# EINSTEIN METRICS WITH PRESCRIBED CONFORMAL INFINITY ON 4-MANIFOLDS

MICHAEL T. ANDERSON

ABSTRACT. This paper considers the existence of conformally compact Einstein metrics on 4-manifolds. A reasonably complete understanding is obtained for the existence of such metrics with prescribed conformal infinity, when the conformal infinity is of positive scalar curvature. We find in particular that general solvability depends on the topology of the filling manifold. The obstruction to extending these results to arbitrary boundary values is also identified. While most of the paper concerns dimension 4, some general results on the structure of the space of such metrics hold in all dimensions.

## 1. INTRODUCTION.

This paper is concerned with the existence of conformally compact Einstein metrics on a given manifold $M$ with boundary $\partial M$. The main results are restricted to dimension 4, although some of the results hold in all dimensions.

This existence problem was raised by Fefferman and Graham in [19] in connection with a study of conformal invariants of Riemannian manifolds. More recently, the study of such metrics has become of strong interest through the AdS/CFT correspondence, relating gravitational theories on $M$ with conformal field theories on $\partial M$, cf. [18], [38] and references therein.

Let $M$ be a compact, oriented manifold with non-empty boundary $\partial M$; $M$ is assumed to be connected, but apriori $\partial M$ may be connected or disconnected. A defining function $\rho$ for $\partial M$ in $M$ is a non-negative $C^{\infty}$ function on the closure $\bar{M} = M \cup \partial M$ such that $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on $\partial M$. A complete Riemannian metric $g$ on $M$ is *conformally compact* if there is a defining function $\rho$ such that the conformally equivalent metric

$$\widetilde{g} = \rho^2 \cdot g \tag{1.1}$$

extends at least continuously to a Riemannian metric $\widetilde{g}$ on $\bar{M}$. The metric $\gamma = \widetilde{g}|_{\partial M}$ induced on $\partial M$ is the boundary metric induced by $g$ and the compactification $\rho$. Any compact manifold with boundary carries many conformally compact metrics; for instance, one may let $\widetilde{g}$ be any smooth metric on $\bar{M}$ and define $g$ by $g = \rho^{-2} \cdot \widetilde{g}$, for any choice of defining function $\rho$.

Defining functions for $\partial M$ are unique only up to multiplication by positive smooth functions on $\bar{M}$ and hence the compactification (1.1) is not uniquely determined by $(M, g)$. On the other hand, the conformal class $[\gamma]$ of a boundary metric is uniquely determined by $g$; the class $[\gamma]$ is called the *conformal infinity* of $(M, g)$.

A conformal compactification is called $C^{m,\alpha}$ or $L^{k,p}$ if the metric $\widetilde{g}$ extends to a $C^{m,\alpha}$ or $L^{k,p}$ metric on the closure $\bar{M}$; here $C^{m,\alpha}$ and $L^{k,p}$ are the usual Hölder and Sobolev function spaces.

In this paper, we consider complete conformally compact Einstein metrics $g$ on $(n+1)$-dimensional manifolds $M$, normalized so that

$$Ric_g = -ng. \tag{1.2}$$

Partially supported by NSF Grant DMS 0604735.

It is easy to see, cf. the Appendix, that any $C^2$ conformally compact Einstein metric $g$ satisfies $|K_g + 1| = O(\rho^2)$, where $K_g$ denotes the sectional curvature of $g$. Hence, such metrics are asymptotically hyperbolic (AH), in that the local geometry tends to that of hyperbolic space at infinity.

Let $E_{AH} = E_{AH}^{m,\alpha}$ be the space of AH Einstein metrics on $M$ which admit a $C^{m,\alpha}$ compactification $\widetilde{g}$ as in (1.1). We require that $m \geq 3$, $\alpha \in (0,1)$ but otherwise allow any value of $m$, including $m = \infty$ or $m = \omega$, (for real-analytic). The space $E_{AH}^{m,\alpha}$ is given the $C^{m,\alpha'}$ topology on metrics on $\bar{M}$, for any fixed $\alpha' < \alpha$, via a fixed compactification as in (1.1). Let $\mathcal{E}_{AH} = E_{AH}/\mathcal{D}_1(\bar{M})$, where $\mathcal{D}_1(\bar{M})$ is the group of orientation preserving $C^{m+1,\alpha}$ diffeomorphisms of $\bar{M}$ inducing the identity on $\partial M$, acting on $E_{AH}$ in the usual way by pullback.

As boundary data, let $Met(\partial M) = Met^{m,\alpha}(\partial M)$ be the space of $C^{m,\alpha}$ metrics on $\partial M$ and $\mathcal{C} = \mathcal{C}^{m,\alpha}(\partial M)$ the corresponding space of pointwise conformal classes, as above endowed with the $C^{m,\alpha'}$ topology. There is a natural boundary map, (for any fixed $(m,\alpha)$),

$$\Pi : \mathcal{E}_{AH} \to \mathcal{C} \qquad (1.3)$$
$$\Pi[g] = [\gamma],$$

which takes an AH Einstein metric $g$ on $M$ to its conformal infinity on $\partial M$.

It is proved in [7] that when $dim M = 4$, and $\pi_1(M, \partial M) = 0$, the space $\mathcal{E}_{AH}^{m,\alpha}$ is a $C^\infty$ infinite dimensional Banach manifold, and the boundary map $\Pi$ is $C^\infty$ of Fredholm index 0, provided $\mathcal{E}_{AH}^{m,\alpha}$ is non-empty. Moreover, the spaces $\mathcal{E}_{AH}^{m,\alpha}$ are essentially independent of $(m,\alpha)$, in that these spaces are all diffeomorphic, with $\mathcal{E}_{AH}^{m',\alpha'}$ dense in $\mathcal{E}_{AH}^{m,\alpha}$ whenever $m' + \alpha' > m + \alpha$. Essentially the same results also hold in in all dimensions $n + 1 \geq 4$, see §2 for a more detailed discussion.

The issues of existence and uniqueness of AH Einstein metrics with a given conformal infinity are thus equivalent to the surjectivity and injectivity of the boundary map $\Pi$.

The main result of the paper which is used to approach the existence question is the following. Let $\mathcal{C}^o$ be the space of non-negative conformal classes $[\gamma]$ on $\partial M$, in the sense that $[\gamma]$ has a *non-flat* representative $\gamma$ of non-negative scalar curvature. Let $\mathcal{E}_{AH}^o = \Pi^{-1}(\mathcal{C}^o)$ be the space of AH Einstein metrics on $M$ with conformal infinity in $\mathcal{C}^o$. Thus, one has the restricted boundary map $\Pi^o = \Pi|_{\mathcal{E}_{AH}^o} : \mathcal{E}_{AH}^o \to \mathcal{C}^o$.

**Theorem A.** *Let $M$ be a 4-manifold satisfying $\pi_1(M, \partial M) = 0$ and for which the inclusion $\iota : \partial M \to \bar{M}$ induces a surjection*

$$H_2(\partial M, \mathbb{F}) \to H_2(\bar{M}, \mathbb{F}) \to 0, \qquad (1.4)$$

*for all fields $\mathbb{F}$. Then for any $(m,\alpha)$, $m \geq 4$, the boundary map*

$$\Pi^o : \mathcal{E}_{AH}^o \to \mathcal{C}^o \qquad (1.5)$$

*is proper.*

Unfortunately (or fortunately), the boundary map $\Pi$ is not proper in general. In fact, a sequence of distinct AH Einstein metrics $\{g_i\}$ on $M = \mathbb{R}^2 \times T^2$ was constructed in [6], whose conformal infinity is an arbitrary fixed flat metric on $\partial M = T^3$, but which has no convergent subsequence to an AH Einstein metric on $M$. These metrics are twisted versions of the AdS $T^2$ black hole metrics, cf. Remark 2.6. The sequence $\{g_i\}$ converges, (in a natural sense), to a complete hyperbolic cusp metric

$$g_C = dr^2 + e^{2r} g_{T^3} \qquad (1.6)$$

on the manifold $N = \mathbb{R} \times T^3$. Since the boundary metric on $T^3$ is flat and $M$ satisfies (1.4), Theorem A is sharp, at least without further restrictions.

We will show in Theorem 5.4 that this behavior is the only way that $\Pi$ is non-proper, in that divergent sequences $\{g_i\}$ of AH Einstein metrics on a fixed 4-manifold $M$ and with a fixed conformal infinity not in $\mathcal{C}^o$, (or a convergent sequence of conformal infinities outside $\mathcal{C}^o$), necessarily converge to AH Einstein manifolds $(N, g)$ with cusps, although the cusps are not necessarily of the simple form (1.6). Thus, the completion $\bar{\mathcal{E}}_{AH}$ of $\mathcal{E}_{AH}$ in a natural topology consists of AH Einstein metrics on $M$, together with AH Einstein metrics with cusps, cf. Theorem 5.4 and Corollary 5.5 for further details. Using this, we show in Corollary 5.6 that the boundary map $\Pi$ is proper onto the subset $\hat{\mathcal{C}}$ of $\mathcal{C}$ consisting of conformal classes which are not the boundary metrics of AH Einstein metrics with cusps.

The infinite sequence of AH Einstein metrics $g_i$ on $\mathbb{R}^2 \times T^2$ above all lie in distinct components of the space $\mathcal{E}_{AH}$, so that this space has infinitely many components. On the other hand, the fact that $\Pi^o$ is proper of course implies that $\mathcal{E}_{AH}^o$ has only finitely many components mapping into any compact region of $\mathcal{C}^o$.

The topological condition (1.4) is used solely to rule out orbifold degenerations of AH Einstein sequences $\{g_i\}$. This condition may not be necessary, but it has been included in order to not overly complicate the paper; some situations where it is not needed are described in §7, cf. Remark 7.4. Similarly it may well be possible to weaken or drop the condition $\pi_1(M, \partial M) = 0$ on the fundamental group, but this will not be pursued further here.

Theorem A implies that at least on $\mathcal{E}_{AH}^o$, the map $\Pi^o$ is a proper Fredholm map of index 0. It follows from a result of Smale [34] that $\Pi^o$ has a well-defined mod 2 degree, $deg_2\Pi^o \in \mathbb{Z}_2$, on each component of $\mathcal{E}_{AH}^o$. In fact, building on the work of Tromba [35] and White [36], [37] we show in Theorem 6.1 that $\Pi^o$ has a $\mathbb{Z}$-valued degree

$$deg\Pi^o \in \mathbb{Z}, \tag{1.7}$$

again on components of $\mathcal{E}_{AH}^o$. Tautologically, if $deg\Pi^o \neq 0$, then $\Pi^o$ is surjective.

We compute $deg\Pi^o$ in certain cases via symmetry arguments. Namely, it is proved in [8], (cf. also Theorem 2.5 below) that any connected group of isometries of the conformal infinity $[\gamma]$ on $\partial M$ extends to a group of isometries of any AH Einstein filling metric $(M, g)$, with $\Pi[g] = [\gamma]$, in any dimension, again provided $\pi_1(M, \partial M) = 0$. Using this, and a straightforward classification of Einstein metrics with large symmetry groups leads to the following result.

**Theorem B.** *Let $M = B^4$ be the 4-ball, with $\partial M = S^3$, and, (abusing notation slightly), let $\mathcal{C}^o$ be the component of the non-negative $C^{m,\alpha}$ conformal classes containing the round metric on $S^3$. Also let $\mathcal{E}_{AH}^o$ be the component of $\Pi^{-1}(\mathcal{C}^o)$ containing the Poincaré metric on $B^4$. Then*

$$deg_{B^4}\Pi^o = 1. \tag{1.8}$$

*In particular, for any $(m, \alpha)$, $m \geq 4$, any conformal class $[\gamma] \in \mathcal{C}^o$ on $S^3$ is the conformal infinity of an AH Einstein metric on $B^4$.*

Explicit examples of AH Einstein metrics on $B^4$, whose the conformal infinity is an arbitrary Berger sphere, were constructed by Pedersen [32]. More generally, Hitchin [26] has constructed such metrics with conformal infinity an arbitrary left-invariant metric on $S^3$.

It remains an open question whether the boundary map $\Pi$ is surjective onto all of $\mathcal{C}$ when $M = B^4$. Based on reasoning from the AdS/CFT correspondence, Witten [38] has remarked that this may not be the case. It appears that physically, the natural class of boundary metrics are those of positive scalar curvature; in this regard, see also the work of Witten-Yau [39] proving that $\partial M$ is necessarily connected when the conformal infinity has a component of positive scalar curvature. In any case, as remarked above in connection with Theorem 5.4, the only obstruction to surjectivity is the possible existence of AH Einstein cusp metrics associated to $B^4$, cf. also Remark 6.8.

Theorem B also holds when $M$ is a disc bundle over $S^2$, so that $\partial M = S^3/\mathbb{Z}_k$, provided $k \geq 10$, with again $\mathcal{C}^o$ the component containing the standard round metric on $S^3/\mathbb{Z}_k$, cf. Proposition 7.6.

We conjecture this result holds for any $k \geq 3$. However, for $M = S^2 \times \mathbb{R}^2$, (i.e. $k = 0$), it is shown in Proposition 7.2, that

$$deg_{\mathbb{R}^2 \times S^2} \Pi^o = 0, \tag{1.9}$$

and further that $\Pi^o$ is not surjective onto $\mathcal{C}^o$. In fact, the conformal class of $S^2(1) \times S^1(L)$, for any $L > 2\pi/\sqrt{3}$ is not the boundary metric of any AH Einstein metric on $S^2 \times \mathbb{R}^2$. This result is based on the work of Hawking-Page [23] on the AdS-Schwarzschild metric. Similarly, for $M$ the degree 1 disc bundle over $S^2$, i.e. $M = \mathbb{CP}^2 \setminus B^4$, as well as the degree 2 disc bundle, $\Pi^o$ is not surjective. These results for $\Pi^o$ on disc bundles use properties of the AdS-Taub Bolt metrics analysed in [24],[31]. These results and others on deg $\Pi^o$ are given in §7.

Theorem B and the related results above represent the first *global* results on an existence theory for complete Einstein metrics, outside the context of Kähler-Einstein metrics.

I would like to thank Piotr Chruściel, Robin Graham, Rafe Mazzeo, Michael Singer for their interest and comments on this work. Thanks also to the referees for a careful examination of the paper leading to improvements in the exposition.

## 2. Background Results

In this section, we discuss a number of background results from [6]-[8], [11] [19], [21] needed for the work to follow.

We begin with the structure of the moduli space $\mathcal{E}_{AH}^{m,\alpha}$ of $C^{m,\alpha}$ conformally compact Einstein metrics on a given $(n+1)$-dimensional manifold $M$. The results discussed below are from [7].

Let $E_{AH}^{m,\alpha}$ be the space of AH Einstein metrics on $M$ which have a $C^{m,\alpha}$ conformal compactification with respect to a smooth (or $C^\omega$) defining function $\rho$, as in (1.1). The space $E_{AH}^{m,\alpha}$ is given the $C^{m,\alpha'}$ topology on the space of $C^{m,\alpha}$ metrics $Met^{m,\alpha}(\bar{M})$ on $\bar{M}$, for some $\alpha' < \alpha$. Thus, a neighborhood of $g \in E_{AH}^{m,\alpha}$ is defined to be the set of metrics $g'$ whose compactification $\widetilde{g}'$ by $\rho$ lie in a neighborhood of $\widetilde{g}$ in $Met^{m,\alpha}(\bar{M})$, where $Met^{m,\alpha}(\bar{M})$ is given the $C^{m,\alpha'}$ topology. Let $Met^{m,\alpha}$ be the corresponding space of metrics on $\partial M$, again with the $C^{m,\alpha'}$ topology. One thus has a natural boundary map

$$\Pi = \Pi^{m,\alpha} : E_{AH}^{m,\alpha} \to Met^{m,\alpha}(\partial M), \tag{2.1}$$

sending $g$ to its boundary metric $\gamma$.

An element $[g]$ in the moduli space $\mathcal{E}_{AH}^{m,\alpha}$ is an equivalence class of a $C^{m,\alpha}$ conformally compact Einstein metric $g$ on $M$, where $g \sim g'$ if $g' = \phi^* g$, with $\phi$ a $C^{m+1,\alpha}$ diffeomorphism of $\bar{M}$, equal to the identity on $\partial M$, i.e. $\phi \in \mathcal{D}_1$. Changing $g$ by a diffeomorphism in $\mathcal{D}_1$, or changing the defining function $\rho$ in (1.1) changes the boundary metric conformally. Hence if $\mathcal{C}^{m,\alpha}$ denotes the space of conformal classes of $C^{m,\alpha}$ metrics on $\partial M$, then the boundary map

$$\Pi = \Pi^{m,\alpha} : \mathcal{E}_{AH}^{m,\alpha} \to \mathcal{C}^{m,\alpha} \tag{2.2}$$

is well defined. When there is no danger of confusion, we will usually work with any given representative $g \in [g]$ and $\gamma \in [\gamma]$.

Suppose $dim M = n + 1 = 4$, and the 4-manifold $M$ satisfies $\pi_1(M, \partial M) = 0$. Then if $\mathcal{E}_{AH}^{m,\alpha}$ is non-empty, the spaces $E_{AH}^{m,\alpha}$ and $\mathcal{E}_{AH}^{m,\alpha}$ are $C^\infty$ infinite dimensional separable Banach manifolds, and the boundary maps $\Pi$ in (2.1) and (2.2) are $C^\infty$ smooth maps of Banach manifolds, of Fredholm index 0. This result holds for any $m \geq 3$, and $\alpha \in (0,1)$, and, replacing Banach by Fréchet, also for $m = \infty$ and $m = \omega$, (i.e. the real-analytic case). Implicit in this statement is the following boundary regularity result: an AH Einstein metric with a $C^2$ conformal compactification which has a $C^{m,\alpha}$ boundary metric, $m \geq 2$, has a $C^{m,\alpha}$ conformal compactification.

In addition, the spaces $\mathcal{E}_{AH}^{m,\alpha}$ are all diffeomorphic and for any $(m',\alpha')$ with $m'+\alpha' > m+\alpha$, the space $\mathcal{E}_{AH}^{m',\alpha'}$ is dense in $\mathcal{E}_{AH}^{m,\alpha}$. In particular,

$$\overline{\mathcal{E}_{AH}^{\omega}} = \mathcal{E}_{AH}^{m,\alpha}, \tag{2.3}$$

where the completion is taken in the $C^{m,\alpha'}$ topology.

If $dim M = n+1 > 4$, then these results hold when $3 \leq m \leq n-1$, using the boundary regularity results in [30]. In addition, the smooth manifold result holds when $m = \infty$, if, when $n$ is even, $\mathcal{E}_{AH}^{\infty}$ is understood to be the space of AH Einstein metrics which are $C^{\infty}$ polyhomogeneous; this uses the boundary regularity result of [16], cf. also the discussion following (2.10) below.

Let $\mathbb{S}^{m,\alpha}(\bar{M})$ denote the space of $C^{m,\alpha}$ symmetric bilinear forms on $\bar{M}$, and similarly for $M$ and $\partial M$. A tangent vector $h$ to the Banach manifold $\mathcal{E}_{AH}^{m,\alpha}$ at a representative $g \in [g]$, i.e. an infinitesimal Einstein deformation of $g$, is a form $h \in \mathbb{S}^{m,\alpha}(\bar{M})$ satisfying the equation

$$L(h) = \tfrac{1}{2}D^*Dh - R(h) = 0, \quad \beta_g(h) = 0, \tag{2.4}$$

where $R$ is the action of the curvature tensor of $g$ on symmetric bilinear forms and $\beta_g = \delta_g + \frac{1}{2}dtr_g$ is the Bianchi operator with respect to $g$. Such $h$ are transverse to the orbits of the diffeomorphism group $\mathcal{D}_1$ acting on $\bar{M}$. Let $T_g\mathcal{E}_{AH}^{m,\alpha}$ denote the space of such tangent vectors.

An Einstein metric $(M,g)$ is *non-degenerate* if the operator $L$ has no kernel in $L^2(M,g)$,

$$K = L^2 - Ker L = 0. \tag{2.5}$$

For $g \in \mathcal{E}_{AH}^{m,\alpha}$, the $L^2$ kernel $K$ equals the kernel of the linear map $D\Pi : T_g\mathcal{E}_{AH}^{m,\alpha} \to T_{\Pi(g)}\mathcal{C}^{m,\alpha}$. Hence, $g$ is non-degenerate if and only if $g$ is a regular point of the boundary map $\Pi$ and thus $\Pi$ is a local diffeomorphism near $g$. Any element $\kappa \in K$ is transverse-traceless, i.e.

$$\delta_g\kappa = 0, \quad tr_g\kappa = 0, \tag{2.6}$$

and so satisfies the Bianchi gauge condition (2.4).

In the next sections, we will frequently consider compactifications $\bar{g}$ of $g$ by a geodesic defining function $t$, for which

$$t(x) = dist_{\bar{g}}(x, \partial M). \tag{2.7}$$

A compactification $\bar{g} = t^2 g$ satisfying (2.7) is called a *geodesic compactification* and $t$ is a *geodesic* defining function. Such compactifications are natural from a number of viewpoints. In particular the curvature of $\bar{g}$ has a particularly simple form; this and related issues are discussed in the the Appendix. It is not difficult to see that given a $C^2$ conformally compact Einstein metric $g$ with boundary metric $\gamma$ in some compactification, there is a unique geodesic compactification $\bar{g}$ of $g$ with boundary metric $\gamma$. Further if $\widetilde{g}$ is some $C^{m,\alpha}$ compactification, then then the geodesic compactification $\bar{g}$ is at least $C^{m-1,\alpha}$ off the cutlocus $\bar{C}$ of $\partial M$ in $(M, \bar{g})$.

Since the integral curves of $\bar{\nabla}t$ are geodesics, the metric $\bar{g}$ splits as

$$\bar{g} = dt^2 + g_t, \tag{2.8}$$

within a collar neighborhood $U$ (inside the cutlocus) of $\partial M$; here $g_t$ is a curve of metrics on the boundary $\partial M$ with $g_0 = \gamma$. Setting $r = -\log t$, a simple calculation shows that the integral curves of $\nabla r$ are also geodesics in $(M,g)$ and so the metric $g$ also splits as $g = dr^2 + g_r$.

