigma\rangle}$. The same statement holds with respect to $\langle\sigma\rangle^\perp$. Since, as above, the vectors $t^r_\gamma$ and $t^r_{b_i}$ have the same form, it follows that the $\sigma$ and $\sigma^\perp$ components of $R^r$ also vanish, as required. This shows that the condition (3.49) is a sufficient condition that $M_\sigma$ is well-defined.

Observe that the action of $\Gamma$ is well-defined at the core $(n-2)$-torus $T^{n-2} = \{r = r_+\}$ where $\lambda(r_+) = 0$, and so

$$\langle\gamma(b_i(r_+)), \sigma\rangle = 0. \tag{3.52}$$

Conversely, the condition (3.52) is necessary for the Dehn filling $M_\sigma$ to be well-defined. Since $\Gamma$ acts by isometries, $\langle\gamma(b_i(r_+)), \sigma\rangle = \langle b_i(r_+), \gamma^{-1}\sigma\rangle$. However, by construction, i.e. (3.7), we know that $\langle b_i(r_+), \sigma\rangle = 0$, $\forall i > 1$. Hence, (3.52) requires the condition (3.49), so that (3.49) is also necessary. ■

Define the Dehn filling along $\sigma$ to be *admissible* if $\Gamma$ and $\sigma$ satisfy the condition (3.49). This leads to the following extension of Theorem 1.1.

**Corollary 3.11.** *Let $M_{\bar\sigma}$ be any manifold obtained by performing a sufficiently large, admissible Dehn filling of the ends $E_j$, $1 \leq j \leq q$, of a complete hyperbolic $(N, g_{-1})$. Then $M_{\bar\sigma}$ admits an Einstein metric $g$ satisfying (1.1).*

**Proof:** Using Lemma 3.10, one constructs the approximate Einstein metric $\widetilde{g}$ exactly as in Proposition 3.1. The rest of the proof proceeds exactly as in the proof of Theorem 1.1. ■

For a given end $E = F \times \mathbb{R}^+$ with $F = T^{n-1}/\Gamma$, not all Dehn fillings will be admissible, unless $E$ is toral. Nevertheless, for many such $F$, there will be an infinite number of admissible fillings; this can be checked by inspection.

## 4. Further Results and Remarks

In this section, we collect a number of remarks on the geometry and topology of the Einstein metrics $(M_{\bar{\sigma}}, g)$ constructed in Theorem 1.1 or Corollary 3.11, and prove the remaining results stated in the Introduction; Theorem 1.2 is proved in §4.1.

**§4.1.** By the Chern-Gauss-Bonnet theorem [15], if $N$ is a complete hyperbolic $n$-manifold of finite volume, then

$$volN = (-4\pi)^m \frac{m!}{(2m)!}\chi(N), \tag{4.1}$$

where $n = 2m$ and $\chi(N)$ is the Euler characteristic of $N$. In particular, the sign of the Euler characteristic is $(-1)^m$. Since the Dehn-filled manifold $M = M_{\bar{\sigma}}$ decomposes as a union of $N$ and a collection of solid tori $D^2 \times T^{n-1}$, an elementary Mayer-Vietoris argument shows that

$$\chi(N) = \chi(M).$$

Since $\chi(N)$ can be arbitrarily large for hyperbolic manifolds, (by passing to covering spaces), $\chi(M)$ can thus be made arbitrarily large when $n$ is even.

Next we verify the claims (1.4) and (1.5). Regarding (1.4), the curvature of the black hole metric is given by (2.8), while that of the approximate Einstein metric $\widetilde{g}$ is as stated in Proposition 3.1. The Einstein metric $g$ on $M$ is close to $\widetilde{g}$ in the $\widetilde{C}^{m,\alpha}$ topology, for any $m$. Hence, the curvature of $g$ is uniformly close to that of $\widetilde{g}$. This gives the estimate (1.4).

Regarding the Weyl curvature estimate (1.5), $W = 0$ on any hyperbolic manifold. For the black hole metric, as noted following (2.8), $W$ decays as $|W| = O(e^{-(n-1)s})$, where $s$ is the distance to the core $T^{n-2}$. On the other hand, the volume of the region $D(s)$ with respect to the approximate solution $\widetilde{g}$ is on the order of $O(e^{(n-1)(s-\ln R)})$, where $R$ is given by (3.4). It follows that the volume of the region where $|W| \geq \delta$ is on the order of $R^{-(n-1)}\delta^{-1}$. This verifies (1.5) for the approximate solution $\widetilde{g}$. Again, since the Einstein metric $g$ is uniformly close to $\widetilde{g}$, (1.5) follows for $g$. On the other hand, there is a fixed constant $c_0 > 0$, depending only on dimension, such that

$$|W|_{L^\infty} \geq c_0, \tag{4.2}$$

since this is the case for the black hole metric $g_{BH}$ near the core torus $T^{n-2}$. Of course (4.2) assumes $n \geq 4$.

An immediate consequence of (1.5) and the Chern-Weil theory is that all Pontryagin numbers of $M$ vanish when $M$ is compact. In particular, by the Hirzebruch signature theorem, the signature $\tau(M) = 0$.

**Remark 4.1.** In a natural sense, most of the Einstein manifolds constructed are not locally isometric. (All hyperbolic manifolds are of course locally isometric). Let $N$ be a complete, noncompact hyperbolic manifold of finite volume, and let $\bar{N}$ be a covering of $N$ of degree $k$. If $M_{\bar{\sigma}}$ is obtained from $N$ by Dehn filling, then $M_{\bar{\sigma}}$ admits a degree $k$ covering $\bar{M}_{\bar{\sigma}}$, such that $\bar{M}_{\bar{\sigma}}$ is obtained from $\bar{N}$ by Dehn filling on cusps of $\bar{N}$; these Dehn fillings are lifts of the Dehn fillings on $M_{\bar{\sigma}}$. However, $\bar{N}$ admits many new Dehn fillings which are not lifts of Dehn fillings on $N$. Hence, "most all" of the Einstein metrics associated with $\bar{N}$ are not lifts of Einstein metrics associated to $N$.

**Remark 4.2.** Let $N$ be as above, and suppose $\pi_1(N)$ admits a homomorphism onto a free group $F_2$ with two generators. The lower bound in (1.3) is achieved by taking coverings of hyperbolic manifolds which admit such a surjection onto $F_2$, cf. [13], [24]. Let $C(N)$ denote the number of cusps of $N$. We claim that many coverings $\bar{N}^k$ of $N$ of degree $k$ have $C(\bar{N}^k)$ growing linearly with $k$, i.e. linearly in the volume. More precisely, there exist constants, $c, d > 0$, depending only on dimension $n$, such that

$$C(\bar{N}^k) \geq d \cdot k, \tag{4.3}$$

for a collection of isometrically distinct coverings $\bar{N}^k$ of cardinality at least $e^{ck \ln k}$. To see this, let $\phi : \pi_1(N) \to F_2$ be the surjective homomorphism onto $F_2$. Any subgroup $H$ of index $k$ in $F_2$ determines a covering space $\bar{N}^k$, with $\pi_1(\bar{N}^k) = (\phi)^{-1}(H)$. Since $F_2$ is free, $\phi$ sends any $\pi_1(T_j^{n-1}) \simeq \mathbb{Z}^{n-1}$ to $\langle a_j \rangle$, for some fixed $a_j \in F_2$. If $a_j \in H$, then the covering $\bar{N}^k$ unwraps $T_j^{n-1}$ into $k$ disjoint copies of $T^{n-1}$, giving rise to $k$ cusp ends, and thus giving (4.3). Hence, one needs to count the number of distinct index $k$ subgroups of $F_2$ containing a given element $a$. Following [20], there are at least $k \cdot k!$ subgroups of $F_2$ of index $k$, and at least $k!$ of these contain a given element $a \in F_2$. Following [13], this gives the lower bound on $c$ above for the number of non-isometric coverings.

The opposite bound to (4.3),

$$C(\bar{N}^k) \leq D \cdot k,$$

for some fixed constant $D = D(n)$, is an immediate consequence of the Margulis Lemma.