The Fefferman-Graham expansion [19] is the formal expansion of the curve $g_t$ in a Taylor-like series: the specific form of the expansion depends on $n$. If $dim M = n+1$ is even, then the expansion reads

$$g_t \sim g_{(0)} + t^2 g_{(2)} + \cdots + t^{n-1} g_{(n-1)} + t^n g_{(n)} + \cdots, \tag{2.9}$$

where the coefficients are symmetric bilinear forms on $\partial M$. The expansion is in even powers of $t$ up to order $n-1$ and the terms $g_{(2k)}$ are intrinsically determined by the boundary metric $\gamma = g_{(0)}$

and its tangential derivatives up to order $2k$, for $2k \leq n-1$. The term $g_{(n)}$ is transverse-traceless; however, beyond this, $g_{(n)}$ is not locally determined by $\gamma$; it depends on global properties of the Einstein metric $(M,g)$. For example, when $n=3$, one has

$$g_{(3)} = \tfrac{1}{6}\bar{\nabla}_N \bar{Ric},$$

where $\bar{Ric}$ is the Ricci curvature and $N = \bar{\nabla}t$ is the unit normal of $\partial M$ in $(\bar{M},\bar{g})$. This term is not computable in terms of $\gamma$; it depends on the global properties of $(M,g)$. On the other hand, the higher terms $g_{(k)}$, $k>n$ depend on two tangential derivatives of $g_{(k-2)}$.

If $dim M = n+1$ is even, then the expansion is

$$g_t \sim g_{(0)} + t^2 g_{(2)} + \ldots + t^n g_{(n)} + t^n(\log t)\mathcal{H} + \cdots , \tag{2.10}$$

where again this is an even expansion up to order $n$, with coefficients up to order $(n-2)$ locally determined by $\gamma$. The term $\mathcal{H}$, called the obstruction tensor in [19], is naturally identified as the stress-energy tensor of the integral $L$ of the conformal anomaly, (see (2.16) below), via the AdS/CFT correspondence, cf. [18], [33]; it is transverse-traceless and also locally determined by $\gamma$. The trace and divergence of $g_{(n)}$ are determined by $\gamma$; in fact there is a symmetric bilinear form $r_{(n)}$ and scalar function $a_{(n)}$, both explicitly computable from $\gamma$ and its derivatives, such that

$$\delta(g_{(n)} + r_{(n)}) = 0, \quad tr(g_{(n)} + r_{(n)}) = a_{(n)}. \tag{2.11}$$

The term $a_{(n)}$ is the conformal anomaly, (so that $\int_{\partial M} a_{(n)} = L$). However, as above, $g_{(n)}$ is otherwise only globally determined by $(M,g)$. The higher order terms in (2.10) contain terms of the form $t^m(\log t)^p$. The expansion is even in powers of $t$, and each coefficient $g_{(k)}$, $k \neq n$, depends on two derivatives of $g_{(k-2)}$. Note that (2.11) also holds when $n$ is odd, in which case $r_{(n)} = a_{(n)} = 0$. In both cases, all terms in the expansion are thus determined by the two terms $g_{(0)}$ and $g_{(n)}$ or $\tau_{(n)}$.

Let

$$\tau_{(n)} = g_{(n)} + r_{(n)}, \tag{2.12}$$

so that $\tau_{(n)}$ is divergence-free. This term will play an important role thoughout much of the paper, in particular in §4. In the AdS/CFT correspondence $\tau_{(n)}$ corresponds, (up to a constant), to the stress-energy tensor of the corresponding CFT on $\partial M$, cf. (2.19) below.

The boundary regularity results of [6], [7], [16] discussed above give the existence of the formal series (2.9) and (2.10); in the $C^\infty$ case, these are well-defined asympotic series for the curve $g_t$. If the data $(g_{(0)}, g_{(n)})$ are real-analytic, a result of Kichenassamy [28] shows that the series (2.9) and (2.10) converge to $g_t$, so that such data determine a solution to the Einstein equations defined in a neighborhood of $\partial M$.

The next local unique continuation result from [8] shows that the terms $g_{(0)}$ and $g_{(n)}$ uniquely determine an AH Einstein metric up to isometry, near the boundary.

**Proposition 2.1.** *Let $g^1$ and $g^2$ be two Einstein metrics defined on a half-ball $U \simeq (\mathbb{R}^{n+1})^+$, which have a $C^{3,\alpha}$ conformal compactification to the closed half-ball $\bar{U}$. If*

$$|g^1 - g^2| = o(t^n), \tag{2.13}$$

*where $t$ is the distance to $\partial U \simeq \mathbb{R}^n$, (in either compactified metric), then $g^1$ is isometric to $g^2$ in $U$.*

Of course (2.13) holds if $g^1$ and $g^2$ are $C^n$ polyhomogeneous conformally compact with identical $g_{(0)}$ and $g_{(n)}$ terms. Since Einstein metrics are real-analytic in local harmonic coordinates, local unique continuation in the interior, away from the boundary, is well-known.

Observe that the terms $g_{(k)}$ depend on the choice of the boundary metric $\gamma \in [\gamma]$. If $\widetilde{\gamma} = \lambda^2\gamma$, then the coefficients $\widetilde{g}_{(k)}$ are determined explicitly by $\lambda$ and $\gamma$, cf. [18]. For $n$ odd, the tranformation rule for the undetermined coefficient $g_{(n)}$ has the simple form

$$\widetilde{g}_{(n)} = \lambda^{-(n-2)} g_{(n)}. \tag{2.14}$$

In particular $|\widetilde{g}_{(n)}|_{\widetilde{g}} = \lambda^{-n} |g_{(n)}|_g$. For $n$ even, the transformation rule is more complicated; it has the form (2.14) at leading order, but has lower order terms depending explicitly on the derivatives of $\gamma$ and $\lambda$. Exact formulas in low dimensions are given in [18], [33].

Next we consider certain global issues associated with AH Einstein metrics which arise via the AdS/CFT correspondence. Let $g$ be an AH Einstein metric on an $(n+1)$-manifold $M$ with boundary metric $\gamma$, and such that the corresponding geodesic compactification is at least $C^n$ in the polyhomogeneous sense. The expansions (2.9)-(2.10) easily lead to an expansion for the volume of the region $B(t) = \{x \in M : t(x) \leq t\}$. As before, the form of the expansion depends on the parity of $n$. For $n$ odd,

$$volB(t) = v_{(n)} t^{-n} + v_{(n-2)} t^{-(n-2)} + \cdots + V + o(1), \tag{2.15}$$

while for $n$ even,

$$volB(t) = v_{(n)} t^{-n} + v_{(n-2)} t^{-(n-2)} + \cdots + L \log t + V + o(1). \tag{2.16}$$

The constant term $V$ is called the renormalized volume. More importantly for our purposes, let $\mathcal{I}_{EH}$ be the Einstein-Hilbert action, (with Gibbons-Hawking-York boundary term), given by

$$\mathcal{I}_{EH} = \int_M (s - 2\Lambda) dvol + \tfrac{1}{2} \int_{\partial M} H dv, \tag{2.17}$$

where $s$ is the scalar curvature, $\Lambda$ is the cosmological constant and $H$ is the mean curvature of the boundary. In the normalization (1.2), $\Lambda = -n(n-1)/2$. Einstein metrics, with scalar curvature $-n(n+1)$, are critical points of $\mathcal{I}_{EH}$, among variations fixing the boundary metric. (This is of course not the case for the volume functional). The action $\mathcal{I}_{EH}$ is infinite on $M$, but since $s$ is constant on Einstein metrics, (i.e. on shell), the cut-off action has an expansion of the same form as (2.15) or (2.16). Subtracting the divergent contributions as in (2.15)-(2.16) gives the renormalized action $\mathcal{I}_{EH}^{ren}$. On the space $E_{AH}$, i.e. on shell, when $n$ is odd one has

$$\mathcal{I}_{EH}^{ren} = -2nV,$$

as functionals on $E_{AH}$; the boundary term in (2.17) renormalizes to 0. Similarly in this case, the renormalized action or volume is in fact independent of the defining function $t$, so these functionals descend to functionals on the moduli space $\mathcal{E}_{AH}$. Neither of these facts hold however for $n$ even.

When $dimM = 4$, it is proved in [5] that on $E_{AH}$, $\mathcal{I}_{EH}^{ren}$ and $V$ are related to the square of the $L^2$ norm of the Weyl curvature $\mathcal{W}$ of $(M, g)$ in the following simple way:

$$\frac{1}{8\pi^2} \int_M |W|^2 = \chi(M) + \frac{1}{8\pi^2} \mathcal{I}_{EH}^{ren} = \chi(M) - \frac{3}{4\pi^2} V. \tag{2.18}$$

The relation (2.18) will play an important role in understanding the structure of the space $E_{AH}$, in particular in §3 and §5.

It is proved in [5], [18], cf. also [33], that as a 1-form on $E_{AH}^{m,\alpha}$ the differential of the renormalized action is given by

$$d_g \mathcal{I}_{EH}^{ren}(h) = c_n \int_{\partial M} \langle \tau_{(n)}, h_{(0)} \rangle dv, \tag{2.19}$$

where $\tau_{(n)}$ is as in (2.12), the inner product and volume form are with respect to $\gamma$ and $c_n$ is a constant depending only on $n$. The term $h_{(0)}$ is the variation of the boundary metric induced by $h$, i.e. $h_{(0)} = \Pi_*(h)$.

For later purposes, we need a slight but important extension of (2.19). Thus, given $g \in E_{AH}$, define an enlarged "tangent" space $\widetilde{T}_g E_{AH}$ roughly as follows: $h \in \widetilde{T}_g E_{AH}$ if $h \in \mathbb{S}^{m,\alpha}(M)$ and $h$ is an infinitesimal Einstein deformation to order at least $n$ at $t=0$, in the sense that the expansion (2.9) or (2.10) holds to order $n$ in $t$, to $1^{\text{st}}$ order in the variation $h$. More precisely, consider the curve $g_u = g + uh$ and assume without loss of generality that the geodesic defining function associated to $g_u$ is the fixed function $t$; (this can always be achieved by modifying $g_u$ by a curve of diffeomorphisms in $\mathcal{D}_1$ if necessary). Then $\bar{g}_u$ has the expansion

$$\bar{g}_u = dt^2 + (g_{(0)} + uh_{(0)}) + t(g_{(1)} + uh_{(1)}) + t^2(g_{(2)} + uh_{(2)}) + \cdots + t^n \log t(\mathcal{H}_{g+uh}) \\ + t^n(g_{(n)} + uh_{(n)}) + O(t^{n+\alpha}). \tag{2.20}$$

Then $h \in \widetilde{T}_g E_{AH}$ means that the coefficients $h_{(i)} = \frac{d}{du}(g_{(i)} + uh_{(i)})|_{u=0}$ are the linearizations of the coefficients $g_{(i)}$ and $\mathcal{H}$ in (2.9)-(2.10) in the direction $h_{(0)}$, for $i \leq n$. Similarly, the linearization of the trace condition in (2.11) should hold in the direction $h_{(0)}$. Here we recall that the coefficients $g_{(i)}$, $\mathcal{H}$ and $trg_{(n)}$ are explicitly determined from the Einstein equations by the boundary metric $\gamma = g_{(0)}$.

**Lemma 2.2.** *For any boundary variation $h_{(0)} \in \mathbb{S}^2(\partial M)$, there exists $h \in \widetilde{T}_g E_{AH}$ such that $h$ induces $h_{(0)}$ at $\partial M$, i.e. for $\Pi$ as in (2.1), $\Pi_* : \widetilde{T}_g E_{AH} \to \mathbb{S}^2(\partial M)$ is surjective.*

*Moreover, at any $g \in E_{AH}$, the equation (2.19) holds for any $h \in \widetilde{T}_g E_{AH}$.*

**Proof:** The first statement follows immediately from the fact that the equations for the terms $h_{(i)}$ in the expansion of $h$ can easily be solved algebraically. For example, the $g_{(2)}$ term is given explicitly by

$$g_{(2)} = \frac{1}{n-2}(Ric_\gamma - \frac{s_\gamma}{2(n-1)}\gamma).$$

The linearization of the right-hand side of this equation in the direction $h_{(0)}$ then defines the term $h_{(2)}$. The same reasoning applies to all the other coefficients, including the log coefficient $\mathcal{H}$, as well as the trace constraint (2.11). Of course $h_{(i)} = 0$ if $i$ is odd, $i < n$.

The second statement follows directly from the proof of (2.19) in [5], to which we refer for some further details. First, it is easy to see that $\mathcal{I}_{EH}^{ren}$ is well-defined, and so finite, for any metric $g_u = t^{-2}\bar{g}_u$ as in (2.20) which is Einstein to $n^{\text{th}}$ order at $\partial M$; see also [5, Remark 1.2]. Further, as noted above, Einstein metrics are critical points of $\mathcal{I}_{EH}^{ren}$, and so its variation in any direction depends only on the variation of the metric at the boundary. Thus, $d\mathcal{I}_{EH}^{ren}$ depends only on the boundary terms $g_{(j)}$ in the Fefferman-Graham expansion (2.9)-(2.10). Since it is exactly the form of these terms which leads to the formula (2.19), and since this form is preserved for the metrics $g_u$, (to 1st order in $u$), it is essentially clear that (2.19) holds for $h \in \widetilde{T}_g \mathcal{E}_{AH}$. Alternately, this may verified directly by an examination of the proof of (2.19) in [5, Lemma 2.1, Theorem 2.2], with $\mathcal{I}_{EH}$ in place of volume.

■

The divergence constraint in (2.11) is not used in the computation of $\mathcal{I}_{EH}^{ren}$ or its first variation. (It does arise however in the second variation). Let $\mathcal{D}_g E_{AH} \subset \mathbb{S}^{m,\alpha}(M)$ be the subspace consisting of forms satisfying the linearized divergence constraint (2.11) at $\partial M$, so that

$$\delta'(\tau_{(n)}) + \delta\tau'_{(n)} = 0, \tag{2.21}$$

where $\delta' = \frac{d}{du}\delta_{(g_{(0)}+uh_{(0)})}$ and $\tau'_{(n)} = g'_{(n)} + r'_{(n)} = h_{(n)} + s_{(n)}$, $s'_{(n)} = \frac{d}{du}(r_{(n)})_{(g_{(0)}+uh_{(0)})}$. Of course $s_{(n)} = 0$ for $n$ odd.

The space $\mathcal{F}_g E_{AH} = \mathcal{D}_g E_{AH} \cap \widetilde{T}_g E_{AH}$ represents the space of formal solutions of the linearized Einstein equations near $\partial M$, in that any $h \in \mathcal{F}_g E_{AH}$ defines uniquely a formal series solution as in (2.9)-(2.10) of the linearized Einstein equations. If $h_{(0)}$ and $h_{(n)}$ are real-analytic on $\partial M$, the result of Kichenassamy [28] mentioned above implies that the series converges to an actual linearized

Einstein deformation $h$ defined near $\partial M$. Note that while the divergence and trace of $h_{(n)}$ are determined by $h_{(0)}$, the transverse-traceless part of $h_{(n)}$ may be freely chosen.

**Proposition 2.3.** *For $g \in E_{AH}$, the map $\Pi_* : \mathcal{D}_g E_{AH} \to \mathbb{S}^2(\partial M)$, $\Pi_*(h) = h_{(0)}$, is surjective.*

**Proof:** Given Lemma 2.2, one needs to show that (2.21) is solvable, for any boundary variation $h_{(0)}$, i.e. $\delta'(\tau_{(n)}) \in Im\delta$, for all variations $h_{(0)}$ of $\gamma$. The space of 1-forms on $\partial M$ has the splitting

$$\Omega^1(\partial M) = Im\delta \oplus Ker\delta^*,$$

so that if $Ker\delta^* = 0$, i.e. $(\partial M, \gamma)$ has no Killing fields, then the result is clear. When $(\partial M, \gamma)$ does have Killing fields, this result is far from clear; using Theorem 2.5 below, this result is proved in [8], to which we refer for details.

∎

We will see later in §4, (in and following (4.45)), that $\Pi_*$ is surjective on the full formal space $\mathcal{F}_g E_{AH}$. It is worth noting that the global boundary map $\Pi_* : T_g \to \mathbb{S}^2(\partial M)$ is not surjective in general, cf. §7. Thus, the formal solutions of the linearized Einstein equations in $\mathcal{F}_g E_{AH}$, even if they give rise to actual linearized solutions, do not extend in general to smooth solutions on the compact manifold $M$.

On the other hand, Proposition 2.3 is false for AH Einstein metrics defined only in a neighborhood or thickening of $\partial M$. The proof of Proposition 2.3 in [8] is global; it requires that $(M, g)$ is conformally *compact*, as does Theorem 2.5 below.

The following result will play an important role in §4, cf. the proof of Proposition 4.6.

**Proposition 2.4.** *For $g \in E_{AH}$ and $h \in \mathcal{F}_g E_{AH}$, let $\sigma_{(n)} = \tau'_{(n)}$, for $\tau'_{(n)}$ as in (2.21). Similarly, let $a'_{(n)}$ be the variation of the conformal anomaly $a_{(n)}$ in the direction $h$. Then for any boundary metric $(\partial M, \gamma)$, $\gamma \in [\gamma] = \Pi[g]$, and smooth vector field $X$ on $\partial M$, one has*

$$\int_{\partial M} \langle \mathcal{L}_X \tau_{(n)} + [(1 - \tfrac{2}{n}) divX]\tau_{(n)}, h_{(0)} \rangle dV = \int_{\partial M} \langle \sigma_{(n)} + \tfrac{1}{2} trh_{(0)}\tau_{(n)}, \hat{\mathcal{L}}_X \gamma \rangle dV + \tag{2.22}$$

$$\tfrac{1}{n} \int_{\partial M} divX (a_{(n)} trh_{(0)} + 2a'_{(n)}) dV,$$

*where $\hat{\mathcal{L}}_X \gamma$ is the conformal Killing operator on $\partial M$, $\hat{\mathcal{L}}_X \gamma = \mathcal{L}_X \gamma - \frac{2divX}{n}\gamma$. The formula (2.22) holds for any variation $h_{(0)}$ of $\gamma$.*

**Proof:** This is a simple consequence of a result proved in [8]. Namely, from [8, Prop.5.4], one has

$$\int_{\partial M} \langle \mathcal{L}_X \tau_{(n)}, h_{(0)} \rangle = -2 \int_{\partial M} \langle \delta'(\tau_{(n)}), X \rangle + \int_{\partial M} \delta X \langle \tau_{(n)}, h_{(0)} \rangle + \langle \tau_{(n)}, \delta^* X \rangle trh_{(0)} dV, \tag{2.23}$$

where $\delta' = \frac{d}{du}\delta_{\gamma + uh_{(0)}}$. Using the definition of $\hat{\mathcal{L}}_X \gamma$, this gives

$$\int_{\partial M} \langle \mathcal{L}_X \tau_{(n)} + divX \tau_{(n)}, h_{(0)} \rangle = -2 \int_{\partial M} \langle \delta'(\tau_{(n)}), X \rangle$$

$$+ \tfrac{1}{2} \int_{\partial M} \langle \tau_{(n)}, \hat{\mathcal{L}}_X \gamma \rangle trh_{(0)} dV + \tfrac{1}{n} \int_{\partial M} a_{(n)} divX trh_{(0)} dV.$$

Next, by (2.21) and Proposition 2.3, $\delta' \tau_{(n)} = -\delta\sigma_{(n)}$, for any variation $h_{(0)}$ of the boundary metric $\gamma$, so that

$$-2 \int_{\partial M} \langle \delta' \tau_{(n)}, X \rangle = 2 \int_{\partial M} \langle \delta\sigma_{(n)}, X \rangle = 2 \int_{\partial M} \langle \sigma_{(n)}, \delta^* X \rangle = \int_{\partial M} \langle \sigma_{(n)}, \mathcal{L}_X \gamma \rangle.$$

Write $\mathcal{L}_X\gamma = \hat{\mathcal{L}}_X\gamma + \frac{2divX}{n}\gamma$. Then $\langle\sigma_{(n)},\gamma\rangle = tr\sigma_{(n)} = -tr'(\tau_{(n)}) + a'_{(n)} = \langle\tau_{(n)}, h_{(0)}\rangle + a'_{(n)}$, where the middle equality follows from the linearization of (2.11) with $h \in \widetilde{T}_g E_{AH}$. Combining these computations gives (2.22).

■

Observe that the first term on the right-hand side of (2.22) vanishes when $X$ is a conformal Killing field of $(\partial M, [\gamma])$. Using (2.14), it is easy to see that both sides of (2.22) are conformally invariant when $n$ is odd; on the other hand, both sides of (2.22) depend on the representative $\gamma \in [\gamma]$ when $n$ is even. Note also that (2.22) is invariant under the addition of transverse-traceless terms to $\sigma_{(n)}$.

The following result, also from [8] and used to prove Proposition 2.3, will be frequently used in §7.

**Theorem 2.5.** *Let $g$ be an AH Einstein metric on an $(n+1)$-manifold $M$, with $C^\infty$ boundary metric $\gamma$, (in some compactification). Then any connected Lie group of isometries of $(\partial M, \gamma)$ extends to an action by isometries on $(M, g)$.*

**Remark 2.6.** Finally, for the work in §5, we mention briefly the class of toral AdS black hole metrics on $M = \mathbb{R}^2 \times T^{n-1}$. These are AH Einstein metrics given explicitly by

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{T^{n-1}}, \tag{2.24}$$

where $V(r) = r^2 - \frac{2m}{r^{n-2}}$, $\theta \in [0, 4\pi/nr_+]$, $r_+ = (2m)^{1/n} > 0$ and $g_{T^{n-1}}$ is any flat metric on the torus $T^{n-1}$, cf. [6, Prop.4.4] and references therein. The boundary metric of $g_m$ is a flat (product) metric on the $n$-torus $T^n$.

Twisted versions of these metrics, obtained by taking suitable covering spaces and then passing to (different) discrete quotients, give rise to infinitely many isometrically distinct AH Einstein metrics $g_i$ on $\mathbb{R}^2 \times T^{n-1}$ with a fixed flat boundary metric $T^{n-1}$. These metrics are also the examples discussed following the statement of Theorem A. It is worth noting that the metrics $g_m$ are all locally isometric.