Next we prove the following expanded version of Theorem 1.2. Let $\mathcal{E}$ denote the class of Einstein metrics constructed via Theorem 1.1 or Corollary 3.11, together with the class of complete, non-compact hyperbolic $n$-manifolds $(N, g_{-1})$ of finite volume.

**Theorem 4.3.** *The space $\mathcal{E}$ is closed with respect to the pointed Gromov-Hausdorff topology or the $C^\infty$ topology. Further the volume functional*

$$vol : \mathcal{E} \to \mathbb{R}^+ \tag{4.4}$$

*is continuous and proper with respect to these topologies. Any limit point $(M, g)$ of a sequence $(M^i, g^i) \in \mathcal{E}$ is obtained by opening a finite number of cusps of $M_i$.*

**Proof:** Let $(M^i, g^i)$ be any sequence in $\mathcal{E}$ of bounded volume. By passing to a subsequence, we may assume that $M^i = M_{\bar{\sigma}^i}$, where $M_{\bar{\sigma}^i}$ is obtained from a fixed complete hyperbolic manifold $N$ by Dehn filling a collection of cusp ends. The sequence $\bar{\sigma}_j$ (partially) diverges to infinity in the Dehn filling space; thus for one and possibly several fixed $j$, $L(\sigma_j^i) \to \infty$ as $i \to \infty$, where $\sigma_j^i$ is a sequence of simple closed geodesics in tori $T_j^{n-1}$ in the $j^{\text{th}}$ cusp end of $N$. By passing to a further subsequence, we may then assume that $M^i$ is obtained by Dehn filling of $a + b$ fixed cusps of $N$, and that $L(\sigma_j^i) \to \infty$ for $1 \leq j \leq a$, while $L(\sigma_j^i)$ remains bounded, for $a + 1 \leq j \leq a + b$. Here $a + b \leq q$, where $q$ is the number of cusps of $N$.

By construction, each Einstein metric $g^i \in B_{\widetilde{g}_i}(\varepsilon_0)$, where $B(\varepsilon_0)$ is the $\varepsilon_0$-ball in the $\widetilde{C}^{m,\alpha}$ topology and $\widetilde{g}_i$ is the approximate metric constructed on $M^i$; see the proof of Theorem 1.1. Further, as in the proof of Theorem 1.1 in the non-compact case, the sequence of metrics $\widetilde{g}_i$ converges, in a subsequence, to a limit metric $\widetilde{g}_\infty$ on a manifold $M_\infty = M_{\bar{\sigma}_\infty}$, where $\bar{\sigma}_\infty = (\infty, ..., \infty, \sigma_{a+1}, ..., \sigma_{a+b})$. Thus, $M_\infty$ is obtained from $M^i$ by opening $a$ cusps. The metric $\widetilde{g}_\infty$ is thus obtained from $N$ by Dehn filling $b$ cusps of $N$, along the curves $\sigma_{a+1}, ..., \sigma_{a+b}$.

Theorem 1.1, (or Corollary 3.11), thus gives the existence of an Einstein metric $g_\infty$ on $M_\infty$, in $B_{\widetilde{g}_\infty}(\varepsilon_0)$. This proves that $\mathcal{E}$ is closed in the pointed $\widetilde{C}^{m,\alpha}$ topology and in fact $\mathcal{E}_V = \{g \in \mathcal{E} : vol_g M \leq V\}$ is compact. The convergence in the $C^\infty$ topology then follows from well-known elliptic regularity associated to the Einstein equation. The $C^\infty$ topology is much stronger than the Gromov-Hausdorff topology, hence $\mathcal{E}$ is also closed in the Gromov-Hausdorff topology.

To see that the volume functional (4.4) is continuous, the sequence $(M^i, g^i)$ or $(M^i, \widetilde{g}^i)$ converges smoothly to its limit, uniformly on compact subsets. Hence, for any compact domain $D \subset M_\infty$,

$vol_{g^i}D \to vol_{g_\infty}D$. Further, if $D$ contains a sufficiently large region of $N$, the volume of the complement is uniformly small, for all $i$; this follows since the volume of the approximate metrics $\widetilde{g}$ at geodesic distance $t$ from the glueing tori is uniformly exponentially small. This proves the continuity of $vol$ on $\mathcal{E}$. The properness of $vol$ follows from the argument above: any sequence in $\mathcal{E}$ of bounded volume has a convergent subsequence in $\mathcal{E}$. Similarly, the fact that limits are obtained by opening cusps has already been proved above. ∎

**§4.2.** In this section, we discuss further aspects of the volume and convergence behavior of the Einstein metrics constructed above in dimension 4. To begin, the Chern-Gauss-Bonnet theorem in dimension 4 states

$$\chi(M) = \frac{1}{8\pi^2}\int_M (|W|^2 - \tfrac{1}{2}|z|^2 + \tfrac{1}{24}s^2)dV, \tag{4.5}$$

where $z = Ric - \frac{s}{4}g$ is the trace-free Ricci curvature. The formula (4.5) holds for all compact manifolds $M$. It also holds for complete non-compact hyperbolic manifolds of finite volume. This follows by using the Chern-Gauss-Bonnet formula for manifolds with boundary [15]; it is easily seen that the boundary contribution decays to 0 as the boundary is taken to infinity.

For an Einstein metric $g$ as in (1.1), $z = 0$, and thus (4.5) gives (1.7), via the normalization (1.1). Further, none of the Einstein metrics constructed above is conformally flat, i.e. the Weyl tensor $W$ does not vanish identically. This is because a conformally flat Einstein metric is of constant curvature; however, none of the manifolds $M_{\bar\sigma}$ admit a negatively curved metric, as noted following (2.5). It follows that for any Dehn filling,

$$volM_{\bar\sigma} < volN,$$

see (1.8). Thus, all Einstein manifolds $(M, g)$ obtained by performing Dehn filling on the ends of a complete hyperbolic 4-manifold $(N, g_{-1})$ have volume less than the volume of $(N, g_{-1})$.

If $N$ is a complete non-compact hyperbolic 4-manifold of finite volume, then (4.1) gives

$$volN = \frac{4\pi^2}{3}k, \tag{4.6}$$

where $k = \chi(N) \in \mathbb{Z}^+$. Thus the volume spectrum of hyperbolic 4-manifolds is contained in the set $(4\pi^2/3)\mathbb{Z}^+$.

Currently, one does not have a complete classification of the hyperbolic 4-manifolds of minimal volume $4\pi^2/3$, i.e. of Euler characteristic 1. However, in [29], an explicit description of 1171 complete non-compact hyperbolic 4-manifolds is given, all of minimal volume $4\pi^2/3$. To be concrete, we base the discussion to follow on this collection of hyperbolic 4-manifolds, although it is easily seen to apply to any initially given hyperbolic 4-manifold.

Let $N_a$, $1 \leq a \leq 1171$ denote the list of complete, non-compact hyperbolic 4-manifolds in [29]; of these, 22 are orientable, while the rest are non-orientable. Most of the manifolds $N_a$ have non-zero first Betti number. Hence, for any $k \in \mathbb{Z}^+$, there are coverings of such manifolds of degree $k$, and thus of Euler characteristic $k$ and volume $4\pi^2k/3$. It follows that the volume spectrum of hyperbolic 4-manifolds is precisely the positive integral multiples of $4\pi^2/3$. Again the number of such distinct manifolds of volume $4\pi^2k/3$ grows super-exponentially in $k$, as in (1.3).

All of the manifolds $N_a$ above have either 5 or 6 cusp ends. However, no $N_a$ has all ends given by 3-tori $T^3$, (although many such $N_a$ have double covers with all ends toral). Thus, one needs to use Corollary 3.11 to perform Dehn filling on a non-toral end. For this, one needs to understand the structure of compact flat 3-manifolds.

The classification of compact flat 3-manifolds, cf. [21] or [34] shows that there are exactly 10 topological types, 6 orientable and 4 non-orientable. The 6 orientable manifolds are labelled $A$-$F$

in [21] and [29], corresponding to $G_1$-$G_6$ in [34], while the remaining 4 non-orientable manifolds are labelled $G$-$J$ in [21], [29] corresponding to $B_1$-$B_4$ in [34]. The 3-torus $T^3$ corresponds to $A = G_1$. Further, the moduli of flat structures on such manifolds is completely classified, cf. [34].