## 3. Compactness I: Interior Behavior.

The purpose of the next two sections is to lay the groundwork for the proof of Theorem A; this section deals with the interior behavior, while §4 is concerned with the behavior near the boundary. The property that the boundary map $\Pi$ is proper is a compactness issue. Thus, given a sequence of boundary metrics $\gamma_i$ converging to a limit metric $\gamma$, one needs to prove that a sequence of AH Einstein metrics $(M, g_i)$ with boundary metrics $\gamma_i$ has a subsequence converging, modulo diffeomorphisms, to a limit AH Einstein metric $g$ on $M$, with boundary metric $\gamma$.

In §3.1, we summarize background material on convergence and degeneration of sequences of metrics in general, as well as sequences of Einstein metrics. This section may be glanced over and then referred to as necessary. The following section §3.2 then applies these results to the interior behavior of AH Einstein metrics.

**§3.1.** In this section, we discuss $L^p$ Cheeger-Gromov theory as well as the convergence and degeneration results of Einstein metrics on 4-manifolds from [1]-[3].

We begin with the $L^p$ Cheeger-Gromov theory. The $L^\infty$ Cheeger-Gromov theory [15], [22], describes the (moduli) space of metrics on a manifold, (or sequence of manifolds), with uniformly bounded curvature in $L^\infty$, i.e.

$$|R_g|(x) \leq \Lambda < \infty, \tag{3.1}$$

in that it describes the convergence or possible degenerations of sequences of metrics satisfying the bound (3.1). The space $L^\infty$ is not a good space on which to carry out analysis, and so we replace

(3.1) by a corresponding $L^p$ bound, i.e.

$$\int_M |R_g|^p dV \leq \Lambda < \infty. \tag{3.2}$$

The curvature involves 2 derivatives of the metric, and so (3.2) is analogous to an $L^{2,p}$ bound on the metric. The critical exponent $p$ with respect to Sobolev embedding $L^{2,p} \subset C^o$ is $(n+1)/2$, where $n+1 = dimM$ and hence we will always assume that

$$p > (n+1)/2. \tag{3.3}$$

In order to obtain local results, we need the following definitions of local invariants of Riemannian metrics, cf. [4].

**Definition 3.1.** If $(M,g)$ is a Riemannian $(n+1)$-manifold, the $L^p$ *curvature radius* $\rho(x) \equiv \rho^p(x)$ at $x$ is the radius of the largest geodesic ball $B_x(\rho(x))$ such that, for all $B_y(s) \subset B_x(\rho(x))$, with $s \leq dist(y, \partial B_x(\rho(x)))$, one has

$$\frac{s^{2p}}{volB_y(s)} \int_{B_y(s)} |R|^p dV \leq c_0, \tag{3.4}$$

where $c_0$ is a fixed sufficiently small constant. Although $c_0$ is an essentially free parameter, we will fix $c_0 = 10^{-2}$ throughout the paper. The left-side of (3.4) is a scale-invariant local average of the curvature in $L^p$.

The *volume radius* $\nu(x)$ of $(M,g)$ at $x$ is given by

$$\nu(x) = \sup\{r : \frac{volB_y(s)}{\omega_{n+1}s^{n+1}} \geq \mu_0, \forall B_y(s) \subset B_x(r)\}, \tag{3.5}$$

where $\omega_{n+1}$ is the volume of the Euclidean unit $(n+1)$-ball and again $\mu_0 > 0$ is a free small parameter, which will be fixed in any given discussion, e.g. $\mu_0 = 10^{-2}$.

The $L^{k,p}$ *harmonic radius* $r_h^{k,p}(x)$ is the largest radius such that on the geodesic ball $B_x(r)$, $r = r_h^{k,p}(x)$, there is a harmonic coordinate chart in which the metric components satisfy

$$C^{-1}\delta_{ij} \leq g_{ij} \leq C\delta_{ij} \tag{3.6}$$

$$r^{kp-(n+1)} \int_{B_x(r)} |\partial^k g_{ij}|^p dV_g \leq C,$$

where $C$ is a fixed constant; again $C$ may be arbitrary, but will be fixed to $C = 2$, (for example).

Observe that $\rho(x)$, $\nu(x)$ and $r_h(x)$ scale as distances, i.e. if $g' = \lambda^2 \cdot g$, for some constant $\lambda$, then $\rho'(x) = \lambda \cdot \rho(x)$, $\nu'(x) = \lambda \cdot \nu(x)$ and $r_h'(x) = \lambda \cdot r_h(x)$. By definition, $\rho$, $\nu$ and $r_h$ are Lipschitz functions with Lipschitz constant 1; in fact for $y \in B_x(\rho(x))$, it is immediate from the definition that

$$\rho(y) \geq dist(y, \partial B_x(\rho(x))), \tag{3.7}$$

and similarly for $\nu$, $r_h$.

One may define $L^{k,p}$, (or $C^{k,\beta}$) curvature radii $\rho^{k,p}$ in a manner completely analogous to (3.4). For the $L^{k,p}$ radius, the curvature $|R|$ term in (3.4) is replaced by $\sum_{j\leq k} |\nabla^j R|$ and the power of $s$ is chosen to make the resulting expression scale invariant. Clearly

$$\rho^{k,p} \geq \rho^{k',p'},$$

whenever $k + p \geq k' + p'$. It is proved in [3] that the curvature radius and harmonic radius are essentially equivalent, given a lower bound on the volume radius. Thus

$$r_h^{2,p}(x) \geq r_0 \rho^p(x), \tag{3.8}$$

where $r_0$ depends only on a lower bound $\nu_0$ for $\nu(x)$. The same statement holds for the radii $r_h^{k+2,p}$ and $\rho^{k,p}$.

A sequence of Riemannian metrics $(\Omega_i, g_i)$ is said to converge in the $L^{k,p}$ topology to a limit $L^{k,p}$ metric $g$ on $\Omega$ if there is an atlas $\mathcal{A}$ for $\Omega$ and diffeomorphisms $F_i : \Omega \to \Omega_i$ such that $F_i^*(g_i)$ converges to $g$ in the $L^{k,p}$ topology in local coordinates with respect to the atlas $\mathcal{A}$. Thus, the local components $(F_i^*(g_i))_{\alpha\beta} \to g_{\alpha\beta}$ in the usual $L^{k,p}$ Sobolev topology on functions on $\mathbb{R}^{n+1}$. Similar definitions hold for $C^{m,\alpha}$ convergence, and convergence in the weak $L^{k,p}$ topology. Any bounded sequence in $L^{k,p}$ has a weakly convergent subsequence, and similarly any bounded sequence in $C^{m,\alpha}$ has a convergent subsequence in the $C^{m,\alpha'}$ topology, for any $\alpha' < \alpha$. By Sobolev embedding,

$$L^{k,p} \subset C^{m,\alpha},$$

for $m + \alpha < k - \frac{n+1}{p}$. In particular, in dimension 4, for $p \in (2,4), L^{2,p} \subset C^{\alpha}, \alpha = 2 - \frac{4}{p}$, and for $p > 4$, $L^{2,p} \subset C^{1,\alpha}$, for $\alpha = 1 - \frac{4}{p}$.

One then has the following result on the convergence and degeneration of metrics with bounds on $\rho$, cf. [4].

**Theorem 3.2.** *Let $(\Omega_i, g_i, x_i)$ be a pointed sequence of connected Riemannian $(n+1)$-manifolds and suppose there are constants $\rho_o > 0$, $d_o > 0$ and $D < \infty$ such that, for a fixed $p > (n+1)/2$,*

$$\rho^p(y) \geq \rho_o, \ diam\Omega_i \leq D, \ dist(x_i, \partial\Omega_i) \geq d_o. \tag{3.9}$$

*Then for any $0 < \epsilon < d_o$, there are domains $U_i \subset \Omega_i$, with $\epsilon/2 \leq dist(\partial U_i, \partial\Omega_i) \leq \epsilon$, for which one of the following alternatives holds.*

*(I). (Convergence) If there is constant $\nu_o > 0$ such that,*

$$\nu_i(x_i) \geq \nu_o > 0,$$

*then a subsequence of $\{(U_i, g_i, x_i)\}$ converges, in the weak $L^{2,p}$ topology, to a limit $L^{2,p}$ Riemannian manifold $(U,\ g,\ x)$, $x = \lim x_i$. In particular, $U_i$ is diffeomorphic to $U$, for $i$ sufficiently large.*

*(II). (Collapse) If instead,*

$$\nu(x_i) \to 0,$$

*then $U_i$ has an F-structure in the sense of Cheeger-Gromov, cf. [15]. The metrics $g_i$ are collapsing everywhere in $U_i$, i.e. $\nu_i(y_i) \to 0$ for all $y_i \in U_i$, and so in particular the injectivity radius $inj_{g_i}(y_i) \to 0$. Any limit of $\{g_i\}$ in the Gromov-Hausdorff topology [22] is a lower dimensional length space.*

**Remark 3.3. (i).** The local hypothesis on $\rho^p$ in (3.9) can be replaced by the global hypothesis

$$\int_{\Omega_i} |R|^p dV \leq \Lambda, \tag{3.10}$$

in case the volume radius satisfies $\nu_i(y_i) \geq \nu_o$, for some $\nu_o > 0$ and all $y_i \in \Omega_i$; in fact under this condition (3.9) and (3.10) are then equivalent, with $\Lambda = \Lambda(\rho_o, D, \nu_o)$.

**(ii).** It is an easy consequence of the definitions that the $L^p$ curvature radius $\rho^p$ is continuous under convergence in the (strong) $L^{2,p}$ topology, i.e. if $g_i \to g$ in the $L^{2,p}$ topology, then

$$\rho_i(x_i) \to \rho(x), \tag{3.11}$$

whenever $x_i \to x$, cf. [4] and references therein. However, (3.11) does *not* hold if the convergence is only in the weak $L^{2,p}$ topology.

If in addition to a bound on $\rho^p$ one has $L^p$ bounds the covariant derivatives of the Ricci curvature up to order $k$, then in Case (I) one obtains convergence to the limit in the $L^{k+2,p}$ topology. Analogous statements hold for convergence in the $C^{m,\alpha'}$ topology.

Next we discuss the convergence and degeneration of Einstein metrics. If $M$ is a closed 4-manifold and $g$ an Einstein metric on $M$, then the Chern-Gauss-Bonnet theorem gives

$$\frac{1}{8\pi^2}\int_M |R|^2 dV = \chi(M), \tag{3.12}$$

where $\chi(M)$ is the Euler characteristic of $M$. This gives apriori control on the $L^2$ norm of the curvature of Einstein metrics on $M$. However the $L^2$ norm is critical in dimension 4 with respect to Sobolev embedding, cf. (3.3) and so one may not expect Theorem 3.2 to hold for sequences of Einstein metrics on $M$. In fact, there is a further possible behavior of such sequences.

**Definition 3.4.** An *Einstein orbifold* $(X, g)$ is a 4-dimensional orbifold with a finite number of cone singularities $\{q_j\}, j = 1, ..., k$. On $X_o = X \setminus \cup\{q_j\}$, $g$ is a smooth Einstein metric while each singular point $q \in \{q_j\}$ has a neighborhood $U$ such that $U \setminus q$ is diffeomorphic to $C(S^3/\Gamma) \setminus \{0\}$, where $\Gamma$ is a finite subgroup of $O(4)$ and $C$ denotes the cone with vertex $\{0\}$. Further $\Gamma \neq \{e\}$ and when lifted to the universal cover $B^4 \setminus \{0\}$ of $C(S^3/\Gamma) \setminus \{0\}$, the metric $g$ extends smoothly over $\{0\}$ to a smooth Einstein metric on the 4-ball $B^4$.

There are numerous examples of sequences of smooth Einstein metrics $g_i$ on a compact manifold $M$ which converge, in the Gromov-Hausdorff topology, to an Einstein orbifold limit $(X, g)$. Such orbifold metrics are not smooth metrics on the manifold $M$, but may be viewed as singular metrics on $M$, in that $M$ is a resolution of $X$.

We then have the following result describing the convergence and degeneration of Einstein metrics on 4-manifolds, cf. [3].

**Theorem 3.5.** *Let $(\Omega_i, g_i, x_i)$ be a pointed sequence of connected Einstein 4-manifolds satisfying*

$$diam\Omega_i \leq D, \quad dist(x_i, \partial\Omega_i) \geq d_o, \tag{3.13}$$

*and*

$$\int_{\Omega_i} |R_{g_i}|^2 dV_{g_i} \leq \Lambda_o, \tag{3.14}$$

*for some constants $d_o > 0$, $D$, $\Lambda_o < \infty$. Then for any $0 < \epsilon < d_o$, there are domains $U_i \subset \Omega_i$, with $\epsilon/2 \leq dist(\partial U_i, \partial\Omega_i) \leq \epsilon$, for which exactly one of the following alternatives holds.*

*(I). (Convergence). A subsequence of $(U_i, g_i, x_i)$ converges in the $C^\infty$ topology to a limit smooth Einstein metric $(U, g, x)$, $x = \lim x_i$.*

*(II). (Orbifolds). A subsequence of $(U_i, g_i, x_i)$ converges to an Einstein orbifold metric $(X, g, x)$ in the Gromov-Hausdorff topology. Away from the singular variety, the convergence $g_i \to g$ is $C^\infty$, while the curvature of $g_i$ blows up in $L^\infty$ at the singular variety.*

*(III). (Collapse). A subsequence of $(U_i, g_i, x_i)$ collapses, in that $\nu(y_i) \to 0$ for all $y_i \in U_i$. The collapse is with "generalized" bounded curvature, in that $inj^2_{g_i}(x)|R|_{g_i}(x) \to 0$, metrically away from a finite number of singular points; however such singularities might be more complicated than orbifold cone singularities.*

The cases (I) and (II) occur if and only if

$$\nu_i(x_i) \geq \nu_o, \tag{3.15}$$

for some $\nu_o > 0$, while (III) occurs if (3.15) fails. One obtains $C^\infty$ convergence in (I), and (II) away from the singularities, since Einstein metrics satisfy an elliptic system of PDE, (in harmonic coordinates).

When $\Omega_i$ is a closed 4-manifold $M$, the bound (3.14) follows immediately from (3.12). When $(\Omega_i, g_i)$ are complete AHE metrics on a fixed 4-manifold $M$, then of course (3.14) does not hold with $\Omega_i = M$. However, in the normalization (1.2), one has $|W|^2 = |R|^2 - 6$ where $W$ is the Weyl tensor of $(M, g)$ and so by (2.18)

$$\frac{1}{8\pi^2}\int_M (|R_g|^2 - 6)dV_g = \chi(M) - \frac{3}{4\pi^2}V. \tag{3.16}$$

Thus, a *lower* bound on the renormalized volume $V$ (or upper bound on the renormalized action) and an upper bound on $\chi(M)$ give a global bound on the $L^2$ norm of $W$. In particular, under these bounds (3.14) holds for $\Omega_i$ a geodesic ball $B_{x_i}(R)$ of any fixed radius $R$ about the base point $x_i \in (M, g_i)$.

We point out that Theorem 3.5 is special to dimension 4. A similar result holds in higher dimensions only if one has a uniform bound on the $L^{n/2}$ norm of curvature in place of (3.14); however there is no analogue of (3.16) for (general) Einstein metrics in higher dimensions.

**Remark 3.6.** Theorems 3.2 and 3.5 are local results. However, we will often apply them globally, to sequences of complete manifolds, and with complete limits. This is done by a standard procedure as follows, in the situation of Theorem 3.2 for example. Suppose $\{g_i\}$ is a sequence of complete metrics on a manifold $M$, or on a sequence of manifolds $M_i$, with base points $x_i$, and satisfying $\rho(y_i) \geq \rho_o$, for all $y_i \in (M, g_i)$. One may then apply Theorem 3.2 to the domains $(B_{x_i}(R), g_i, x_i)$, where $B_{x_i}(R)$ is the geodesic $R$-ball about $x_i$ to obtain a limit manifold $(U(R), g_\infty, x_\infty)$, in the non-collapse case. Now take a divergent sequence $R_j \to \infty$ and carry out this process for each $j$. There is then a diagonal subsequence $B_{x_i}(R_i)$ of $B_{x_i}(R_j)$ which converges to a complete limit $(N, g_\infty, x_\infty)$. Similar arguments apply in the case of collapse and the cases of Theorem 3.5.

The convergence in these situations is also convergence in the pointed Gromov-Hausdorff topology, [22].

**§3.2.** In this section, we study the behavior of sequences of AH Einstein metrics in the interior, i.e. away from $\partial M$. We first discuss the hypotheses, and then state and prove the main result, Theorem 3.7.

Essentially for the rest of the paper, we will assume the topological condition on $M = M^4$ that

$$H_2(\partial M, \mathbb{F}) \to H_2(\bar{M}, \mathbb{F}) \to 0, \tag{3.17}$$

where the map is induced by inclusion $\iota : \partial M \to \bar{M}$ and $\mathbb{F}$ is any field. As will be seen, this serves to rule out any orbifold degenerations of AH Einstein sequences. While many of the results of this paper carry over to the orbifold setting, we prefer for simplicity to exclude this possibility here.

Next, let $g_i \in E_{AH}$ be a sequence of AH Einstein metrics on $M$ with $C^{m,\alpha}$ boundary metrics $\gamma_i$, $m \geq 3$. As noted in §2, the geodesic compactification $\bar{g}_i$ of $g_i$ with boundary metric $\gamma_i$ is at least a $C^{m-1,\alpha}$ compactification. We assume the boundary behavior of $\{g_i\}$ is controlled, in that

$$\gamma_i \to \gamma, \tag{3.18}$$

in the $C^{m,\alpha'}$ topology on $\partial M$, for some $\alpha' \leq \alpha$. Next assume that the inradius of $(M, \bar{g}_i)$ has a uniform lower bound, i.e.

$$In_{\bar{g}_i}(\partial M) = dist_{\bar{g}_i}(\bar{C}_i, \partial M) \geq \bar{\tau}, \tag{3.19}$$

for some constant $0 < \bar{\tau} \leq 1$, where $\bar{C}_i$ is the cutlocus of $\partial M$ in $(\bar{M}, \bar{g}_i)$, and also assume an upper diameter bound

$$diam_{\bar{g}_i} S(t_1) \leq T, \tag{3.20}$$

where $t_1 = \bar{\tau}/2$ and $S(t_1) = \{x \in (M, \bar{g}_i) : t_i(x) = t_1\}$ is the $t_1$-level set of the geodesic defining function $t_i$ for $(g_i, \gamma_i)$.

Finally, we assume that the Weyl curvature of $g_i$ is uniformly bounded in $L^2$, i.e.

$$\int_M |W_{g_i}|^2 dV_{g_i} \leq \Lambda_0 < \infty. \tag{3.21}$$

All of these assumptions will be removed in §4 and at the beginning of §5. The main result of this subsection is the following:

**Theorem 3.7.** *Let $\{g_i\}$ be a sequence of metrics in $E_{AH}$ on $M$, satisfying* (3.17)-(3.21) *and let $x_i$ be a base point of $(M, g_i)$ satisfying*

$$d \leq dist_{\bar{g}_i}(x_i, \partial M) \leq D, \tag{3.22}$$

*for some constants $d > 0$ and $D < \infty$, where $\bar{g}_i$ is the geodesic compactification associated to $\gamma_i$.*

*Then a subsequence of $(M, g_i, x_i)$, converges to a complete Einstein metric $(N, g, x_\infty)$, $x_\infty = \lim x_i$. The convergence is in the $C^\infty$ topology, uniformly on compact subsets, and the manifold $N$ weakly embeds in $M$,*

$$N \subset\subset M, \tag{3.23}$$

*in the sense that smooth bounded domains in $N$ embed as such in $M$.*

**Proof:** By Theorem 3.5 and the discussion following (3.16), together with Remark 3.6, a subsequence of $\{(M, g_i)\}$ based at $x_i$ either, (i): converges smoothly to a complete limit Einstein manifold $(N, g, x_\infty)$, (ii): converges to an Einstein orbifold, smoothly away from the singular variety, or (iii): collapses uniformly on domains of arbitrary bounded diameter about $\{x_i\}$.

The first (and most important) task is to rule out the possibility of collapse. To do this, we first prove a useful volume monotonicity formula in the Lemma below; this result will also be important in §4 and holds in all dimensions.

Let $(M, g)$ be any AH Einstein manifold of dimension $n+1$, with $C^{2,\alpha}$ geodesic compactification

$$\bar{g} = t^2 g \tag{3.24}$$

and boundary metric $\gamma$. Let $\bar{E}$ be the inward normal exponential map of $(M, \bar{g})$ at $\partial M$, so that $\sigma_x(t) = \bar{E}(x, t)$ is a geodesic in $t$, for each fixed $x \in \partial M$. As discussed in §2, for $r = -\log t$, the curve $\sigma_x(r)$ is a geodesic in the Einstein manifold $(M, g)$. Let $\bar{J}(x, t)$ be the Jacobian of $\bar{E}(x, t)$, so that $\bar{J}(x, t) = d\bar{V}(x, t)/d\bar{V}(x, 0)$, where $d\bar{V}$ is the volume form of the 'geodesic sphere' $S(t)$, i.e. the $t$-level set of $t$ as following (3.20). Also, let $\tau_0 = \tau_0(x)$ be the distance to the cutlocus of $\bar{E}$ at $x \in \partial M$.

**Lemma 3.8.** *In the notation above, the function*

$$\frac{\bar{J}(x, t)}{\tau_0^n (1 - (\frac{t}{\tau_0})^2)^n} \tag{3.25}$$

*is monotone non-decreasing in $t$, for any fixed $x$.*

**Proof:** As above, let $r = \log(\frac{1}{t})$ and, for any fixed $x$, set $r_0 = r_0(x) = \log(\frac{1}{\tau_0})$. Then the curve $\sigma_x(r)$ is a geodesic in $(M, g)$ and $S(r)$ is the $r$-rlevel set of the distance function $r$. Let $J(x, r)$ be the corresponding Jacobian for $S(r)$ along $\sigma_x(r)$, so that $J(x, r) = t^{-n}\bar{J}(x, t)$ by (3.24). Since $Ric_g = -ng$, the infinitesimal form of the Bishop-Gromov volume comparison theorem [22] implies the ratio

$$\frac{J(x, r)}{\sinh^n(r - r_0)} \downarrow$$

i.e. the ratio is monotone non-increasing in $r$, as $r \to \infty$. Converting this back to $(M, \bar{g})$, it follows that, in $t$,

$$\frac{t^{-n}\bar{J}(x, t)}{t^n \sinh^n(\log(\frac{\tau_0}{t}))} \uparrow,$$

is increasing, since as $r$ increases to $\infty$, $t$ decreases to 0. But $\sinh^n(\log(\frac{\tau_0}{t})) = \frac{1}{2^n}(\frac{\tau_0}{t})^n(1 - (\frac{t}{\tau_0})^2)^n$, which gives (3.25). ∎

Let $q$ be any point in $\partial M$ and consider the metric $s$-ball $B_x(s) = \{y \in \bar{M} : dist_{\bar{g}}(y,q) \leq s\}$ in $(\bar{M}, \bar{g})$. Let $D_q(s) = B_q(s) \cap \partial M$, so that $D_q(s)$ is the metric $s$-ball in $(\partial M, \gamma)$, since $\partial M$ is totally geodesic. Observe that there are constants $\mu_0 > 0$ and $\mu_1 < \infty$, which depend only on the $C^0$ geometry of the boundary metric $\gamma$, such that

$$\mu_0 \leq \frac{vol_\gamma(D_q(s))}{s^n} \leq \mu_1. \tag{3.26}$$

Let $D_q(s,t) = \{x \in M : x = \bar{E}(y,t)$, for some $y \in D_q(s)\}$, so that $D_q(s,t) \subset S(t)$. It follows immediately from (3.25) by integration over (a domain in) $\partial M$ that, for any fixed $s$,

$$\frac{vol_{\bar{g}} D(s,t)}{(s\bar{\tau})^n(1-(\frac{t}{\bar{\tau}})^2)^n} \uparrow, \ \text{ and } \ \frac{vol_{\bar{g}} S(t)}{\bar{\tau}^n(1-(\frac{t}{\bar{\tau}})^2)^n} \uparrow, \tag{3.27}$$

for $\bar{\tau}$ as in (3.19). Now let $t_1 = \bar{\tau}/2$, so that (3.27) implies in particular that

$$vol_{\bar{g}} S(t_1) \geq (\frac{3}{4})^n vol_\gamma \partial M, \tag{3.28}$$

and hence, with respect to $(M, g)$,

$$vol_g S(t_1) \geq (\frac{3}{4})^n t_1^{-n} vol_\gamma \partial M. \tag{3.29}$$

This leads easily to the following lower bound on volumes of balls.

**Corollary 3.9.** *Let $(M, g)$ be an AH Einstein metric, satisfying* (3.19)-(3.20), *and choose a point $x$ satisfying* (3.22), *i.e. $d \leq dist_{\bar{g}}(x, \partial M) \leq D$. Then*

$$vol_g B_x(1) \geq \nu_0 > 0, \tag{3.30}$$

*where $\nu_0$ depends only on $d, D, \bar{\tau}, T$ and the $C^0$ geometry of the boundary metric $\gamma$.*

**Proof:** Again, by the Bishop-Gromov volume comparison theorem on $(M, g)$, one has for any $R \geq r$,

$$\frac{vol B_x(r)}{\sinh^n(r)} \geq \frac{vol B_x(R)}{\sinh^n(R)}. \tag{3.31}$$

Suppose $r(x) = D_1$, so that $D_1 \in [-\log D, -\log d]$. Then for any $y \in S(t_1)$, the triangle inequality and (3.20) imply that

$$dist_g(x,y) \leq |D_1| + t_1^{-1} \cdot T \equiv D_2.$$

Hence, choose $R$ so that $R = D_2 + 1$, which implies that $S(t_1) \subset B_x(R-1)$. But $vol B_x(R) \geq vol A(R-2, R) \geq \frac{1}{2} vol S(t_1)$, where the second inequality follows from the coarea formula, (changing $t_1$ slightly if necessary). Combining this with the comparison estimate above implies

$$vol B_x(1) \geq c_2 vol S(t_1) \sinh^{-n}(D_2) \geq c_3 t_1^{-n} \sinh^{-n}(D_2) vol_\gamma \partial M,$$

where the last inequality follows from (3.29). This gives (3.30). ∎

Corollary 3.9 gives a uniform lower bound on the volume radius of each $g_i$ at $x_i$ satisfying (3.22) and so there is no possibility of collapse. Theorem 3.5 and Remark 3.6 then imply that $(M, g_i, x_i)$ has a subsequence converging in the Gromov-Hausdorff topology either to a complete Einstein manifold $(N, g, x_\infty)$ or to a complete Einstein orbifold $(V, g, x_\infty)$.

Next, we use the hypothesis (3.17) to rule out orbifold limits. Thus, suppose the second alternative above holds, so that $(M, g_i)$ converges in the Gromov-Hausdorff topology to a complete Einstein orbifold $(X, g)$. With each orbifold singularity $q \in X$ with neighborhood of the form $C(S^3/\Gamma)$, there is associated a (preferred) smooth complete Ricci-flat 4-manifold $(E, g_0)$, which is asymptotically locally Euclidean (ALE), in that $(E, g)$ is asymptotic to a flat cone $C(S^3/\Gamma_E)$, $\Gamma_E \neq \{e\}$, cf. [1,§5], [3,§3]. The complete manifold $(E, g_o)$ is obtained as a limit of blow-ups or rescalings of the metrics $g_i$ restricted to small balls $B_{y_i}(\delta_i)$ at the *maximal* curvature scale; thus one does not necessarily have $\Gamma = \Gamma_E$, since there may be other Ricci-flat ALE orbifolds arising from blow-ups at other curvature scales. In any case, the manifold $E$ is embedded in the ambient manifold $M$; in fact for any $\delta > 0$ and points $y_i \to q$, $E$ is topologically embedded in $(B_{y_i}(\delta), g_i)$. The Einstein metrics $g_i$ crush the topology of $E$ to a point in that $E \subset (B_{y_i}(\delta), g_i)$ and $B_{y_i}(\delta)$ converges to the cone $C(S^3/\Gamma)$, in the Gromov-Hausdorff topology for any fixed $\delta > 0$ sufficiently small.

Next we point out that any such ALE space $E$ has non-trivial topology. This result corrects a minor inaccuracy in [1, Lemma 6.3].

**Proposition 3.10.** *Let $(E, g_0)$ be a complete, non-flat, oriented Ricci-flat ALE manifold. Then*

$$H_2(E, \mathbb{F}) \neq 0, \tag{3.32}$$

*for some field $\mathbb{F}$. Moreover, if $E$ is simply connected, then $H_2(E, \mathbb{Z})$ is torsion-free, so that $H_2(E, \mathbb{R}) \neq 0$, while $H_2(E, \mathbb{F}) = 0$, for any finite field $\mathbb{F}$.*

**Proof:** It is well-known that $|\pi_1(E)| < \infty$, so that $b_1(E) = 0$, and also $b_3(E) = 0$, (cf. [1] for example). Hence

$$\chi(E) = 1 + b_2(E) \geq 1.$$

If $b_2(E) > 0$, then of course (3.32) follows, so suppose $b_2(E) = 0$, so that $\chi(E) = 1$. The Euler characteristic may be computed with respect to homology with coefficients in any field $\mathbb{F}$, so that

$$\chi(E) = 1 - H_1(E, \mathbb{F}) + H_2(E, \mathbb{F}) = 1,$$

since again $H_3(E, \mathbb{F}) = H_4(E, \mathbb{F}) = 0$. Now if (3.32) does not hold, then it follows that $H_1(E, \mathbb{F}) = 0$, for all $\mathbb{F}$. By the universal coefficient theorem, this implies $H_1(E, \mathbb{Z}) = H_2(E, \mathbb{Z}) = 0$.

Thus, $E$ is an integral homology ball, with finite $\pi_1$. It then follows by the arguments in [1, Lemma 6.3] that $E$ is flat, giving a contradiction. This proves (3.32). The proof of the second statement is an immediate consequence of the universal coefficient theorem. ∎

Next, observe that there is an injection

$$0 \to H_2(E, \mathbb{F}) \to H_2(M, \mathbb{F}). \tag{3.33}$$

To see this, the Mayer-Vietoris sequence for (a thickening of) the pair $(E, M \setminus E)$ gives

$$H_2(S^3/\Gamma, \mathbb{F}) \to H_2(E, \mathbb{F}) \oplus H_2(M \setminus E, \mathbb{F}) \to H_2(M, \mathbb{F}).$$

Thus, it suffices to show that $i_* : H_2(S^3/\Gamma, \mathbb{F}) \to H_2(E, \mathbb{F})$ is the zero-map. If $E$ is simply connected, this follows immediately from Proposition 3.10, since $H_2(S^3/\Gamma, \mathbb{Z})$ is torsion, while $H_2(E, \mathbb{Z})$ is torsion-free. In general, let $\widetilde{E}$ be the universal cover of $E$, so that the covering map $\pi : \widetilde{E} \to E$ is finite-to-one. One has $\partial \widetilde{E} = S^3/\widetilde{\Gamma}$, for some finite group $\widetilde{\Gamma}$. Now any 2-cycle in $S^3/\Gamma$ with coefficients in $\mathbb{F}$ may be represented by a map, (or more precisely a collection of maps), $f : S^2 \to S^3/\Gamma$. This lifts to a map $\widetilde{f} : S^2 \to S^3/\widetilde{\Gamma} = \partial \widetilde{E}$. As noted above, the 2-cycle $\widetilde{f}$ bounds a 3-chain $F$ in $\widetilde{E}$. Composing with the projection map $\pi$, it follows that $f$ bounds a 3-chain in $E$, which proves the claim.

Now returning to (3.17) and orbifold limits, let $\Sigma$ be any non-zero 2-cycle in $H_2(E, \mathbb{F})$; ($\Sigma$ exists by (3.32)). By (3.17), there is a 2-cycle $\Sigma_\infty$ in $H_2(\partial M, \mathbb{F})$ homologous to $\Sigma$, so that there is a 3-chain $W$ in $M$ such that $\partial W = \Sigma - \Sigma_\infty$. But $Z = W \cap (S^3/\Gamma) \neq \emptyset$ and so $Z$ represents a 2-cycle in $H_2(S^3/\Gamma, \mathbb{F})$. By the preceding argument, $Z$ bounds a 3-chain $W'$ in $E$. Hence $(W \cap E) + W'$ is a 3-chain in $E$, with boundary $\Sigma$, i.e. $[\Sigma] = 0$ in $H_2(E, \mathbb{F})$, giving a contradiction.

Thus, the hypothesis (3.17) rules out any possible orbifold degeneration of the sequence $(M, g_i)$. Having ruled out the possibilities (ii) and (iii), it follows that (i) must hold. The fact that $N$ is weakly embedded in $M$ follows immediately from Remark 3.6 and the definition of smooth convergence preceding (3.8). This completes the proof of Theorem 3.7. ∎

**Remark 3.11. (i).** Note that Theorem 3.7 does not imply that the limit manifold $N$ topologically equals $M$. For instance, it is possible at this stage that some of the topology of $M$ escapes to infinity, and is lost in the limit $N$. All of the results and discussion in §3.2 hold without change if the 4-manifold $M$ is replaced by a sequence of 4-manifolds $M_i$, with a common boundary $\partial M_i = \partial M$, so that one has $(M_i, g_i)$ in place of $(M, g_i)$.

**(ii).** We also observe that the proof shows that there is a uniform bound $\Lambda = \Lambda(\Lambda_0, d, D, T, \{\gamma_i\}) < \infty$ such that, for $x$ satisfying (3.22),

$$|R_{g_i}|(x) \leq \Lambda, \tag{3.34}$$

i.e. there is a uniform bound on the sectional curvatures of $\{g_i\}$ in this region.

**(iii).** The existence and behavior of $(M, g_i)$ at base points $x_i$ where $dist_{\bar{g}_i}(x_i, \partial M) \to \infty$ will be discussed at the end of §4.

## 4. Compactness II: Boundary Behavior.

Theorem 3.7 essentially corresponds to a uniform interior regularity result in that one has uniform control of the AHE metrics $g_i$ in the interior of the 4-manifold $M$, i.e. on compact subsets of $M$. In this subsection, we extend this to similar control in a neighborhood of definite size about the boundary. The Chern-Gauss-Bonnet theorem in (3.12), cf. also (3.21), plays a crucial role in obtaining control in the interior. A significant point in this section is to prove that such an apriori $L^2$ bound on the curvature is not necessary to obtain control near the boundary; thus we show that control of the boundary metric itself gives control of the Einstein metric in a neighborhood of the boundary of definite size, cf. Corollary 4.10. Much of the proof of this result holds in fact in all even dimensions and conjecturally in all dimensions.

We begin with the following result, most of which was proved in [6], cf. also [7]. We fix $p > 4$ and denote the $L^p$ curvature radius by $\rho$ instead of $\rho^p$.

**Proposition 4.1.** *On a fixed 4-manifold $M$, let $g \in E_{AH}^{m,\alpha}$, $m \geq 4$, with boundary metric $\gamma$ satisfying $||\gamma||_{C^{m,\alpha}} \leq K$ in a fixed coordinate atlas for $\partial M$. Let $U_\delta = \{x \in \bar{M} : dist_{\bar{g}}(x, \partial M) \leq \delta\}$ where $\bar{g}$ is the geodesic compactification of $(M, g)$ with boundary metric $\gamma$. Suppose the $L^p$ curvature radius of $\bar{g}$ satisfies*

$$\rho(x) \geq \rho_0, \tag{4.1}$$

*for some constant $\rho_0 > 0$, for all $x \in U_{\delta_0}$, some $\delta_0 > 0$.*

*Then there is a $\delta_1 = \delta_1(\rho_0, \delta_0, K) > 0$ such that the $C^{m-1,\alpha}$ geometry of the geodesic compactification $\bar{g}$ is uniformly controlled in $U_{\delta_1}$, in that*

$$r_h^{m-1,\alpha} \geq r_0, \tag{4.2}$$

*where $r_0$ depends only on $\rho_0$, $\delta_0$, and $K$. The same statement holds if $m = \infty$ or $m = \omega$.*

**Proof:** By the discussion in §2, boundary regularity implies that $(M, g)$ has a $C^{m,\alpha}$ conformal compactification $\widetilde{g}$, with boundary metric $\gamma$. The metric $\widetilde{g}$ may be chosen to be of constant scalar curvature $\widetilde{s} = const$ and $\widetilde{g}$ is $C^{m,\alpha}$ in local harmonic coordinates. Again as noted in §2, the geodesic compactification $\bar{g}$ of $g$ with boundary metric $\gamma$ is then at least $C^{m-1,\alpha}$. Moreover, by [6, Prop. 2.1, Thm. 2.4 and Prop. 2.7], the $C^{m-1,\alpha}$ geometry of the metric $\bar{g}$ is uniformly controlled in $U_{\delta_1}$, i.e. (4.2) holds, for $\delta_1 = \delta_1(\rho_o, \gamma)$, provided the $C^{m,\alpha}$ geometry of the boundary metric $\gamma$ is uniformly bounded, (4.1) holds, and provided there is a uniform lower bound on the volume radius of $\bar{g}$ in $U_{\delta_1}$.

The control on the boundary metric and (4.1) are given by hypothesis. To obtain a lower bound on the volume radius $\nu$, observe that Lemma 3.8, cf. also (3.27), gives a lower bound on the volume ratios of balls, $volB_q(s)/s^4$ centered at points $q \in \partial M$, depending only on the $C^0$ geometry of the boundary metric. In particular, there is a constant $\nu_0 > 0$ such that $volB_q(\rho_0)/\rho_0^4 \geq \nu_0$, $q \in \partial M$, for all $(M, \bar{g}_i)$. Within the $L^p$ curvature radius, i.e. for balls $B_x(r) \subset B_q(\rho_0)$, standard volume comparison results imply then a lower bound $volB_x(r)/r^4 \geq \nu_1$, where $\nu_1$ depends only on $\nu_0$, cf. [4,§3]. This gives a uniform lower bound on the volume radius within $U_{\rho_0}$, and hence the result follows.

$\blacksquare$

Proposition 4.1 also holds locally, in that if (4.1) holds at all points in an open set $V$ containing a domain of fixed size in $\partial M$, then the conclusion holds in $V \cap U_{\delta_1}$. Also, this result holds for $m \geq 3$. Of course the arguments concerning the lower bound on the volume radius hold in all dimensions.

Proposition 4.1 should be understood as a natural strengthening of the boundary regularity property that AH Einstein metrics with $C^{m,\alpha}$ boundary metric have a $C^{m,\alpha}$ conformal compactification. Thus, it gives uniform control or stability of the metric $g \in E_{AH}^{m,\alpha}$ near the boundary, depending only on the boundary metric and a lower bound on $\rho$ as in (4.1). The main result of this section, Theorem 4.4 below, removes the assumption (4.1) and proves the estimate (4.2) independent of $\rho_0$ and $\delta_0$. Of course this could not be true locally; Theorem 4.4 is necessarily a global result.

The weak hypothesis (4.1) on the curvature radius may obviously be replaced by a stronger assumption. Thus, Proposition 4.1 implies, for instance, that in the geodesic compactification,

$$r_h^{k,\beta} \geq r_0 \Rightarrow r_h^{m-1,\alpha} \geq r_1, \tag{4.3}$$

for any $2 < k + \beta < m - 1 + \alpha$ with $r_1 = r_1(r_0, \delta_0, K)$. Thus, control of the metric in a weak norm, such as $L^{k,p}$ or $C^{k,\beta}$ implies control of the metric in a stronger norm, governed by the regularity of the metric at the boundary. We formalize this as follows:

**Definition 4.2.** In any dimension, an AH Einstein metric $g \in E_{AH}$ satisfies the *strong control property* if (4.3) holds, for some $(k, \beta)$ and $(m, \alpha)$ with $k + \beta < m - 1 + \alpha$, and $m - 1 \geq n$.

The proof of boundary regularity in [6], (cf. also [7]), uses the fact that AH Einstein metrics and their conformal compactifications satisfy the conformally invariant Bach equation in dimension 4. In a suitable gauge, this is a non-linear elliptic system of PDE and boundary regularity follows essentially from standard boundary regularity for such elliptic systems. The strong control property (4.3), (or its original version in Proposition 4.1), then follows from the fact that concurrent with boundary regularity, one has uniform elliptic estimates for solutions of the Bach equations near and up to the boundary, given weak control in such a region.

**Remark 4.3.** It will be noted in the proofs of the results below that all of the results of this section hold in all dimensions in which the strong control property holds. A recent result of Helliwell [25] proves boundary regularity for AH Einstein metrics in all even dimensions, by generalizing the proof in dimension 4 in [6], [7]. The idea is to use the Fefferman-Graham obstruction tensor in place of the Bach equations in higher dimensions; again in a suitable gauge, this gives a non-linear elliptic system. The proof given in [25] requires that $g$ be $C^{n,\alpha}$ conformally compact, although we suspect that it suffices for $g$ to be $C^{2,\alpha}$ conformally compact. In any case, it is straightforward to show from [25] that the strong control property (4.3) holds in all even dimensions with $n < k + \beta < m - 1 + \alpha$.

In odd dimensions, conformal compactifications of AH Einstein metrics are not $C^n$ up to the boundary, exactly due to the presence of the obstruction tensor $\mathcal{H}$. Nevertheless, one can define the spaces $C^{m,\alpha}$, $m \geq n$ in a polyhomogeneous sense. We conjecture that with such a suitable modification of the definition of harmonic radius, the strong control property holds in all odd dimensions; this is an interesting open question.

The main result of this section is the following:

**Theorem 4.4.** *Let $(M, g)$ be an AH Einstein metric on a 4-manifold with a $L^{2,p}$ conformal compactification, $p > 4$, and boundary metric $\gamma$. Suppose that $\gamma \in C^{m,\alpha}$, $m \geq 4$ and that, in a fixed coordinate system for $\partial M$, $||\gamma||_{C^{m,\alpha}} \leq K$, for some $\alpha > 0$. If $\bar{g}$ is the associated $C^{m-1,\alpha}$ geodesic* *compactification of $g$ determined by $\gamma$, then there are constants $\mu_0 > 0$ and $\rho_0 > 0$, depending only on $K$, $\alpha$, and $p$, such that*

$$\rho(x) \geq \rho_0, \quad \text{for all } x \text{ with } \; t(x) \leq \mu_0. \tag{4.4}$$

*The same result holds in all dimensions in which the strong control property holds, with $m \geq n+1$.*

The proof is rather long, and so is broken into several Propositions and Lemmas. Before beginning, it is worth pointing out that the curvature of $\bar{g}$ *at* $\partial M$ is uniformly bounded by the $C^2$ geometry of the boundary metric $\gamma$, cf. (A.8)-(A.10) in the Appendix. Thus, the geometry of $g$ at $\partial M$ is well-controlled to $2^{\text{nd}}$ order; one needs to extend this to control of $g$ near $\partial M$.

We will give the proof in all dimensions, assuming the strong control property holds. Also, we work with the $L^p$ curvature radius $\rho = \rho^p$, $p > n$, but one may work equally well with the $L^{k-2,q}$ curvature or $L^{k,q}$ harmonic radius; the arguments are exactly the same for these radii.

Let $\tau_0$ be the distance to the cutlocus of the normal exponential map from $\partial M$; $\tau_0(x) = dist_{\bar{g}}(x, \bar{C})$, where $\bar{C}$ is the cutlocus, as in (3.19). The first step is to prove that the function $\rho$ is bounded below by $\tau_0$ near $\partial M$. The second, more difficult, step is to prove that $\tau_0$ itself is uniformly bounded below near $\partial M$.

We begin with the following result.

**Proposition 4.5.** *For $(M, g)$ as in Theorem 4.4, there is a constant $c_0 > 0$, depending only on $K$, $\alpha$ and $p$, such that*

$$\rho(x) \geq c_0\tau_0(x), \quad \text{for all } x \text{ with } \; t(x) \leq 1. \tag{4.5}$$

**Proof:** The proof is by contradiction. If (4.5) is false, then there must exist a sequence of AH Einstein metrics $g_i$ on $M_i$, with $C^{n,\alpha}$ (polyhomogeneous) geodesic compactifications $\bar{g}_i$, for which the boundary metrics $\gamma_i$ satisfy

$$||\gamma_i||_{C^{n,\alpha}} \leq K, \tag{4.6}$$

but for which the $L^p$ curvature radius $\rho = \rho_i$ of $\bar{g}_i$ satisfies

$$\rho(x_i)/\tau_0(x_i) \rightarrow 0 \;\text{ as }\; i \rightarrow \infty, \tag{4.7}$$

on some sequence of points $x_i \in (\bar{M}_i, \bar{g}_i)$. Note that the ratio in (4.7) is scale-invariant. Without loss of generality, assume that $x_i$ realizes the minimal value of $\rho/\tau_0$ on $(M_i, \bar{g}_i)$. Of course $x_i$ may occur at $\partial M$.