Using the criterion (3.49), a straightforward inspection in [34] shows that, among the 10 flat manifolds, only the manifolds $A, B, G, H$, (corresponding to $G_1, G_2, B_1, B_2$), admit an infinite sequence of admissible Dehn fillings. In the notation of [34], $\sigma$ may be any primitive (integer coefficient) vector in the plane $\langle a_2, a_3\rangle$ in the case of $G_2$, while it may be any such vector in the plane $\langle a_1, a_2\rangle$ in the case of $B_1$ or $B_2$.

Thus, by Corollary 3.11, infinite sequences of Dehn fillings may be applied to any of the cusp ends of the form $A$, $B$, $G$ or $H$, of any of the manifolds $N_a$, to give complete finite volume Einstein metrics. For concreteness, let us illustrate the volume and convergence behavior on a specific seed manifold.

Take for instance $N_{23}$ from [29]. This manifold has five cusp ends, of the type AAGGH, i.e. two of the cusp ends are 3-tori, two are of type $G$ and one is of type $H$. The first Betti number of $N_{23}$ is given by $b_1(N_{23}) = 4$.

There are now a number of ways to close off the cusps by Dehn filling.

(1). Close off any one cusp end of $N_{23}$. This gives an infinite sequence of complete Einstein manifolds $(M_i^1, g_i^1)$, with 4 cusp ends, converging to $N_{23}$ in the pointed Gromov-Hausdorff topology. Formally, to each toral end there are $3 \cdot \infty$ admissible Dehn fillings, while to each end of type $G$ or $H$, there are $1 \cdot \infty$ admissible Dehn fillings. With respect to a suitable labeling, the volume of $(M_i^1, g_i^1)$ increases to $volN_{23} = 4\pi^2/3$.

(2). Next, close off any two cusp ends of $N_{23}$, giving a (bi)-infinite sequence of complete Einstein manifolds $(M_i^2, g_i^2)$, with 3 cusp ends. If one chooses a subsequence $(M_{i'}^2, g_{i'}^2)$ of $(M_i^2, g_i^2)$ for which both Dehn fillings tend to infinity, then $(M_{i'}^2, g_{i'}^2)$ converges to $N_{23}$ again and the volumes of $(M_{i'}^2, g_{i'}^2)$ increase to $volN_{23}$, (w.r.t. a suitable labeling).

However, if the Dehn filling is fixed on one end, and taken to infinity on the other, then the corresponding subsequence of $(M_i^2, g_i^2)$ converges to a complete finite volume Einstein metric $(M_\infty^1, g_\infty^1)$ on a manifold with 4 cusp ends. By Theorem 4.3, $(M_\infty^1, g_\infty^1)$ is one of the manifolds constructed in (1) above. While the volumes of the subsequence of $(M_i^2, g_i^2)$ converge to the volume of the limit $(M_\infty^1, g_\infty^1)$, it is not known if this convergence can be made monotone, unless the limit $(M_\infty^1, g_\infty^1)$ is hyperbolic, (see below). Since there are infinitely many possibilities for the limit $(M_\infty^1, g_\infty^1)$, most of these limits cannot be hyperbolic.

(3). Next, close off any three cusp ends of $N_{23}$, giving a (tri)-infinite sequence of complete Einstein manifolds $(M_i^3, g_i^3)$, with 2 cusp ends. Limits of sequences in this family are then complete, finite volume Einstein manifolds with 3, 4 or 5 ends, of the type in (2), (1) or $N_{23}$ respectively.

(4). Close off any four cusp ends of $N_{23}$, giving a family $(M_i^4, g_i^4)$ with the same features as before.

(5). Finally, one may close off all 5 cusp ends of $N_{23}$ at once, giving a (5-fold)-infinite sequence of compact Einstein manifolds $(M_i^5, g_i^5)$. By taking various subsequences, one obtains limits of the form in (1)-(4) above, or again $N_{23}$.

Thus, one sees that there is a large number of sequences of Einstein manifolds, compact or non-compact, converging to the initial seed manifold $N_{23}$, as well as many other sequences converging to other Einstein limits. The same structure of convergence holds with respect to any initial hyperbolic seed manifold $N^k$.

The discussion above proves the following:

**Proposition 4.4.** *Let $N^k$ be a complete non-compact hyperbolic 4-manifold, with volume $V^k = (4\pi^2/3)k$, and with $q$ cusps, each of type $A$, $B$, $G$, or $H$. Then $N^k$ is a $q$-fold limit point of elements of $\mathcal{E}$, while $V^k$ is a $q$-fold limit point of elements in $\mathcal{V} = vol\mathcal{E} \subset \mathbb{R}^+$.*

■

An obvious modification of Proposition 4.4 holds when some ends of $N^k$ are not of the type $A$, $B$, $G$, or $H$.

Unlike the situation with the Thurston theory in 3 dimensions, it is not clear that the volume spectrum $\mathcal{V}$ is well-ordered, (as a subset of $\mathbb{R}^+$), or finite to one. For the approximate metrics $\widetilde{g}$, although a Dehn-filled end has less volume than the corresponding hyperbolic cusp with the same boundary, the difference is on the order of $O(R^{-(n-1)})$, which is of the same order as the deviation of the Einstein metric $g$ from $\widetilde{g}$, cf. (3.11). Hence, more refined estimates are needed to see if the volume is essentially monotone on sequences which open a cusp.

**§4.3.** Similar results regarding the volume behavior hold at least in all even dimensions $n = 2m$. Thus, the Chern-Gauss-Bonnet formula in this case states that

$$\chi(M) = \frac{(-1)^m}{4^m \pi^m m!} \int_M \sum \varepsilon_{i_1 \dots i_n} R_{i_1 i_2} \wedge \dots \wedge R_{i_{n-1} i_n}, \tag{4.7}$$

where the sum is over all permutations of $(1, ..., n)$ and $R$ denotes the curvature tensor. This formula holds for all compact manifolds, and non-compact manifolds of finite volume of the type considered here. For Einstein metrics of the form (1.1), the trace-free part of the Ricci curvature vanishes, and $R$ may be written as

$$R_{i_a i_b} = \theta_{i_a} \wedge \theta_{i_b} + W_{i_a i_b}, \tag{4.8}$$

where $W$ is the Weyl tensor and $\{\theta_i\}$ run over an orthonormal basis. (Here the sign convention is such that $\langle R_{i_a i_b}, \theta_{i_b} \wedge \theta_{i_a} \rangle$ gives the sectional curvature $K_{i_a i_b}$). Substituting (4.8) in (4.7) gives

$$\chi(M) = \frac{(-1)^m 2m!}{4^m \pi^m m!} volM + \int_M P^m(W), \tag{4.9}$$

where $P^m(W)$ is a polynomial of order $m$ in the Weyl tensor $W$. By the same arguments as in §4.2, the term $P^m(W)$ is small, by construction, and becomes arbitrarily small whenever all Dehn fillings are sufficiently large. In particular, as the Dehn fillings of each end are taken to infinity, one has

$$volM_{\bar{\sigma}} \to volN = (-4\pi)^m \frac{m!}{2m!} \chi(N). \tag{4.10}$$

However, in contrast to the situation in 4-dimensions (1.8), it is not known if the term $P^m(W)$ has a sign. Hence, it is not known if the convergence (4.10) is monotone increasing or decreasing.

The analogue of Proposition 4.4 regarding the structure of $\mathcal{E}$ holds in all dimensions, while the analogue regarding the structure of $\mathcal{V}$ holds at least in all even dimensions.

**Remark 4.5.** An analogue of Theorem 1.1 also holds for complete, conformally compact hyperbolic manifolds $(N, g_{-1})$ with a finite number of cusp ends, cf. [16]. Such manifolds are of infinite volume, with a finite number of expanding ends in addition to the cusp ends. Each expanding end may be conformally compactified by a smooth defining function $\rho$ as in (2.9). The conformal infinity is then a compact manifold $\partial N$, possibly disconnected, with a conformally flat metric $g_\infty$. In the terminology of Kleinian groups, such manifolds are geometrically finite hyperbolic manifolds with a finite number of parabolics.