Now blow-up or rescale the metrics $\bar{g}_i$ at $x_i$ to make $\rho(x_i) = 1$, i.e. set

$$g_i' = \lambda_i^2 \bar{g}_i, \tag{4.8}$$

where $\lambda_i = \rho(x_i)^{-1} \rightarrow \infty$. Let $\rho' = \lambda_i \rho$ and $\tau_0' = \lambda_i \tau_0$ be the $L^p$ curvature radius and distance to the cutlocus, with respect to $g_i'$. Then

$$\rho'(x_i) = 1, \tag{4.9}$$

and by the minimality property of $\rho(x_i)$,

$$\rho'(y_i) \geq \rho'(x_i)\frac{\tau_0'(y_i)}{\tau_0(x_i)} = \frac{\tau_0'(y_i)}{\tau_0(x_i)}. \tag{4.10}$$

for any $y_i \in \bar{M}$. Since (4.7) implies $\tau_0'(x_i) \rightarrow \infty$, it follows that

$$\rho'(y_i) \geq \frac{1}{2}, \tag{4.11}$$

for all $y_i$ with $dist_{g_i'}(x_i, y_i) \leq D$, for any given $D$, provided $i$ is sufficiently large. Of course one also has $\tau_0'(y_i) \rightarrow \infty$ for any such $y_i$.

It follows that the metrics $g_i'$ have uniformly bounded curvature, on the average in $L^p$ on all unit balls of $g_i'$-bounded distance to $x_i$. For clarity, we divide the proof into two cases, according to whether $dist_{g_i'}(x_i, \partial M)$ remains bounded or not.

**Case I. Bounded distance.**

Suppose $dist_{g_i'}(x_i, \partial M) \leq D$, for some $D < \infty$. We may then apply Proposition 4.1, or more precisely its local version as following the statement of Proposition 4.1, to conclude that a subsequence of the metrics $(M_i, g_i', x_i)$ converges in the $C^{n,\alpha'}$ (or stronger) topology to a limit $(N, g', x_\infty)$. (Of course the convergence is modulo diffeomorphisms). Since $\tau_0'(y_i) \to \infty$ for any $y_i$ within bounded $g_i'$-distance to $x_i$, the limit $(N, g')$ is a complete manifold with boundary. Moreover, since the unscaled boundary metrics $\gamma_i$ are uniformly bounded in $C^{n,\alpha}$, the limit $(\partial N, \gamma') = (\mathbb{R}^n, \delta)$, where $\delta$ is the flat metric.

Since the $L^p$ curvature radius is continuous in the $C^{n,\alpha'}$ topology, it follows from (4.9) that

$$\rho'(x_\infty) = 1. \tag{4.12}$$

The rest of the proof in this case is to prove that $(N, g')$ is flat. This clearly contradicts (4.12), and so will complete the proof.

Let

$$t'(x) = dist_{g'}(x, \partial N) = \lim_{i \to \infty} t_i'(x_i), \tag{4.13}$$

where $t_i'$ is the geodesic defining function for $(\partial M_i, g_i')$ and $x_i \to x$. Since $\partial M$ is totally geodesic in $(M, g_i')$, the smooth convergence implies $\partial N = \mathbb{R}^n$ is totally geodesic in $(N, g')$. Since $\tau_0' = \infty$ on $N$, $t'$ is globally defined and smooth on $N$.

We now use some of the curvature properties of geodesic compactifications given in the Appendix. The equations (A.1)-(A.3) for the curvatures $R'$, $Ric'$ and $s'$ also hold on $(N, g')$, with $t'$ in place of $t$. We note that the Hessian $D^2 t' = A$, where $A$ is the $2^{\text{nd}}$ fundamental form of the level sets of $t'$, and $\Delta t' = H$, the mean curvature of the level sets.

By (A.3), the scalar curvature $s'$ of $(N, g')$ is given by

$$s' = -2n\frac{\Delta t'}{t'}, \tag{4.14}$$

and by (A.11) satisfies

$$\dot{s}' \geq \frac{1}{2n^2} t'(s')^2, \tag{4.15}$$

where $\dot{s}' = \partial s'/\partial t'$. Since $s_{\gamma'} = 0$, (A.8) implies that $s' = 0$ on $\partial N$, and hence by (4.15), $s' \geq 0$ everywhere on $N$. Moreover, elementary integration of (4.15) implies that if, along a geodesic $\sigma = \sigma(t')$ normal to $\partial N$, $s'(t_0) > 0$, for some $t_0 > 0$, then $(t')^2 \leq t_0^2 + 4n^2/(s'(t_0))$, which is impossible, since $t' \to \infty$ along all geodesics normal to $\partial N$. It follows that one must have

$$s' \equiv 0 \ \text{ on } \ (N, g'). \tag{4.16}$$

From (4.14) and (4.16), we then have

$$\Delta t' = 0, \tag{4.17}$$

so that $t'$ is a smooth harmonic function on $N$, with $|\nabla t'| = 1$.

Now the Riccati equation (A.7) for the $t'$ geodesics on $(N, g')$ gives

$$|A|^2 + Ric(\nabla t', \nabla t') = 0, \tag{4.18}$$

On the other hand, the formula (A.2) for the Ricci curvature holds, so that on $(N, g')$,

$$Ric = -(n-1)t^{-1}D^2 t - t^{-1}(\Delta t)g' = -(n-1)t^{-1}D^2 t. \tag{4.19}$$

Here and below, we drop the prime from the notation. Clearly $D^2t(\nabla t, \nabla t) \equiv 0$ on $N$, and so by (4.18), $|A|^2 = 0$, i.e. all the level sets are totally geodesic. Since $A = D^2t$, (4.19) implies that $Ric \equiv 0$, so that $(N, g')$ is Ricci-flat. The vector field $\nabla t$ is thus a parallel vector field, and so $(N, g')$ splits as a product along the flow lines of $\nabla t$. Since $\partial N = \{t = 0\}$ is flat $\mathbb{R}^n$, it follows that $(N, g')$ is flat, as claimed.

**Case II. Unbounded distance.**
Suppose $t_i'(x_i) = dist_{g_i'}(x_i, \partial M) \to \infty$, as $i \to \infty$. Using (4.7) again, exactly the same arguments as in (4.14)-(4.19) applied to $(M_i, g_i', x_i)$ show that, for $i$ sufficiently large, $(M_i, g_i')$ almost splits in regions of bounded diameter about $x_i$, in that such regions are topologically products and the flow by the integral curves of $t_i'$ are almost isometries of $g_i'$.

More precisely, suppose first that $(M_i, g_i', x_i)$ does not collapse near $x_i$. Then, exactly as in Case I, by the strong control property, $(M_i, g_i', x_i)$ is close in the $C^{n,\alpha'}$ topology, to a product metric in regions of $g_i'$-bounded diameter about $x_i$. Suppose instead $(M_i, g_i', x_i)$ collapses near $x_i$. The strong control property implies the collapse is with bounded curvature. Clearly the collapse, which is caused by short geodesic loops, is transverse to the flow lines of $t_i'$. By the general collapse theory, cf. [15], the collapse may be unwrapped in local covering spaces. Thus, there are covering spaces of balls of fixed but small diameter about $x_i$ such that the lifted metrics on the covers do not collapse, and so converge in $C^{n,\alpha'}$ to a local limit. All of the estimates (4.14)-(4.19) remain valid on such local covers, and since these are pointwise estimates transverse to the collapsing directions, it follows again that the original metrics $(M_i, g_i', x_i)$ are close in $C^{n,\alpha'}$ to a product metric in regions of $g_i'$-bounded diameter about $x_i$.

This shows that $\rho_i'$ is almost constant in such regions, and, (again by the strong control property),

$$\frac{\partial \rho_i'}{\partial t_i'} \sim 0. \tag{4.20}$$

Note that $\partial\rho/\partial t$ is scale-invariant, (as are the ratios $\rho/t$ and $\rho/\tau_0$). The argument above was carried out at the base points $x_i$. However, exactly the same reasoning shows that (4.20) holds at all points $p_i$ where $(\rho/\tau_0)(p_i) << 1$. Now integrate (4.20) along integral curves $\sigma_i$ of $t_i'$ starting at points $y_i$ of bounded $g_i'$-distance to $x_i$ in the direction toward $\partial M$. Since $\rho_i'(y_i) \sim 1$ and $t_i'(y_i) \to \infty$, this implies in particular that the scale-invariant ratio $\rho/t$ on $(M_i, g_i')$ satisfies

$$\frac{\rho_i'}{t_i'} << 1, \tag{4.21}$$

along such curves, all the way to $\partial M$, since $\rho/\tau_0 << 1$ along such curves. However, this is clearly impossible, since again the arguments in Case I imply that $(M_i, g_i')$ is almost flat within bounded $g_i'$-distance to $\partial M$, so that $\rho_i' \geq (1 - \delta)t_i'$ in such regions, where $\delta$ is small for $i$ large. In other words, (4.20) and (4.21) prevent the ratio $\rho/t$, which is arbitrarily small around $x_i$, from increasing to near 1 near $\partial M$. This contradiction completes the proof.

■

Note that the opposite inequality to (4.5) also holds, i.e.

$$\rho(x) \leq \tau_0(x). \tag{4.22}$$

This is essentially a tautology, since the metric $\bar{g}$ is singular at the cutlocus and so $\rho$-balls are not defined past the cutlocus. The proof of Proposition 4.5 does not use the strong control property with $m - 1 \geq n$. Basically, it suffices to have this property in any norm stronger than the $L^{2,p}$ norm, so that the $L^p$ curvature radius is continuous. However, the condition $m - 1 > n$ will be crucial in the next result.

The next step in the proof of Theorem 4.3 is to prove that $\tau_0$ is uniformly bounded below near $\partial M$.

**Proposition 4.6.** *For $(M, g)$ as in Theorem 4.4, there is a constant $\mu_1 > 0$, depending only on $K$, $\alpha$ and $p$, such that, for any $x \in \partial M$,*

$$\tau_0(x) \geq \mu_1. \tag{4.23}$$

**Proof:** The proof of this estimate is much more subtle than that of Proposition 4.5. Propositions 4.1 and 4.5 are essentially local results; they hold for AH Einstein metrics defined only in a geodesic collar neighborhood of $\partial M$, $t \in [0, \varepsilon_0]$, for some $\varepsilon_0 > 0$. However, Proposition 4.6, (and hence Theorem 4.4), is global; it requires $g$ to be an AH Einstein metric defined on a compact manifold $M$, with conformal boundary $\partial M$ at infinity.

We first prove some preliminary lemmas, which are basically straightforward consequences of the strong control property. The actual proof begins after the proof of Lemma 4.9; on a first reading, one might want to start at this point to understand how the Lemmas are used. The crux of the proof is the use of the relation (2.22) in Proposition 2.4. It is this relation in particular, (as well as Lemma 4.7), which requires $(M, g)$ is global.

To begin, we examine the geometry of the geodesic compactification $\bar{g}$ in regions where $\tau_0$ is very small, in fact possibly arbitrarily small. Thus suppose $(M_i, g_i)$ is a sequence of AH Einstein metrics with boundary metrics $\gamma_i$ on $\partial M_i = \partial M$ uniformly controlled in $C^{m,\alpha}$, $m \geq n+1$. Hence, a subsequence of $\gamma_i$ converges in $C^{m,\alpha'}$ to a limit $C^{m,\alpha}$ boundary metric $\gamma$. Let $x_i$ be any sequence of points in $M_i$ such that

$$\tau_0(x_i) \to 0. \tag{4.24}$$

Rescale the compactified metrics $\bar{g}_i$ to

$$g_i' = (\tau_0(x_i))^{-2} \bar{g}_i, \tag{4.25}$$

so that

$$\tau_0'(x_i) = 1. \tag{4.26}$$

By Proposition 4.5 and (4.22), $\rho'(x_i)$ is uniformly bounded above and below. Hence as in the proof of Proposition 4.5, a subsequence of $(M, g_i', x_i)$ converges in the $C^{n,\alpha'}$ topology (modulo diffeomorphisms), to a maximal limit $(N, g', x_\infty)$; maximal here means the maximal connected domain on which the convergence is $C^{n,\alpha'}$. The manifold $N$ has boundary-at-infinity $\partial N$ a domain in $\mathbb{R}^n$, with boundary metric the flat metric $\gamma' = \delta$, and containing the ball $B_{x_\infty}(1)$ in $\mathbb{R}^n$. However, in this case, $N$ is not complete away from its boundary, since $\tau_0'(x_\infty) \leq 1$.

Consider now also the sequence of AH Einstein metrics $g_i$ on $M_i$, with conformal compactification $\bar{g}_i$, with base points $x_i$ chosen above. Of course $dist_{\bar{g}_i}(x_i, \partial M) \to 0$. The metrics $\bar{g}_i$ are rescaled up to $g_i' = \lambda_i^2 \bar{g}_i$, $\lambda_i = (\tau_0(x_i))^{-1}$, and converge, (in a subsequence), in $C^{n,\alpha'}$ to the maximal limit $(N, g', x)$. Now the rescaling of $\bar{g}_i$ just corresponds to changing the defining function $t_i$ to $t_i' = \lambda_i t_i$. Thus it does not change $g_i$ itself; in fact $g_i = (t_i')^{-2} g_i'$. Since $t_i' \to t'$ and $g_i' \to g'$ on $N$, it follows that the pointed sequence of AH Einstein metrics $(M_i, g_i, x_i)$ converges to a limit AH Einstein metric $(\mathcal{N}, g_\infty, x_\infty)$. Again here $\mathcal{N}$ is the maximal connected domain containing $x_\infty$ on which the convergence is $C^{n,\alpha'}$; in fact the convergence is now $C^\infty$ smooth on compact subsets, by regularity properties of Einstein metrics. The limit $(\mathcal{N}, g_\infty)$ is of course not conformally *compact* in general; the "compactification" of $(\mathcal{N}, g_\infty)$ by $t$ gives the manifold $(N, g')$, i.e.

$$g_\infty = t^{-2} g', \tag{4.27}$$

where $N \subset \mathcal{N}$ is the region where $t$ is smooth.

The first result is an analogue of (4.12) in this setting.

**Lemma 4.7.** *Any blow-up limit manifold $(N, g', x_\infty)$ is not flat.*

**Proof:** If $(N, g')$ is flat, then by (4.27), $(\mathcal{N}, g_\infty)$ is hyperbolic, i.e. of constant curvature -1, and so locally embeds in the hyperbolic space $\mathbb{H}^{n+1}(-1)$. By construction, the maximal limit $(\mathcal{N}, g_\infty)$ is not complete; it has a "boundary" corresponding to the points where $\rho_{g_i} \to 0$. However, the pointed sequence $(M_i, g_i, x_i)$ of complete manifolds has a uniform lower bound on Ricci curvature, since the metrics are Einstein, and is non-collapsing at points within bounded distance to $x_i$. By the Gromov weak compactness theorem [22], a subsequence of $(M_i, g_i, x_i)$ converges in the pointed Gromov-Hausdorff topology to a limit $(\bar{\mathcal{N}}, d, x)$, where $(\bar{\mathcal{N}}, d)$ is a *complete*, non-compact, length space. In particular, $(\bar{\mathcal{N}}, d)$ is geodesically complete; any two points may be joined by a minimizing geodesic and all geodesic balls $B_x(r)$ in $\bar{\mathcal{N}}$ have compact closure strictly contained in $\bar{\mathcal{N}}$.

Clearly the smooth domain $(\mathcal{N}, g_\infty)$ embeds in $(\bar{\mathcal{N}}, d)$. By a result of Cheeger-Colding [13], the singular set $\partial\mathcal{N} = \bar{\mathcal{N}} \setminus \mathcal{N}$ of $\bar{\mathcal{N}}$ is of codimension 2 in $\bar{\mathcal{N}}$. In particular the metric boundary $\partial\mathcal{N}$ is not a topological boundary, (which would have codimension 1).

By the volume comparison theorem (3.31) on $(M_i, g_i)$, the ratio $volB_z^i(r)/volB_{-1}(r)$ is monotone non-increasing in $r$, where $B_z^i(r)$ is the geodesic $r$-ball about any $z$ in $(M_i, g_i)$ and $volB_{-1}(r)$ is the volume of the geodesic $r$-ball in $\mathbb{H}^{n+1}(-1)$. Further, for any given $i$, $\lim_{r\to 0} volB_z^i(r)/volB_{-1}(r) = 1$. A fundamental result of Colding [14], cf. also [13], shows that the volume of geodesic balls is continuous under Gromov-Hausdorff limits so that in $(\bar{\mathcal{N}}, d)$, $volB_z(r)/volB_{-1}(r)$ is also monotone non-decreasing, and hence $\lim_{r\to 0} volB_z(r)/volB_{-1}(r) \leq 1$, with equality for $z \in \mathcal{N}$.

Now suppose first for simplicity of argument that $\mathcal{N}$ is simply connected. Since $\mathcal{N}$ is of constant curvature $-1$, the developing map based at any point in $\mathcal{N}$ gives an isometric immersion $F : \mathcal{N} \to H^{n+1}(-1)$. Observe that the image of $F$ is of full measure in $\mathbb{H}^{n+1}(-1)$; this is because $\bar{\mathcal{N}}$ is complete, so that geodesics do not terminate, and $\partial\mathcal{N}$ is of codimension 2 and so of measure 0. In fact $F$ extends to a continuous map of $\bar{\mathcal{N}}$ onto $\mathbb{H}^{n+1}(-1)$.

It follows that $F$ maps $B_z(r)$ onto $B_{F(z)}(r) \subset \mathbb{H}^{n+1}(-1)$ modulo sets of measure 0. Moreover, $F$ preserves the volumes of these balls when counting multiplicities of the image, so that

$$\frac{volB_z(r)}{volB_{-1}(r)} \geq \frac{volF(B_z(r))}{volB_{-1}(r)} = 1. \tag{4.28}$$

Since by the monotonicity above one has

$$\frac{volB_z(r)}{volB_{-1}(r)} \leq 1, \tag{4.29}$$

for all $r$, it is then immediate that

$$\frac{volB_z(r)}{volB_{-1}(r)} = 1, \tag{4.30}$$

for all $r$. This of course implies that $\bar{\mathcal{N}}$ is isometric to $\mathbb{H}^{n+1}(-1)$, by the volume rigidity theorem for Ricci curvature. Moreover, by [2, Thm. 3.2], the sequence $(M_i, g_i, x_i)$ now converges smoothly to $(\bar{\mathcal{N}}, g_\infty, x_\infty)$ everywhere. In particular, $\tau_0 = \infty$, contradicting the fact that $\tau_0(x_\infty) = 1$ from (4.26).

The proof is similar when $\mathcal{N}$ is not simply connected. Thus, let $\widetilde{\mathcal{N}}$ be the universal cover of $\mathcal{N}$. (For illustration, it is useful to picture $\widetilde{\mathcal{N}}$ as a branched cover of $\bar{\mathcal{N}}$ branched over the singular locus $\partial\mathcal{N}$). Then the developing map $F$ is an isometric immersion of $\widetilde{\mathcal{N}}$ into $\mathbb{H}^{n+1}(-1)$. Let $D$ be a Dirichlet fundamental domain for the action of $\pi_1(\mathcal{N})$ on $\widetilde{\mathcal{N}}$, based at a point $z \in \mathcal{N}$. Modulo sets of measure 0, $D$ may be identified with $\mathcal{N}$, and $F$ gives an isometric immersion $F : D \to \mathbb{H}^{n+1}(-1)$. Now however $\partial D$ is of codimension 1 and $D$, (or $\bar{D}$), is not geodesically complete; the action of $\pi_1(\mathcal{N})$ identifies subsets of $\partial D$ to obtain the manifold $\mathcal{N}$ or its completion $\bar{\mathcal{N}}$. Nevertheless, letting $S_z(r)$ be the geodesic $r$-sphere about $z$ in $N$, or equivalently in $D$ modulo sets of measure 0, and letting $S_{F(z)}(r)$ be the corresponding sphere in $\mathbb{H}^{n+1}(-1)$, we claim that

$$\limsup_{r\to\infty} \frac{volS_z(r)}{vol_{-1}(S_{F(z)}(r))} \geq 1. \tag{4.31}$$

To prove (4.31), recall the construction of the blow-up limit $\mathcal{N}$ and its compactification $N$ with conformal infinity given by the domain $\partial N \subset \mathbb{R}^n$. View the codimension 2 singular set $S = \partial\mathcal{N}$ as a subset of the compactification $\bar{N}$, and suppose first for clarity that $S$ intersects the full conformal boundary $\mathbb{R}^n$ in a closed set $S_\infty$ also of codimension 2 in $\mathbb{R}^n$. Then $\partial N = \mathbb{R}^n \setminus S_\infty$, and so in particular $\partial N$ is of full measure in $\mathbb{R}^n$. As $r \to \infty$, the geodesic spheres $S_z(r)$ in $N$, (or $D$), tend to $\partial N$. Then, as in the proof of (4.28), (4.31) follows from the fact that $\partial N$ is a set of full measure in the conformal compactification $S^n = \mathbb{R}^n \cup \{\infty\}$ of $\mathbb{H}^{n+1}(-1)$.

Now the same argument holds regardless of the exact structure of the singular set $S_\infty \subset \mathbb{R}^n$, since it is still the case that $S_z(r) \subset N$ is of full measure in $\bar{S}_z(r) \subset \bar{N}$ and $\lim_{r\to\infty} \bar{S}_z(r) = \mathbb{R}^n$.

By integration, the estimate (4.31) also holds for balls $B_z(r)$ and the relation (4.30) then follows from (4.29) as before. ■

We point out that the same proof shows that $(N, g', x_\infty)$ cannot be conformally flat. This is because (4.27) would then imply that the metric $(\mathcal{N}, g_\infty)$ is conformally flat and Einstein, and hence again hyperbolic; the proof then proceeds just as before.