Theorem 1.1 generalizes to this context to give the following: a sufficiently large Dehn filling of the cusp ends of $(N, g_{-1})$ carries a conformally compact Einstein metric $(M, g)$, with the same conformal infinity as $(N, g_{-1})$. Consequently, for any such $N$, there exist infinitely many conformally compact Einstein manifolds $M = M_{\bar{\sigma}}$, of distinct topological type, which have the same conformal infinity $(\partial N, g_\infty)$. We refer to [16] for further details.

## Appendix A

In this Appendix, we describe the form of $T^{n-1}$-invariant infinitesimal Einstein deformations of the hyperbolic cusp metric; this is used to verify the statement (3.28) and computations in Lemma 3.4 in the proof of Proposition 3.2.

Recall that the hyperbolic cusp metric $(C, g_{-1})$ is given by

$$g_C = r^{-2}dr^2 + r^2 g_{T^{n-1}}. \tag{A.1}$$

An infinitesimal Einstein deformation of $g_{-1}$ is a symmetric bilinear form $h$ such that $h \in \mathrm{Ker}L$, i.e.

$$L(h) = D^*Dh - 2R(h) = 0. \tag{A.2}$$

By Lemma 2.1, we need only consider $h$ such that $trh = 0$. Hence, from (2.13) one has

$$R(h) = h. \tag{A.3}$$

Since $h$ is $T^{n-1}$ invariant, $h$ has the form

$$h = \sum h_{ij}(r)\theta_i \cdot \theta_j, \tag{A.4}$$

where $\theta_i$ is a local orthonormal coframing, dual to $e_i$, defined as follows: $e_1 = \nabla s$, where $ds = r^{-1}dr$, so the integral curves of $\nabla s$ are geodesics, while $e_i$, $i \geq 2$ are tangent to $T^{n-1}$ . If one writes $r^2 g_{T^{n-1}} = r^2(d\phi_2^2 + ... + d\phi_n^2)$, then $e_i = r^{-1}\partial/\partial\phi_i$ and so $\theta_i = rd\phi_i$.

Now we compute $D^*Dh = -\nabla_{e_i}\nabla_{e_i}h + \nabla_{\nabla_{e_i}e_i}h$. From (A.4), one has

$$-\nabla_{e_i}\nabla_{e_i}h = -\nabla_{e_i}\nabla_{e_i}(h_{ab}\theta_a \cdot \theta_b) = -e_ie_i(h_{ab})\theta_a \cdot \theta_b - 2e_i(h_{ab})\nabla_{e_i}(\theta_a \cdot \theta_b) - h_{ab}\nabla_{e_i}\nabla_{e_i}(\theta_a \cdot \theta_b),$$

while

$$\nabla_{\nabla_{e_i}e_i}h = \nabla_{\nabla_{e_i}e_i}(h_{ab}\theta_a \cdot \theta_b) = (\nabla_{e_i}e_i)(h_{ab}) \cdot (\theta_a \cdot \theta_b) + h_{ab}\nabla_{\nabla_{e_i}e_i}(\theta_a \cdot \theta_b).$$

By (A.3), one needs only to consider the $\theta_a \cdot \theta_b$ component of this. Clearly, by orthogonality of the basis

$$\langle \nabla_{e_i}(\theta_a \cdot \theta_b), \theta_a \cdot \theta_b \rangle = 0 \text{ and } \langle \nabla_{\nabla_{e_i}e_i}(\theta_a \cdot \theta_b), \theta_a \cdot \theta_b \rangle = 0.$$

Combining this with (A.3) and (A.2) then gives

$$-\Delta h_{ab} - h_{ab}\langle \nabla_{e_i}\nabla_{e_i}(\theta_a \cdot \theta_b), \theta_a \cdot \theta_b \rangle - 2h_{ab} = 0. \tag{A.5}$$

For $h = h_{ab}$, $h = h(r) = h(s)$, with $dr/ds = r$. Thus

$$\Delta h(s) = (dh/ds)\Delta s + (dh^2/ds^2).$$

But $dh/ds = h' \cdot (dr/ds) = h'r$, and $dh^2/ds^2 = h'r + h''r^2$, with $' = d/dr$. Also

$$\Delta s = \langle \nabla_{e_i}e_1, e_i \rangle = n - 1.$$

Thus,

$$\Delta h(s) = (n-1)rh' + (rh' + r^2h'') = r^2h'' + nrh'.$$

Next, one easily computes that:

$$\nabla_{e_1}\theta_a = 0, \text{ for any } a, \tag{A.6}$$

$$\nabla_{e_i}\theta_a = -\delta_{ia}\theta_1, \text{ for any } a, i > 1, \text{ while } \nabla_{e_i}\theta_1 = \theta_i,\ i > 1. \tag{A.7}$$

The latter equations come from fact that the tori are totally umbilic, with 2$^{\text{nd}}$ fundamental form $A = g$, while the intrinsic connection on tori is the flat connection, so tangential covariant derivatives vanish.

To compute $\langle \nabla_{e_i}\nabla_{e_i}(\theta_a \cdot \theta_b), \theta_a \cdot \theta_b \rangle$, one has $\nabla_{e_i}(\theta_a \cdot \theta_b) = (\nabla_{e_i}\theta_a) \cdot \theta_b + \theta_a \cdot \nabla_{e_i}\theta_b$, and so

$$\nabla_{e_i}\nabla_{e_i}\theta_a \cdot \theta_b = (\nabla_{e_i}\nabla_{e_i}\theta_a) \cdot \theta_b + 2\nabla_{e_i}\theta_a \cdot \nabla_{e_i}\theta_b + \theta_a \cdot \nabla_{e_i}\nabla_{e_i}\theta_b.$$

Suppose first $a > 1$. Then $\nabla_{e_i}\theta_a = -\delta_{ia}\theta_1$, so $\nabla_{e_i}\nabla_{e_i}\theta_a = -\delta_{ia}\nabla_{e_i}\theta_1 = -\delta_{ia}\theta_i = -\theta_a$, while $\nabla_{e_i}\theta_a = -\delta_{ia}\theta_1$. This then gives

$$\langle \nabla_{e_i}\nabla_{e_i}\theta_a \cdot \theta_b, \theta_a \cdot \theta_b \rangle = -2, \ a, b > 1.$$

Thus, the last two terms in (A.5) cancel and, for $h = h_{ab}$, $a, b > 1$, one is left with

$$\Delta h = 0, \text{ i.e. } r^2 h'' + nrh' = 0. \tag{A.8}$$

The general solution of (A.8) is

$$h = c_1 r^{-(n-1)} + c_2, \tag{A.9}$$

as in (3.34).

Next suppose $a = 1$, $b > 1$, and let $h = h_{1b}$. Then $\nabla_{e_i}\nabla_{e_i}\theta_1 = -(n-1)\theta_1$ and $\nabla_{e_i}\nabla_{e_i}\theta_b = -\theta_b$. Using (A.6)-(A.7) for the middle term in (A.5) then gives

$$\langle \nabla_{e_i}\nabla_{e_i}\theta_1 \cdot \theta_b, \theta_1 \cdot \theta_b \rangle = -(n+2).$$

This gives the Euler equation

$$r^2 h'' + nrh' - nh = 0, \tag{A.10}$$

which has the general solution

$$h = h_{1b} = c_1 r + c_2 r^{-n}, \tag{A.11}$$

for some constants $c_1$, $c_2$.

Performing similar calculations on $h = h_{11}$ gives the Euler equation

$$r^2 h + nrh' - 2(n-1)h = 0, \tag{A.12}$$

with general solution

$$h_{11} = c_1 r^{\alpha_+} + c_2 r^{\alpha_-}, \tag{A.13}$$

where $\alpha_{\pm} = \frac{1}{2}(-(n-1) \pm \sqrt{(n-1)^2 + 8(n-1)})$.

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