In the following, we write

$$\phi \sim 1, \tag{4.32}$$

if there is a constant $C < \infty$ such that $C^{-1} \leq \phi \leq C$. The data $\phi$ will be determined by $(M, g)$ as in Theorem 4.4, but $C$ is required to be independent of $(M, g)$. The next result, which is of independent interest, relates the curvature radius $\rho$ with the stress-energy term $\tau_{(n)}$ in (2.12).

**Lemma 4.8.** *For $(M, g)$ as in Theorem 4.4, and for any $x \in \partial M$ with $\tau_0(x)$ sufficiently small, one has the estimate*

$$\rho^n(x) \cdot \sup_{y\in B_x(\frac{1}{2}\rho(x))} |\tau_{(n)}|(y) \sim 1. \tag{4.33}$$

*The same estimate holds for $g_{(n)}$ in place of $\tau_{(n)}$.*

**Proof:** Observe that the product in (4.33) is scale invariant, cf. also (2.14). Suppose first that there exists a sequence of metrics $(M_i, g_i)$ and points $x_i \in \partial M_i$, such that

$$\rho^n(x_i) \cdot \sup_{y\in B_{x_i}(\frac{1}{2}\rho(x_i))} |\tau_{(n,i)}|(y) \to 0. \tag{4.34}$$

We work in the scale $\hat{g}_i = \rho(x_i)^{-2}\bar{g}_i$ where

$$\hat{\rho}(x_i) = 1, \tag{4.35}$$

so that $\hat{\tau}_0(x_i) \sim 1$. The estimate (4.34) implies $|\hat{\tau}_{(n,i)}|(y) << 1$, for all $y \in \hat{B}_{x_i}(\frac{1}{2})$. By Proposition 4.1, (or the strong control property), the rescaled metrics $\hat{g}_i$ converge (in a subsequence) in the $C^{n,\alpha'}$ topology to a limit metric $\hat{g}$ on a maximal connected domain $(U, x_\infty)$ where $\hat{\rho}$ does not converge to 0, (i.e. the region where the curvature of $\hat{g}_i$ does not blow-up in $L^p$). Of course $B_{x_\infty}(1) \subset U$. As noted prior to Lemma 4.7, the blow-up limit of the metric $\gamma$ on $\partial M$ is the flat metric on $\mathbb{R}^n$.

Now the $C^{n,\alpha'}$ convergence implies that

$$\tau_{(n)} = 0, \tag{4.36}$$

on $\partial U$. Since the boundary metric is flat, this implies that $g_{(n)} = 0$, cf. [18], [33]. The unique continuation property, Proposition 2.1, then implies that $(U, \hat{g})$ is flat, which however contradicts Lemma 4.7.

The proof of the opposite inequality in (4.33) is similar. Thus, suppose there exist $(M_i, g_i)$ and points $x_i \in \partial M$ such that

$$\rho^n(x_i) \cdot \sup_{y \in B_{x_i}(\frac{1}{2}\rho(x_i))} |\tau_{(n,i)}|(y) >> 1.$$

Now work in the scale where $\sup_{y \in B_{x_i}(\frac{1}{2}\rho(x_i))} |\tau_{(n,i)}|(y) = 1$, and at points $y_i$ realizing this supremum. In this scale, (using the fact that $\rho$ is Lipschitz, with Lipschitz constant 1), $\rho(y_i) >> 1$. Hence, via (4.22), $\tau_0(y_i) >> 1$ in this scale. Then the same arguments as in (4.14)-(4.19) together with the strong control property imply that the metric is $C^{n,\alpha'}$ close to the flat metric, in bounded domains about $y_i$. It follows that $\tau_{(n)}$ and $g_{(n)}$ are (arbitrarily) small in bounded domains about $y_i$ for $i$ sufficiently large, which gives a contradiction to the scale normalization above.

■

Note that the strong control property implies further that the product in (4.33) is bounded away from 0 and $\infty$ in $C^{\alpha'}$.

Now consider a conformal rescaling of the metric $\bar{g}$ in place of the constant rescalings used before. Thus, set

$$\widetilde{g} = \rho(x)^{-2}\bar{g}. \tag{4.37}$$

Apriori, the function $\rho$ may not be smooth in $x$, and so (4.37) should be replaced by the expression $\widetilde{g} = \rho_s(x)^{-2}\bar{g}$, where $\rho_s$ is a $C^\infty$ smoothing of $\rho$ satisfying $\frac{1}{2}\rho \leq \rho_s \leq \rho$. Thus, in the following, (4.37) is understood with respect to $\rho_s$ in place of $\rho$; however, to keep the notation reasonable, and also because this difference is only of minor significance, we continue to work the notation $\rho$.

If one sets $\hat{g} = \tau_0^{-2}(x)\bar{g}$, then by (4.5) and (4.22) the two metrics $\widetilde{g}$ and $\hat{g}$ are $C$-quasi-isometric, i.e.

$$\hat{g} \sim \widetilde{g}, \tag{4.38}$$

in the sense of (4.32).

Let $\widetilde{\rho}$ be the $L^p$ curvature radius of $(M, \widetilde{g})$ and similarly for $\widetilde{r}_h^{m-1,\alpha}$. Also, let $\widetilde{g}_{(n)}$ be the $n^{\text{th}}$ term in the Fefferman-Graham expansion (2.9)-(2.10) for the geodesic compactification of $(M, g)$ determined by the boundary metric $\widetilde{\gamma} = \widetilde{g}|_{\partial M}$, and similarly for the stress-energy term $\widetilde{\tau}_{(n)}$.

**Lemma 4.9.** *Let $(M, g)$ be as in Theorem 4.4. Then on $(M, \widetilde{g})$, for all $x \in \partial M$, one has the estimates*

$$\widetilde{\rho}(x) \sim 1, \quad \text{and} \quad \widetilde{r}_h^{m-1,\alpha}(x) \sim 1. \tag{4.39}$$

*These estimates also hold for the geodesic compactification determined by $\widetilde{\gamma}$. Further, for $\rho(x)$ sufficiently small,*

$$\sup_{y \in B_x(\frac{1}{2})} \widetilde{\tau}_{(n)}(y) \sim 1. \tag{4.40}$$

**Proof:** We first prove $\widetilde{\rho}$ is bounded below on $\partial M$. To do this, standard formulas for the behavior of the curvature under conformal changes give

$$\widetilde{R} = \rho^{-2}[R - g \wedge (D^2 \log \rho^{-1} - (d \log \rho)^2 + \tfrac{1}{2}|d \log \rho|^2 g)],$$

where $\wedge$ denotes the Kulkarni-Nomizu product, cf. [10]. Hence,

$$|\widetilde{R}|_{\widetilde{g}} \leq c(\rho^2|R| + \rho|D^2\rho| + |d\rho|^2), \tag{4.41}$$

where the right side is taken with respect to $\bar{g}$ and we recall the remark on smoothing following (4.37). Note also that the right side of (4.41) is scale invariant. Thus, it suffices to obtain an $L^p$ or $L^\infty$ bound of the terms on the right in (4.41), in the scale where $\rho = 1$, on balls of radius $r_0$, for a uniform $r_0 > 0$. Since the function $\rho$ is Lipschitz, with Lipschitz constant 1, the last term is bounded in $L^\infty$. By the strong control property, the first two terms are also bounded in $L^\infty$ on the ball of radius $\frac{1}{2}$. This gives a uniform upper bound on $|\widetilde{R}|_{\widetilde{g}}$ on the ball of radius $\frac{1}{2}$ and hence

$$\widetilde{\rho} \geq \rho_0,$$

for a fixed $\rho_0$, (depending only on the constant $c_0$ in (3.4)). Also, since $(M, \bar{g})$ satisfies the strong control property, in the scale where $\rho(x) = 1$ one has a uniform lower bound on $r_h^{m-1,\alpha}$, $r_h^{m-1,\alpha}(x) \geq r_0$, which again via (4.41) gives a lower bound on $\widetilde{r}_h^{m-1,\alpha}$ on $\partial M$.

The opposite estimate $\widetilde{\rho}(x) \leq \rho_1 \rho(x)$, for a fixed $\rho_1 < \infty$, also holds. For if, on some sequence $(M_i, \bar{g}_i, x_i)$, $\widetilde{\rho}(x_i) \to \infty$ while $\rho(x_i) = 1$, then the metric $\widetilde{g}_i$ is almost flat in $L^p$ in large balls $B_{x_i}((1-\delta)\widetilde{\rho}(x_i))$, for any fixed $\delta > 0$. It follows that any maximal limit $(N, g'_\infty, x_\infty)$, as following (4.25), is conformally flat. As noted following Lemma 4.7, this contradicts (the proof of) Lemma 4.7. Combining the arguments above proves (4.39).

Next, the estimate (4.39) gives uniform control in $C^{m-1,\alpha}$ of the conformal factor relating the geodesic compactification with respect to $\widetilde{g}$ and the rescaling of $\bar{g}$ by the constant factor $\rho(x)^{-2}$, for any fixed $x$. This implies that (4.39) also holds for the geodesic compactification with respect to $\widetilde{g}$. In particular, this gives an upper bound on the term $\widetilde{\tau}_{(n)}$ in (4.40).

Regarding the lower bound, suppose first $n$ is odd. Then the lower bound on $\widetilde{\tau}_{(n)}$ in (4.40) follows immediately from (4.33) and the conformal transformation rule (2.14). A similar argument holds also for $n$ even. Thus, although (2.14) does not hold exactly, it does hold modulo terms which involve lower order derivatives of the boundary metric; for $\rho(x)$ sufficiently small, these terms scale at a lower power of $\rho(x)^{-1}$ than $g_{(n)}$, so that (2.14) holds to leading order.

■

We are now in position to begin the proof of Proposition 4.6 per se. Suppose then (4.23) is false. Then there exist AH Einstein metrics $(M_i, g_i)$ with $C^{m,\alpha}$ controlled boundary metrics $\gamma_i$ and points $x_i$ such that $\tau_0(x_i) \to 0$, and so

$$\rho_i(x_i) \to 0, \tag{4.42}$$

where $\rho_i$ is the $L^p$ curvature radius of the geodesic compactification $\bar{g}_i$ of $g_i$ with boundary metric $\gamma_i$. From now on, we work in the conformally rescaled metric (4.37). The Lipschitz property of $\rho$, together with (4.42) implies that

$$vol_{\widetilde{\gamma}_i}(\partial M) \to \infty. \tag{4.43}$$

The main point now is to use the relation (2.22) in the $\widetilde{\gamma}_i$ metric:

$$\int_{\partial M} \langle \mathcal{L}_X \tau_{(n)} + [(1-\tfrac{2}{n}) divX]\tau_{(n)}, h_{(0)} \rangle dV_{\widetilde{\gamma}_i} = \int_{\partial M} \langle \sigma_{(n)} + \tfrac{1}{2} tr h_{(0)} \tau_{(n)}, \hat{\mathcal{L}}_X \gamma \rangle dV_{\widetilde{\gamma}_i} + b_{(n)}. \tag{4.44}$$

Here we have dropped the $i$ and tilde from the notation; the inner product in (4.44) as well as $\tau_{(n)}$ and $\sigma_{(n)}$ are taken with respect to $\widetilde{\gamma}_i$; also the term $b_{(n)}$ is given by the last term in (2.22). Recall from (2.12) that $\tau_{(n)}$ is determined by $g_{(n)}$ and the boundary metric. By (2.21) and Lemma 2.2, $\sigma_{(n)}$ is any solution of the system

$$\delta \sigma_{(n)} = -\delta'(\tau_{(n)}), \tag{4.45}$$

$$tr \sigma_{(n)} = -tr'(\tau_{(n)}) + (a_{(n)})',$$

where $\delta'$, $tr'$ and $(a_{(n)})'$ are the variations of $\delta$, $tr$ and $a_{(n)}$ at $(\partial M, \widetilde{\gamma}_i)$ in the direction $h_{(0)}$. Of course solutions $\sigma_{(n)}$ of (4.45) are invariant under the addition of transverse-traceless terms, as is (4.44). We point out that the system of equations (4.45) is conformally invariant, since these equations define formal solutions to the conformally compactified linearized Einstein equations. In more detail, a conformal change of the boundary metric induces a conformal change of $h_{(0)}$; as described in and following (2.14), such conformal changes in turn induce a transformation of $\tau_{(n)}$,

$a_{(n)}$ and $\sigma_{(n)}$. The space of solutions of (4.45) is thus transformed into itself under changes in the conformal compactification of solutions of the linearized Einstein equations.

Consider then the equations (4.45) on $(\partial M, \gamma_i)$, and set $\gamma_i = \gamma$. We first claim that the system (4.45) is solvable for any variation $h_{(0)}$ of $(\partial M, \gamma)$, and hence, by Proposition 2.4, (4.44) also holds for all such variations $h_{(0)}$. To see this, by Lemma 2.2 and Proposition 2.3, we know that each equation in (4.45) is individually solvable, and thus need to show that the equations are simultaneously solvable. As is well-known, any symmetric bilinear form $\sigma$ on $(\partial M, \gamma)$ can be written uniquely as $\sigma = \delta^* V + f\gamma + k$, where $k$ is transverse-traceless. Setting $-\delta'(\tau_{(n)}) = \phi_1$, $-tr'(\tau_{(n)}) + a'_{(n)} = \phi_2$, the system (4.45) becomes

$$\delta\delta^* V - \frac{1}{n} d\delta V = \phi_1 + \frac{1}{n} d\phi_2, \tag{4.46}$$

$$-\delta V + nf = \phi_2.$$

By Proposition 2.3, the term $\phi_1 \in Im\delta$, and similarly $d\phi_2 \in Im\delta$, since for any Killing field $X$ on $(\partial M, \gamma)$, $\langle d\phi_2, X\rangle = \langle \phi_2, \delta X\rangle = 0$. Thus, the first equation is solvable for $V$, and hence the second equation is solvable for $f$, which proves the claim. Note that the first equation in (4.46) is an elliptic equation for $V$.

Next, we claim there is a solution $\sigma_{(n)} = \delta^* V + f\gamma$ of (4.45) on $(\partial M, \gamma)$, $(\gamma = \gamma_i)$, such that

$$||\sigma_{(n)}||_{C^{m-n,\alpha}} \leq C||h_{(0)}||_{C^{m-n,\alpha}}||\tau_{(n)}||_{C^{m-n,\alpha}}, \tag{4.47}$$

where $C$ depends only on $K$ in Theorem 4.4. This follows directly from standard elliptic estimates for $V$ associated to the equation (4.46), (choosing $V$ orthogonal to the kernel of the operator), together with the fact that $\gamma_i$ are uniformly bounded in $C^{m,\alpha}$.

Transforming then to the conformally related metrics $\widetilde{\gamma}_i$ and using the fact (4.39) that $\tau_{(n)}$ is uniformly controlled in $C^{\alpha}$ then gives

$$||\sigma_{(n)}||_{C^{\alpha}} \leq C||h_{(0)}||_{C^{\alpha}}, \tag{4.48}$$

for a fixed constant $C$, independent of $i$, where $\sigma_{(n)}$ and the norms are taken with respect to the $\widetilde{\gamma}_i$ metric.

Let $Z_i = \{x \in \partial M : \rho_i(x) \leq \varepsilon_i\}$, for some sequence $\varepsilon_i \to 0$. For any $x_i \in Z_i$, by Lemma 4.9 the pointed sequence $(\partial M, \widetilde{\gamma}_i, x_i)$ converges in $C^{n,\alpha'}$, and uniformly on compact sets, to a complete conformally flat metric $\widetilde{\gamma}_\infty$ on a domain $V \subset \partial M$. The metric $\widetilde{\gamma}_\infty$ is conformally flat because one is conformally blowing up the sequence of smoothly controlled metrics $\gamma_i$ on $\partial M$. Of course, the convergence is modulo diffeomorphisms which blow-up small regions in $\partial M$ to unit size. Thus, there exist embeddings $F_i : V \to \partial M$ such that $(F_i)^*\widetilde{\gamma}_i \to \widetilde{\gamma}_\infty$. For example, if $x_i$ realizes the minimal value of $\rho_i$ in (4.42), then $\widetilde{\gamma}_\infty$ is a complete conformally flat metric on $\mathbb{R}^n$. Other choices of base points may lead to complete conformally flat metrics on $\mathbb{R} \times S^{n-1}$ for example. In general, since the conformal class $[\gamma_i]$ is controlled in $C^{m,\alpha}$, the limit domain $V$ is contained in $\mathbb{R}^n$, with closure $\bar{V} = \mathbb{R}^n$. Similarly, the forms $\tau_{(n,i)}$ and $\sigma_{(n,i)}$ converge in $C^{\alpha'}$ to limit forms $\tau_{(n)}$ and $\sigma_{(n)}$ on $V$; recall that the tilde has been dropped from the notation.

For the same reasons, since the metrics $\widetilde{g}_i$ satisfy (4.39), the pointed sequence $(M_i, \widetilde{g}_i, x_i)$ also converges, in $C^{n,\alpha'}$, to a maximal connected limit $(N, \widetilde{g}_\infty, x_\infty)$, with boundary data $(V, \widetilde{\gamma}_\infty)$.

Let $Y$ be any $C^2$ smooth vector field on $\partial M$. The vector fields $X_i = (F_i)^*Y$ are blow-ups of $Y$ based at $x_i$ and converge to a conformal Killing field $X_\infty$ on $(V, \widetilde{\gamma}_\infty)$, so that

$$\hat{\mathcal{L}}_{X_i}\widetilde{\gamma}_i \to \hat{\mathcal{L}}_{X_\infty}\widetilde{\gamma}_\infty = 0. \tag{4.49}$$

Similarly, since the metrics $\gamma_i$ are bounded in $C^{n,\alpha}$, the terms $a_{(n)}$ are also bounded, and hence, wirh respect to the boundary metrics $\widetilde{\gamma}_i$, $(a_{(n)})_i \to 0$ pointwise. Combining then (4.44) with (4.43) and (4.48), it thus follows that for $i$ large,

$$\sup_{|Y|\leq 1}\sup_{|h_{(0)}|\leq 1}|\int_{\partial M}\langle\widetilde{\mathcal{L}}_{X_i}\tau_{(n,i)},h_{(0)}\rangle dV_{\widetilde{\gamma}_i}|<<vol_{\widetilde{\gamma}_i}(\partial M),\tag{4.50}$$

where $\widetilde{\mathcal{L}}_X\tau_{(n)}=\mathcal{L}_X\tau_{(n)}+[(1-\frac{2}{n})divX]\tau_{(n)}$, $|h_{(0)}|=||h_{(0)}||_{C^\alpha(\widetilde{\gamma}_i)}$, and $|Y|=||Y||_{C^2(\gamma_i)}\sim||Y||_{C^2(\gamma_\infty)}$. Since $h_{(0)}$ is arbitrary, it follows from (4.50) that there exist (possibly many) regions where $\rho_i\to 0$, for which, on a complete, conformally flat limit $(V,\widetilde{\gamma}_\infty,x_\infty)$ of the boundary, one has

$$\widetilde{\mathcal{L}}_{X_\infty}\tau_{(n)}=0,\tag{4.51}$$

for any conformal Killing field $X_\infty$ on $(V,\widetilde{\gamma}_\infty)$. We claim that the only solution of (4.51), (for all conformal Killing $X_\infty$), is

$$\tau_{(n)}=0.\tag{4.52}$$

To see this, a well-known relation gives $\mathcal{L}_X\tau_{(n)}=\nabla_X\tau_{(n)}+2\delta^*X\circ\tau_{(n)}$, so that if $X=X_\infty$ is conformal Killing, then $\mathcal{L}_{X_\infty}\tau_{(n)}=\nabla_{X_\infty}\tau_{(n)}+\frac{2}{n}(divX_\infty)\tau_{(n)}$. Hence (4.51) gives

$$\nabla_{X_\infty}\tau_{(n)}=-(divX_\infty)\tau_{(n)},\tag{4.53}$$

for all conformal Killing fields $X_\infty$. Pairing (4.53) with $\tau_{(n)}$ gives

$$\frac{1}{2}X_\infty(|\tau_{(n)}|^2)=-(divX_\infty)|\tau_{(n)}|^2.\tag{4.54}$$

Hence, if $\tau_{(n)}\neq 0$, then $X_\infty(\log|\tau_{(n)}|)=-divX_\infty$, so that by (4.53),

$$\nabla_{X_\infty}(|\tau_{(n)}|^{-1}\tau_{(n)})=0.$$

Since $(V,\widetilde{\gamma}_\infty)$ is conformally flat and $X_\infty$ is an arbitrary conformal Killing field, this easily implies that $|\tau_{(n)}|$ is constant, which from (4.54) implies (4.52).

Since the convergence of $(M_i,\widetilde{g}_i,x_i)$ to $(N,\widetilde{g}_\infty,x_\infty)$ is $C^{n,\alpha'}$, (4.52) contradicts (4.39), which proves Proposition 4.6. (Equivalently, (4.52) can be shown to imply that the the blow-up limit $(N,g',x_\infty)$ of $(M_i,\bar{g}_i,x_i)$ as preceding Lemma 4.7 is flat, contradicting Lemma 4.7). Propositions 4.5 and 4.6 together imply Theorem 4.4, which also completes the proof of this result.

■

Combining Proposition 4.1 and Theorem 4.4 gives the following main result of this section.

**Corollary 4.10.** *On a fixed 4-manifold $M$, suppose $g\in E^{m,\alpha}_{AH}$, $m\geq 4$, has boundary metric $\gamma$ satisfying $||\gamma||_{C^{m,\alpha}}\leq K$ in a fixed coordinate atlas for $\partial M$. Then there exists $\delta_1=\delta_1(K)>0$ such that the $C^{m-1,\alpha}$ geometry of the geodesic compactifications $\bar{g}$ is uniformly controlled in $U_{\delta_1}$, in that* (4.2) *holds, with $r_0$ depending only on $K$.*

*The same result holds in all dimensions in which the strong control property holds.*

■

## 5. Properness of the Boundary Map.

In this section, we combine the results of §3 and §4 to prove Theorem A, i.e. the properness of $\Pi$. Following this, in §5.2, we also analyse the behavior when $\Pi$ is not proper and prove this is solely due to the formation of Einstein cusp metrics, cf. Theorem 5.4 and the discussion following it.

**5.1.** Corollary 4.10 gives uniform $C^{m-1,\alpha}$ control of the geodesic compactification of an AH Einstein metric $(M,g)$ in a neighborhood of definite size about $\partial M$, in terms of $C^{m,\alpha}$ control of the boundary metric $\gamma$. Theorem 3.7 addresses control in the interior, away from the boundary. Regarding the hypotheses (3.17)-(3.21) of Theorem 3.7, the assumption (3.19) on a lower bound on the inradius of $\bar{g}$ and the assumption (3.20) on an upper bound on the diameter $T$ of $S(t_1)$ are now immediate consequences of Corollary 4.10. Next we show that there is a global $L^2$ bound on $W$, i.e. (3.21) holds.

**Proposition 5.1.** *Let $(M, g)$ be an AH Einstein metric on a 4-manifold $M$, with boundary metric $\gamma$. Then there is a constant $\Lambda$, depending only on the topology of $M$ and the $C^{4,\alpha}$ norm of $\gamma$, such that*

$$\int_M |W_g|^2 dV_g \leq \Lambda. \tag{5.1}$$

**Proof:** The integral (5.1) is conformally invariant. In the tubular neighborhood $U = U_{\mu_1}$ about $\partial M$, (cf. (4.23)), we compute the $L^2$ integral (5.1) with respect to the compactification $\bar{g}$, while in $M \setminus U$, the integral is computed with respect to the Einstein metric $g$.

Thus, by Corollary 4.10, the curvature $\bar{R}$ of $\bar{g}$ is uniformly bounded in $U$. Since $vol_\gamma \partial M$ is also uniformly bounded, the curvature bound also gives a uniform upper bound on the volume of $(U, \bar{g})$. Hence, by conformal invariance, the integral over $U$ in (5.1) is uniformly bounded.

For the integral over $M \setminus U$, we use the Chern-Gauss-Bonnet theorem for manifolds with boundary, as in [5]. Let $S(t_0) = \partial U_{\mu_0}$, $\mu_0 = \mu_1/2$, viewed as a boundary in $(M, g)$, and let $\Omega = M \setminus U_{\mu_0}$. Since $(\Omega, g)$ is Einstein, one has,

$$\frac{1}{8\pi^2}\int_\Omega |W|^2 dV = \frac{1}{8\pi^2}\int_\Omega (|R|^2 - 6) = \chi(\Omega) - \frac{3}{4\pi^2} vol\Omega + \int_{S(t_1)} B(R, A) \leq \chi(M) + \int_{S(t_1)} B(R, A);$$

compare also with (3.16). Here $B(R, A)$ is a boundary term, depending on the curvature $R$ and second fundamental form $A$ of $S(t_0)$ in $(M, g)$. But again by Corollary 4.10, $R$ and $A$ are uniformly controlled on $S(t_0) \subset (M, g)$, as is $vol_g S(t_0)$. Hence, the $L^2$ norm of $W$ over $\Omega$ is uniformly bounded and so (5.1) follows.

■

We now assemble the work above to obtain the following result, which represents the major part of Theorem A. Recall from [6] that if $(M, g)$ is an AH Einstein metric and $\bar{g}$ is a geodesic compactification, then its width $Wid_{\bar{g}} M$ is defined by

$$Wid_{\bar{g}} M = \sup\{t(x) : x \in M\}. \tag{5.2}$$

The width depends on a choice of the boundary metric $\gamma$ for the conformal infinity. However, if $\gamma$ and $\gamma'$ are representatives in $[\gamma]$, then

$$C^{-1} Wid_{\bar{g}} M \leq Wid_{\bar{g}'} M \leq C Wid_{\bar{g}} M,$$

where $C$ depends only on the $C^1$ norm of $\gamma^{-1}\gamma'$ and $(\gamma')^{-1}\gamma$ in a fixed coordinate system on $\partial M$.

**Theorem 5.2.** *Let $\{g_i\}$ be a sequence of AH Einstein metrics on $M$, with boundary metrics $\gamma_i$ and suppose $\gamma_i \to \gamma$ in the $C^{m,\alpha'}$ topology on $\partial M$, $m \geq 4$. Suppose further that*

$$H_2(\partial M, \mathbb{R}) \to H_2(\bar{M}, \mathbb{R}) \to 0, \tag{5.3}$$

*and there is a constant $D < \infty$ such that*

$$Wid_{\bar{g}_i} M \leq D. \tag{5.4}$$

*Then a subsequence of $\{g_i\}$ converges smoothly and uniformly on compact subsets to an AH Einstein metric $g$ on $M$ with boundary metric $\gamma$. The geodesic compactifications $\bar{g}_i$ converge in the $C^{m-1,\alpha'}$ topology to the geodesic compactification $\bar{g}$ of $g$, within a fixed collar neighborhood $U$ of $\partial M$.*

**Proof:** Corollary 4.10 proves the last statement. As noted above, together with Proposition 5.1, it follows that the all the hypotheses (3.17)-(3.21) of Theorem 3.7 are satisfied. Thus if one chooses base points $x_i$ satisfying (3.22), then a subsequence of $(M, g_i, x_i)$ converges smoothly and uniformly on compact sets, to a limit AH Einstein metric $(N, g, x)$.

For a fixed $d > 0$ small, let

$$(M_i)_d = \{x \in (M, g_i) : t_i(x) \leq d\}, \tag{5.5}$$

and define $N_d$ in the same way. Then the smooth convergence of the compactifications $\bar{g}_i$ implies that $(M_i)_{2d}$ is diffeomorphic to $N_{2d}$ and each is a collar neighborhood of $\partial M$. Further, the limit metric $g$ on $N_{2d}$ is AH, with boundary metric $\gamma$.

On the other hand, (5.4) implies that the complementary domains

$$(M_i)^d = \{x \in (M, g_i) : t_i(x) \geq d\} \tag{5.6}$$

have uniformly bounded diameter with respect to $g_i$, and so Theorem 3.7 implies that $(M_i)^{d/2}$ is diffeomorphic to the limit domain $N^{d/2}$ in $(N, g)$. It follows that $N = M$ and so $g$ is an AH Einstein metric on $M$, with boundary metric $\gamma$.

■

Combining the results above leads easily to the proof of Theorem A.

**Proof of Theorem A.**

Let $\gamma_i$ be a sequence of boundary metrics in $\mathcal{C}^o$, with $\gamma_i \to \gamma \in \mathcal{C}^o$ in the $C^{m,\alpha'}$ topology on $\partial M$, $m \geq 4$, with $\Pi(g_i) = [\gamma_i]$. Since only the conformal classes are uniquely determined, one may choose for instance $\gamma_i$ to be metrics of constant scalar curvature. To prove $\Pi^o$ is proper, one needs to show that $\{g_i\}$ has a convergent subsequence in $\mathcal{E}_{AH}$ to a limit metric $g \in \mathcal{E}_{AH}$ with $\Pi[g] = [\gamma]$.

Suppose first there is a constant $s_0 > 0$ such that

$$s_{\gamma_i} \geq s_0.$$

where $s_{\gamma_i}$ is the (intrinsic) scalar curvature of the boundary metric $\gamma_i$. It is then proved in [6, Prop.5.1] that

$$Wid_{\bar{g}_i} M \leq \sqrt{3}\pi/\sqrt{s_0}, \tag{5.7}$$

cf. also (A.12). Hence, in this case Theorem A follows directly from Theorem 5.2.

Next, suppose only $s_{\gamma_i} \geq 0$. If there is some constant $D < \infty$ such that $Wid_{\bar{g}_i} M \leq D$, then again Theorem 5.2 proves the result. Suppose instead

$$Wid_{\bar{g}_i} M \to \infty. \tag{5.8}$$

In this case, there is a rigidity result associated with the limiting case of (5.7) as $s_0 \to 0$, proved in [6, Rmk.5.2, Lem.5.5]. Namely, $s_{\gamma_i} \geq 0$ and (5.8) imply that the sequence $(M, g_i, x_i)$, for $x_i$ as in (3.22), converges in the Gromov-Hausdorff topology to a hyperbolic cusp metric

$$g_C = dr^2 + r^2 g_F, \tag{5.9}$$

where $g_F$ is a flat metric on $\partial M$. It follows from the proof of Theorem 5.2, as following (5.5), that $\gamma = g_F$, so that, by definition, $\gamma \notin \mathcal{C}^o$. This implies that necessarily $Wid_{\bar{g}_i} \leq D$, for some $D < \infty$, which completes the proof.

■

**Remark 5.3.** Theorem A implies that the space $\mathcal{E}^o_{AH}$ on a given 4-manifold $M$ satisfying (5.3) has only finitely many components whose conformal infinities intersect any given compact set in $\mathcal{C}^o$.

A similar result holds if the filling manifold $M$ is also allowed to vary. Thus, there are only finitely many 4-manifolds $M_i$ having a common boundary $\partial M$, which satisfy (5.3) and $\chi(M_i) \leq C$, for some constant $C < \infty$, for which

$$\cap_i \Pi(\mathcal{E}^o_{AH}(M_i)) \neq \emptyset.$$

The proof of this is exactly the same, using Theorem 3.5 and Remark 3.6 with varying manifolds $M_i$.

**§5.2.** In this section, we characterize the possible degenerations when $\Pi$ is not proper. Observe first that Theorem 5.2 implies that if the manifold $M$ satisfies (5.3), then the "enhanced" boundary map

$$\Psi : E_{AH} \to Met(\partial M) \times \mathbb{R}, \tag{5.10}$$
$$\Psi(g) = (\Pi(g), Wid_{\Pi(g)} M)$$

is proper. Thus, degenerations of AH Einstein metrics with controlled conformal infinity can only occur when the width diverges to $\infty$. On the other hand, as indicated in the Introduction, in general the boundary map $\Pi$ is not proper.

Define an AH *Einstein metric with cusps* $(N, g)$ to be a complete Einstein metric $g$ on a 4-manifold $N$ which has two types of ends, namely AH ends and cusp ends. A cusp end of $(N, g)$ is an end $E$ such that $vol_g E < \infty$. Thus, $N$ has a compact (possibly disconnected) hypersurface $H$, disconnecting $N$ into two non-compact connected components $N = N_1 \cup N_2$ where $(N_1, g)$ is an AH Einstein metric with boundary $H$ and $(N_2, g)$ has finite volume, so that each end of $N_2$ is a cusp end. A natural choice for $H$ is the level set $t^{-1}(1)$, where $t$ is a geodesic defining function for the conformally compact boundary $\partial_{AH} N$ of $N$. Then $N_1 = \{x \in N : t(x) \leq 1\}$, $N_2 = \{x \in N : t(x) \geq 1\}$.

Note that any cusp end $E$ is not conformally compact. As one diverges to infinity in $E$, the metric $g$ is collapsing, in that $vol_g B_x(1) \to 0$ as $x \to \infty$ in $E$, and hence the injectivity radius satisfies $inj_g(x) \to 0$, as $x \to \infty$ in $E$, see §3.1.

**Theorem 5.4.** *Let $M$ be a 4-manifold satisfying* (1.4), *and let $g_i \in E_{AH}$ be AH Einstein metrics with boundary metrics $\gamma_i \in C^{m,\alpha}$, with $\gamma_i \to \gamma$ in the $C^{m,\alpha'}$ topology on $\partial M$, $m \geq 4$. Let $t_i$ be the geodesic defining function associated with $\gamma_i$ and choose base points $x_i \in H_i = t_i^{-1}(1)$.*

*Then a subsequence of $\{g_i\}$ converges, modulo diffeomorphisms in $\mathcal{D}_1$, either to an AH Einstein metric $g$ on $M$, or to an AH Einstein metric with cusps $(N, g, x)$, $x = \lim x_i$. The convergence is smooth and uniform on compact subsets of $M$, $N$ respectively. In both cases, the conformal infinity is given by $(\partial M, [\gamma])$. Further, the manifold $N$ weakly embeds in $M$, as in* (3.23).

**Proof:** For any $D < \infty$, let $(M_i)_D = \{x \in (M, g_i) : t_i(x) \leq D\}$, as in (5.5). Theorem 5.2 implies that a subsequence of $((M_i)_D, g_i, x_i)$ converges smoothly to a limit AH Einstein metric $g$ on a domain $N_D$, with $N_D$ diffeomorphic to $(M_i)_D$, and with conformal infinity of $N_D$ given by $(\partial M, [\gamma])$.

If there is a fixed $D < \infty$ such that (5.4) holds, then the result follows from the proof of Theorem 5.2 or Theorem A. Thus, we may suppose

$$Wid_{\bar{g}_i} M \to \infty. \tag{5.11}$$

In this case, it follows from [6, Lemma 5.4] that there is a constant $V_0 < \infty$, depending only on $\{\gamma_i\}$ and the Euler characteristic $\chi(M)$, such that

$$vol_{g_i} (M_i)^1 \leq V_0, \tag{5.12}$$

where $(M_i)^1 = \{x \in (M, g_i) : t_i(x) \geq 1\}$ is the complementary domain to $(M_i)_1$. (The estimate (5.12) is a straightforward consequence of (2.18), or more precisely the bound $V \leq \frac{4\pi^2}{3}\chi(M)$, given uniform control of the metrics $g_i$ and $\bar{g}_i$ on $(M_i)_1$).

By Theorem 3.5 and Remark 3.6, the pointed manifolds $(M, g_i, x_i)$, for $x_i$ base points in $S_i(1) = t_i^{-1}(1)$, converge in a subsequence and in the Gromov-Hausdorff topology to a complete Einstein manifold $(N, g, x)$, $x = \lim x_i$. The convergence is also in the $C^\infty$ topology, uniform on compact sets. The domain $N^1 = \{x \in (N, g) : t(x) \geq 1\}$, with $t(x) = \lim t_i(x)$, is the limit of the domains $(M_i)^1$.

The bound (5.12) implies that $N^1$ is of finite volume while (5.11) implies that $N^1$ is non-compact. It follows that the full limit $N = N^1 \cup N_1$ with limit metric $g$ is a complete AH Einstein manifold with conformal infinity $(\partial M, [\gamma])$ and with a non-empty collection of cusp ends. The fact that $N$ weakly embeds in $M$ follows exactly as in the proof of Theorem 3.7.

∎

The fact that the AH cusp metric $(N, g)$ has conformal infinity $(\partial M, [\gamma])$ and that it weakly embeds in $M$ implies that there is a sequence $t_j \to \infty$ such that $N_{t_j} \subset N$ embeds in $M$, for $N_{t_j}$ as above. Hence, for any $j$ large, the manifold $M$ may be decomposed as

$$M = N_{t_j} \cup (M \setminus N_{t_j}). \tag{5.13}$$

With respect to a suitable diagonal subsequence $j = j_i$, the metrics $g_i$ on $M$ push the region $M \setminus N_{t_{j_i}}$ off to infinity as $i \to \infty$ and $t_{j_i} \to \infty$, giving rise to cusp ends in the limit $(N, g)$.

Theorem 5.4 suggests the construction of a natural completion $\bar{\mathcal{E}}_{AH}$ of $\mathcal{E}_{AH}$. Thus, for any $R < \infty$ large, let $\mathcal{C}(R)$ denote the space of conformal classes $[\gamma]$ on $\partial M$ which contain a representative $\gamma$ satisfying $||\gamma||_{C^{4,\alpha}} \leq R$, with respect to some fixed coordinate system for $\partial M$. Let $\mathcal{E}_{AH}(R) = \Pi^{-1}(R)$ and for any $g \in \mathcal{E}_{AH}(R)$, choose a base point $x \in t^{-1}(1) \equiv H_g$, where $t$ is the geodesic defining function associated to the boundary metric $\gamma$.

Then Theorem 5.4 implies that the completion $\bar{\mathcal{E}}_{AH}(R)$ of $\mathcal{E}_{AH}(R)$ in the pointed Gromov-Hausdorff topology based at points $x \in H_g$ is the set of AH Einstein metrics on $M$, together with AH Einstein metrics with cusps $(N, g)$. If $\gamma_i$ are metrics in $\mathcal{C}(R)$, then a subsequence of $\gamma_i$ converges in $C^{m,\alpha'}$ to a limit metric $\gamma \in \mathcal{C}(R)$. If $g_i \in \mathcal{E}_{AH}(R)$ satisfy $\Pi(g_i) = \gamma_i$, then the corresponding subsequence of $g_i$ converges in the Gromov-Hausdorff topology based at $x_i$ to $(N, g) \in \bar{\mathcal{E}}_{AH}(R)$, and the conformal infinity of $(N, g)$ is $\gamma = \lim \gamma_i$. The convergence is also in the $C^\infty$ topology, uniform on compact sets. Of course one may have $N = M$, in which case the limit is an AH Einstein metric on $M$.

For $R < R'$, $\mathcal{E}_{AH}(R) \subset \mathcal{E}_{AH}(R')$ and so we may form the union $\bar{\mathcal{E}}_{AH} = \cup_R \bar{\mathcal{E}}_{AH}(R)$ with the induced topology. It is clear that in this topology, the boundary map $\Pi$ extends to a continuous map

$$\bar{\Pi} : \bar{\mathcal{E}}_{AH} \to \mathcal{C}. \tag{5.14}$$

Further, the following corollary is an immediate consequence of Theorem 5.4.

**Corollary 5.5.** *Let $M$ be a 4-manifold satisfying* (1.4)*. Then the extended map $\bar{\Pi} : \bar{\mathcal{E}}_{AH} \to \mathcal{C}$ is proper.*

∎

It follows in particular that the image

$$\bar{\Pi}(\bar{\mathcal{E}}_{AH}) \subset \mathcal{C}$$

is a closed subset of $\mathcal{C}$. However, it is not known if $\bar{\mathcal{E}}_{AH}$ is a Banach manifold. Even if $\bar{\mathcal{E}}_{AH}$ is not a Banach manifold, it would be interesting to understand the 'size' of the set of AH Einstein metrics with cusps. The set of such metrics corresponds of course exactly to

$$\partial\bar{\mathcal{E}}_{AH} = \bar{\mathcal{E}}_{AH} \setminus \mathcal{E}_{AH}.$$

In particular, with regard to the work to follow in §6-§7, one would like to know if the image $\bar{\Pi}(\partial\mathcal{E}_{AH})$ disconnects $\mathcal{C}$ or not, cf. also Remark 7.8 below. Again the fact that $\bar{\Pi}$ is proper implies that $\bar{\Pi}(\partial\mathcal{E}_{AH})$ is a closed subset of $\mathcal{C}$.

Let

$$\hat{\mathcal{C}} = \mathcal{C} \setminus \bar{\Pi}(\partial\mathcal{E}_{AH}); \tag{5.15}$$

this is the space of conformal classes on $\partial M$ which are not the boundaries of AH Einstein metrics with cusps associated to the manifold $M$. Also let $\hat{\mathcal{E}}_{AH} = \bar{\Pi}^{-1}(\hat{\mathcal{C}})$, so that $\hat{\mathcal{E}}_{AH}$ is the class of AH Einstein metrics on $M$ whose conformal infinity is not the conformal infinity of any AH Einstein metric with cusps associated to $M$. If one gives $\hat{\mathcal{C}}$ the relative topology, as a subset of $\mathcal{C}$, then the following result is also an immediate consequence of the results above.

**Corollary 5.6.** *Let $M$ be a 4-manifold satisfying* (1.4). *Then the map*

$$\hat{\Pi} : \hat{\mathcal{E}}_{AH} \to \hat{\mathcal{C}} \tag{5.16}$$

*is proper.*

■

**Remark 5.7.** The first examples where $\Pi$ is not proper, for example $\Pi^{-1}(pt)$ is non-compact in $\mathcal{E}_{AH}$, is the sequence of AH Einstein metrics $g_i$ on $\mathbb{R}^2 \times T^{n-1}$, converging to the hyperbolic cusp metric on $\mathbb{R} \times T^n$, discussed in Remark 2.6; cf. [6, Prop.4.4] for the explicit construction of $\{g_i\}$.

However, these metrics lie in distinct components of the moduli space $\mathcal{E}_{AH}$ on $\mathbb{R}^2 \times T^2$, and so cannot be connected by a curve of metrics on $\mathbb{R}^2 \times T^2$. Thus $\mathcal{E}_{AH}$ has infinitely many components, and this is the cause of $\Pi$ being non-proper. Currently, there are no known examples where $\Pi$ restricted to a component of $\mathcal{E}_{AH}$ is not proper. On the other hand, if one passes to the quotient of $\mathcal{E}_{AH}$ by the full diffeomorphism group, then the metrics $g_i$ above may be joined by a curve.

More recently, G. Craig [17] has constructed infinite sequences of AH Einstein metrics $(M_i, g_i)$ on non-diffeomorphic manifolds $M_i$, which all have the same conformal infinity $(\partial M, [\gamma_0])$. These limit on a complete hyperbolic manifold $(N, g_{-1})$, with conformal infinity given by $(\partial M, [\gamma_0])$, but with additional cusp, i.e. parabolic, ends analogous to the hyperbolic cusp end above. In fact, the Einstein metrics are constructed by Dehn filling each cusp end of $N$.

## 6. Degree of the Boundary Map.

In this section, we prove that the boundary map $\Pi^o$ has a well-defined $\mathbb{Z}$-valued degree, following Smale [34] and White [37], cf. also [34]. Throughout this section, we work componentwise on $\mathcal{E}^o_{AH}$ and $\mathcal{C}^o$, but we will not distinguish components with extra notation. Thus, we assume

$$\Pi^o : \mathcal{E}^o_{AH} \to \mathcal{C}^o$$

where $\mathcal{E}^o_{AH}$ and $\mathcal{C}^o$ are connected, that is connected components of the full spaces. The results of this section also hold for the restricted boundary map $\hat{\Pi} : \hat{\mathcal{E}}_{AH} \to \hat{\mathcal{C}}$ in (5.16), but since there is no intrinsic characterization of $\hat{\mathcal{C}}$, we work with $\mathcal{C}^o$; see also Remark 7.8.

Since $\Pi^o$ is a proper Fredholm map of index 0, the Sard-Smale theorem [34] implies that the regular values of $\Pi^o$ are open and dense in $\mathcal{C}^o$. For $\gamma$ a regular value, the fiber $(\Pi^o)^{-1}(\gamma)$ thus consists of a finite number of points, i.e. (equivalence classes of) AH Einstein metrics on $M$. By [34]

$$deg_2\Pi^o = \#(\Pi^o)^{-1}([\gamma]) \mod 2, \tag{6.1}$$

is well-defined, for any regular value $[\gamma]$ in $\mathcal{C}^o$. We recall that if $[\gamma] \notin \mathrm{Im}\Pi^o$, then $[\gamma]$ is tautologically a regular value of $\Pi^o$.

Next, we show that $\Pi^o$ has a well-defined degree in $\mathbb{Z}$, essentially following [37]. Thus, for $[\gamma] \in \mathrm{Im}\Pi^o$, let $g$ be any AH Einstein metric on $M$, with $\Pi^o(g) = [\gamma]$. Consider the linearization of the Einstein equations, i.e. as in (2.4), the elliptic operator

$$L = \tfrac{1}{2}D^*D - R,$$

acting on $L^{2,2}(M, g)$. The operator $L$ is bounded below on $L^{2,2}$ and as in the theory of geodesics or minimal surfaces, let

$$ind_g \in \mathbb{Z}, \tag{6.2}$$

be the $L^2$ index of the operator $L$ at $(M, g)$, i.e. the maximal dimension of the subspace of $L^{2,2}(M, g)$ on which $L$ is a negative definite bilinear form, with respect to the $L^2$ inner product. The nullity of $(M, g)$ is the dimension of the $L^2$ kernel $K$.

The main result of this section is the following:

**Theorem 6.1.** *Let $\gamma$ be a regular value of $\Pi^o$ on $(M, g)$ and define*

$$deg\Pi^o = \sum_{g_i \in (\Pi^o)^{-1}([\gamma])} (-1)^{ind_{g_i}}. \tag{6.3}$$

*Then deg $\Pi^o$ is well-defined, i.e. independent of the choice of $[\gamma]$ among regular values of $\Pi^o$.*

**Proof:** In [37], White presents general results guaranteeing the existence of a $\mathbb{Z}$-valued degree, and we will show that the current situation is covered by these results. Thus, we refer to [37] for some further details.

Let $[\gamma_1]$ and $[\gamma_2]$ be regular values of $\Pi^o$ and let $[\bar{\sigma}(t)], t \in [0, 1]$ be an oriented curve in $\mathcal{C}^o$ joining them. Choose representatives $\gamma_1 \in [\gamma_1]$ and $\gamma_2 \in [\gamma_2]$ and let $\bar{\sigma}(t)$ be a curve in $Met(\partial M)$ joining $\gamma_1$ to $\gamma_2$. By [34], we may assume that $\bar{\sigma}$ is transverse to $\Pi$, so that the lift $\sigma = \Pi^{-1}(\bar{\sigma})$ is a collection of curves in $E_{AH}$, with boundary in the fibers over $\gamma_1$ and $\gamma_2$. Define an orientation on $\sigma$ by declaring that $\Pi$ is orientation preserving at any regular point of $\sigma$ which has even index, while $\Pi$ is orientation reversing at regular points of $\sigma$ of odd index. Thus, provided this orientation is well-defined, the map $\Pi|_\sigma : \sigma \to \bar{\sigma}$ has a well-defined mapping degree, as a map of 1-manifolds. By construction, this 1-dimensional degree is given by (6.3) at any regular point of $\sigma$ and hence it follows that (6.3) is well-defined. Thus, it suffices to prove that the orientation constructed above is well-defined.

If $\Pi_*(d\sigma/dt) \neq 0$ for all $t$, so that all points of $\sigma$ are regular, then the index of $\sigma(t)$ is constant, and so there is nothing more to prove. Suppose instead that $\Pi_*(\sigma'(t_0)) = 0$, so that $\sigma(t_0)$ is a critical point of $\Pi$; (without loss of generality, from here on assume $\sigma$ is connected). Hence $\sigma'(t_0) = \kappa_0 \in K$, and for $t$ near $t_0$, $\sigma(t) = \sigma(t_0) + (t - t_0)\kappa_0 + O((t - t_0)^2)$. Without loss of generality, (cf. [37]), one may assume that $K = \langle \kappa_0 \rangle$ is the span of $\kappa_0$. Thus, $\sigma(t)$ may be viewed as a graph over $K_0$.

Let $S^{m,\alpha}_{AH}$ be the space of complete metrics $g$ on $M$ which have a $C^{m,\alpha}$ conformal compactification and which satisfy the slice condition $\beta_{\sigma(t_0)}(g) = 0$. It is proved in [7, Thm.4.1] that the map

$$S^{m,\alpha}_{AH} \to \mathbb{S}^{m-2,\alpha}, \quad g \to Ric_g + 3g, \tag{6.4}$$

is a submersion at any $g \in E^{m,\alpha}_{AH}$. Since $Ric_g + 3g = 0$ on the curve $\sigma(t)$, the implicit function theorem implies that there is a 2-parameter family $\sigma(t, s) \in S^{m,\alpha}_{AH}$, with $\Pi(\sigma(t, s)) = \Pi(\sigma(t)) = \gamma(t)$, for $t$ near $t_0$ and $s$ near 0, such that

$$Ric_{\sigma(t,s)} + 3\sigma(t, s) \in K. \tag{6.5}$$

It is well-known that forms $\kappa \in K$ satisfy $|\kappa| = O(t^n)$, i.e. $O(t^3)$ in the case at hand, cf. [11], [30]. This, together with the fact that such $\kappa$ are transverse-traceless, (cf. (2.6)), implies that the metrics $\sigma(t, s)$ are Einstein to order 3 at conformal infinity $\partial M$, in that $\frac{d\sigma}{ds} \in \widetilde{T}_{\sigma(t,s)}E_{AH}$, cf. (2.20).

By Lemma 2.2, the renormalized action $I^{ren}_{EH}$ is then well-defined on the family $\sigma(t, s)$. When viewed as a function of $t, s$, one has

$$\frac{d}{ds}I^{ren}_{EH}(\sigma(t, s)) = \int_M \langle Ric_{\sigma(t,s)} + 3\sigma(t, s), \frac{d\sigma}{ds}\rangle dV, \tag{6.6}$$

since $d\sigma/ds$ has 0 boundary values on $\partial M$. (The Euler-Lagrange equations for $I^{ren}_{EH}$ are the same as those for the usual Einstein-Hilbert action). By (2.6) and (6.5),

$$\frac{d}{ds}(Ric_{\sigma(t,s)} + 3\sigma(t, s)) = L(\frac{d\sigma(t, s)}{ds}) \in K,$$

which implies that $\frac{d}{ds}\sigma(t,s) \in K$. From this, it follows that the integral in (6.6) vanishes only on the curve $\sigma(t)$; thus the curve $\sigma(t)$ in the plane $P$ spanned by $t, s$ is the set where $dI_{EH}^{ren}/ds = 0$. (The functional $I_{EH}^{ren}$ plays the role of $g$ in [37]). It follows that $\sigma(t)$ is (locally) the boundary of the open set $\{dI_{EH}^{ren}/ds > 0\}$ in $P$:

$$\sigma(t) = \partial\{dI_{EH}^{ren}/ds > 0\}. \tag{6.7}$$

If $ind_{\sigma(t)}$ is even, for $t$ near $t_0$, $t \neq t_0$, give $\sigma(t)$ the boundary orientation induced by this open domain, while if $ind_{\sigma(t)}$ is odd for $t$ near $t_0$, $t \neq t_0$, give $\sigma(t)$ the reverse orientation. It is now straightforward to see that this definition coincides with the orientation defined at the beginning of the proof. Thus, the point $t_0 \to \sigma(t_0)$ is a critical point for the map $\pi_1 \circ \sigma$, where $\pi_1 : (t,s) \to t$ is projection onto the first factor. If this critical point is a folding singularity for $\pi_1 \circ \sigma$, then the index of $\sigma(t)$ changes by 1 in passing through $\sigma(t_0)$ and reverses the orientation of $\lambda(t) = \pi_1(\sigma(t))$, (exactly as is the case with the standard folding singularity $x \to x^2$). On the other hand, if $\pi_1 \circ \sigma$ does not fold with respect to $\pi_1$, (so that one has an inflection point), then the index of $\sigma(t)$ does not change through $t_0$ and $\pi_1$ maps $\sigma(t)$ to $\lambda(t)$ in an orientation preserving way. We refer to [37] for further details, and also to §7 for concrete examples of such folding behavior.

■

**Remark 6.2.** As discussed in §2, the inclusion $\mathcal{E}_{AH}^{(m',\alpha')} \subset \mathcal{E}_{AH}^{(m,\alpha)}$, is dense, for any $(m',\alpha') \geq (m,\alpha)$ and these spaces are diffeomorphic. Hence, the degree of $\Pi^o$ is independent of $(m,\alpha)$ and defined on the spaces $\mathcal{E}_{AH}^{(m,\alpha)}$ for all $m \geq 4$. Moreover, the degree is also defined on the spaces $\bar{\mathcal{E}}_{AH}$ and $\hat{\mathcal{E}}_{AH}$ in (5.14) and (5.16), and is independent of $(m,\alpha)$.

## 7. Computations of the Degree.

We conclude the paper with computations of $deg\Pi^o$ for several interesting examples of 4-manifolds. In all cases, the evaluation of the degree is made possible by symmetry arguments, using Theorem 2.5.

Recall that if $deg\Pi^o \neq 0$, then $\Pi$ is surjective onto $\mathcal{C}^o$, and so any conformal class $[\gamma]$ in $\mathcal{C}^o$ on $\partial M$ has a filling by an AH Einstein metric $g$ with conformal infinity $[\gamma]$. Of course, $deg\Pi^o = 0$ does not imply that $\Pi$ cannot be surjective.

We begin with the proof of Theorem B.

**Proof of Theorem B.**

As seed metric, choose the hyperbolic (Poincaré) metric $g_0$ on $B^4$. This has conformal infinity $[\gamma_0]$, where $\gamma_0$ is the round metric on $S^3 = \partial B^4$. Now the boundary metric $\gamma_0$ admits a large connected group $SO(4)$ of isometries. Theorem 2.5 implies that for any AH Einstein metric $g$ on $B^4$, $SO(4)$ acts effectively by isometries on $(B^4, g)$. It is then standard that $g$ must be the Poincaré metric on $B^4$.

It follows that up to isometry, $g_0$ is the unique metric with conformal infinity $[\gamma_0]$. (This also follows from a rigidity theorem in [9, Thm.5.2]). Further, it is well-known that the $L^2$ kernel $K$ of $g_0$ is trivial, i.e. $K = \{0\}$; in fact, this holds for any Einstein metric of negative sectional curvature, cf. [10], [11]. Thus, $g_0$ is a regular point of $\Pi$ and since $\Pi^{-1}[\gamma_0] = g_0$, the point $[\gamma_0]$ is a regular value of $\Pi$. The result then follows from (6.3).

■

We point out the following immediate consequence of the proof.

**Corollary 7.1.** *Let $M$ be any 4-manifold satisfying $\pi_1(M,\partial M) = 0$, and (1.4), with $\partial M = S^3$ and $M \neq B^4$. Then on any component of $\mathcal{E}_{AH}^o$,*

$$deg\Pi^o = 0, \tag{7.1}$$

*and $\Pi$ is not surjective. In fact the conformal class of the round metric $\gamma_0$ on $S^3$ is not in $Im\Pi$.*

**Proof:** Suppose the class $[\gamma_0] \in \mathrm{Im}\Pi$, so that there is an AH Einstein metric $g$ on $M$, with boundary metric $\gamma_0$. The proof of Theorem B above implies that necessarily $g = g_0$, where $g_0$ is the Poincaré metric on $B^4$. Hence, $M = B^4$, a contradiction.

■

Next, we have:

**Proposition 7.2.** *Let $M = \mathbb{R}^2 \times S^2$, so that $\partial M = S^1 \times S^2$. Then*

$$deg\Pi^o = 0, \tag{7.2}$$

*and $\Pi$ is not surjective.*

**Proof:** As seed metric(s) in this case, we take the remarkable 1-parameter family of AdS Schwarzschild metrics, discussed in detail in [23], cf. also [38]. Thus, on $\mathbb{R}^2 \times S^2$, consider the metric

$$g_m = F^{-1}dr^2 + Fd\theta^2 + r^2 g_{S^2(1)}, \tag{7.3}$$

where $F = F(r) = 1 + r^2 - \frac{2m}{r}$. The mass parameter $m > 0$ and $r \in [r_+, \infty)$, where $r_+$ is the largest root of the equation $F(r_+) = 0$. The locus $\{r_+ = 0\}$ is a totally geodesic round 2-sphere $S^2$, of radius $r_+$. Smoothness of the metric at $\{r_+ = 0\}$ requires that the circle parameter $\theta$ run over the interval $[0, \beta)$, where $\beta$ is given by

$$\beta = \frac{4\pi r_+}{1 + 3r_+^2}. \tag{7.4}$$

It is easily seen that as $m$ varies from 0 to $\infty$, $r_+$ varies monotonically from 0 to $\infty$.

The metrics $g_m$ are isometrically distinct, for distinct values of $m$, and form a smooth curve in $\mathcal{E}_{AH}$, with conformal infinity given by the conformal class of the product metric $\gamma_m = S^1(\beta) \times S^2(1)$. Notice however that the length $\beta$ has a maximum value as $m$ ranges over $(0, \infty)$, namely

$$\beta \leq \beta_{max} = 2\pi/\sqrt{3}, \tag{7.5}$$

achieved at $r_+ = 1/\sqrt{3}$, $m = m_0 = 2/\sqrt{3}$. As $m \to 0$ or $m \to \infty$, one has $\beta \to 0$.

Thus, the boundary map $\Pi$ on the curve $g_m$ is a fold map, folding the ray $m \in (0, \infty)$ onto the $\beta$-interval $(0, \beta_{max}]$. The map $\Pi$ restricted to the curve $g_m$ is a 2-1 map, except at the point $g_{m_0}$. The metric $g_{m_0}$ is a critical point of $\Pi$, and, (as will be seen below), the tangent vector $(dg_m/dm)_{m=m_0}$ spans the $L^2$ kernel $K_{g_{m_0}}$.

Next, we claim that the metrics $g_m$ are the only AH Einstein metrics on $\mathbb{R}^2 \times S^2$ with conformal infinity given by a product $\gamma_L = S^1(L) \times S^2(1)$. The isometry group of $\gamma_L$ contains $G = SO(2) \times SU(2)$ and by Theorem 2.5, any AH Einstein metric $g_L$ on $M$ with boundary metric $\gamma_L$ has $G$ acting effectively by isometries. As in (2.7), we may choose a geodesic defining function for $g_L$ so that $g_L$ has the form

$$g_L = ds^2 + g_s,$$

where $g_s$ is a curve of metrics on $S^1 \times S^2$ invariant under the $G$-action. Thus, the Einstein metric $g_L$ has cohomogeneity 1. It then follows from the classification given in [31] for instance that the metric $g_L$ on $\mathbb{R}^2 \times S^2$ is isometric to $g_m$, for some $m = m(L)$.

A similar argument shows that the metrics $g_m$, for $m \neq m_0$ are regular points of $\Pi$. For suppose there exists $\kappa \in K = K_{g_m}$. Then $\kappa$ is the tangent vector to a curve $\sigma(s) \in E_{AH}$ with $\sigma(0) = g_m$ and $\Pi_*(\kappa) = 0$, so that the boundary metric of $\sigma(s)$ is fixed to first order in $s$. The proof of Theorem 2.5 in [8] then shows that $\kappa$ inherits the symmetries of $g_m$, i.e. $\kappa$ is $G$-invariant. Hence, $\kappa$ is tangent to the curve $g_m$, so that, without loss of generality, $\sigma(s) = g_{m(s)}$. Since $\Pi_*(\frac{d}{dm}g_m) \neq 0$, when $m \neq m_0$, this proves the claim.

We may thus compute $deg\Pi^o$ by evaluating the formula (6.3) on a pair of distinct metrics $g_{m_1}$ and $g_{m_2}$ with $\Pi(g_{m_1}) = \Pi(g_{m_2})$. From [23, §3], one has

$$ind_{g_{m_1}} = +1, \quad ind_{g_{m_2}} = 0, \tag{7.6}$$

for $m_1 < m_0 < m_2$, which proves (7.2). Alternately, since $\Pi$ is a 2-1 fold map on $g_m$, the proof of Theorem 7.1 shows directly that (7.2) holds.

Further, the symmetry argument above implies that the metrics $S^1(L) \times S^2(1)$ are not in Im$\Pi$, whenever

$$L > \beta_{max}, \tag{7.7}$$

and hence $\Pi$ is not surjective. ■

This result should be compared with the following:

**Proposition 7.3.** *Let $M = S^1 \times \mathbb{R}^3$, so that $\partial M = S^1 \times S^2$. Then*

$$deg\Pi^o = 1, \tag{7.8}$$

*and $\Pi^o$ is surjective.*

**Proof:** As seed metrics in this situation, we take a family of hyperbolic metrics, namely the metrics $\mathbb{H}^4(-1)/\mathbb{Z}$, where the $\mathbb{Z}$-quotient is obtained by a hyperbolic or loxodromic translation of length $L$ along a geodesic in $\mathbb{H}^4(-1)$. The conformal infinity is the product metric $S^1(L) \times S^2(1)$ in the case of a hyperbolic translation, and the bent product metric $S^1(L) \times_\alpha S^2(1)$ on the same space when the translation is loxodromic; the angle $\alpha$ between the factors $S^1(L)$ and $S^2(1)$ corresponds to the loxodromic rotation.

As in Proposition 7.2, any AH Einstein metric on $S^1 \times \mathbb{R}^3$ with boundary metric $S^1(L) \times S^2(1)$ has $SO(2) \times SO(3)$ acting effectively by isometries. Again, the classification in [31] implies that, on this manifold, the only such metrics are hyperbolic. Hence the result follows as in the proof of Theorem B. ■

Next, we turn to non-trivial disc bundles over $S^2$. For the disc bundle of degree 1 over $S^2$, i.e. $M = \mathbb{CP}^2 \setminus B^4$, with $\partial M = S^3$, Corollary 7.1 implies that $\Pi$ is not surjective so that

$$deg\Pi^o = 0. \tag{7.9}$$

**Remark 7.4.** Actually, to justify this statement one needs to remove the hypothesis (1.4), since in this situation $H_2(\partial M)$ does not surject onto $H_2(\bar{M})$. Recall that (1.4) was only used to rule out orbifold degenerations in the proof of Theorem 3.7. However, it is sometimes possible to rule out orbifold degenerations in the case of sufficiently low Euler characteristic directly, without the use of (1.4). This situation was treated in [1] in the case of orbifold degenerations on compact manifolds, and the argument for AH metrics is similar; thus we refer to [1], [3] for some further details.

Let $M$ be the disc bundle of degree $k$ over $S^2$, so that $\chi(M) = 2$, and let $\{g_i\}$ be a sequence of AH Einstein metrics on $M$ which converge to an AH Einstein orbifold $(X, g)$, with boundary metrics $\gamma_i$ converging in $C^{m,\alpha}$ to the boundary metric $\gamma$ for $(X, g)$. By (2.18), one has

$$\frac{1}{8\pi^2}\int_M |W_{g_i}|^2 = 2 - \frac{3}{4\pi^2}V(g_i).$$

The results of Theorem 3.7, Corollary 4.10 and Proposition 5.1 show that $V(g_i) \to V(g)$, where $V(g)$ is the renormalized volume of the Einstein orbifold $(X, g)$. Now as discussed following Proposition 3.10, orbifold singularities arise by crushing essential 2-cycles in $M$ to points. Since $M$ has only one essential 2-cycle, (up to multiplicity), given by the zero-section, there can only be one singular point on $X$. For the same reason, there is only one Ricci-flat ALE space $(E, g_\infty)$ associated to the singularity. It is then easy to see, cf. [1, §6], [3, §3.3] and references therein, that

$$\lim_{i\to\infty}\int_M |W_{g_i}|^2 = \int_X |W_g|^2 + \int_E |W_{g_\infty}|^2.$$

The formula (2.18) for Einstein orbifolds $(X, g)$ gives

$$\frac{1}{8\pi^2}\int_X |W_g|^2 = \chi(X_0) + \frac{1}{|\Gamma_0|} - \frac{3}{4\pi^2}V(g),$$

where $X_0$ is the regular set of $X$ and the orbifold singularity of $X$ is $C(S^3/\Gamma_0)$, (cf. [1, (6.2)]). Also

$$\int_E |W_{g_\infty}|^2 = \chi(E) - \frac{1}{|\Gamma|},$$

where $E$ is asymptotic to $C(S^3/\Gamma)$ at infinity, (cf. [1, (6.3)]). Combining these equations gives $\Gamma_0 = \Gamma = \mathbb{Z}_k$, and hence $\chi(X_0) = 0$, so that $E$ is diffeomorphic to $M$. In particular, this rules out orbifold degenerations in the case $k = 1$, where $M = \mathbb{CP}^2 \setminus B^4$, since $\Gamma$ must be non-trivial, $\Gamma \neq \{e\}$. This gives (7.9).

For general $k$, one thus has

$$\int_E |W_{g_\infty}|^2 = 2 - \frac{1}{k}. \tag{7.10}$$

On the other hand, the signature formula for ALE metrics, cf. [27] for instance, gives

$$\tau(E) = \frac{1}{12\pi^2}\int_E |W_+|^2 - |W_-|^2 + \eta(S^3/\mathbb{Z}_k), \tag{7.11}$$

and

$$\eta(S^3/\mathbb{Z}_k) = \frac{(k-1)(k-2)}{3k}.$$

Since $\tau(E) = 1$, simple arithmetic as in the Hitchin-Thorpe inequality gives the estimate

$$4k - 2 \geq |3k - (k-1)(k-2)|. \tag{7.12}$$

The estimate (7.12) implies $k \leq 9$, so that orbifold degenerations are not possible if $k \geq 10$.

We conjecture that there are in fact no orbifold degenerations for $k \geq 3$. However, for $k = 2$, it will be seen below that there are orbifold degenerations. In fact, in this case, (7.12) shows that $(E, g_\infty)$ must be self-dual, and hence by Kronheimer's classification [29], $(E, g_\infty)$ is the Eguchi-Hanson metric.

**Example 7.5.** It is interesting to compare the result (7.9) with an explicit family of AH Einstein metrics on $M = \mathbb{CP}^2 \setminus B^4$, namely the AdS Taub-Bolt family, cf. [24], [31]. This is a 1-parameter family of metrics given by

$$g_s = E_s\{(r^2-1)F^{-1}(r)dr^2 + (r^2-1)^{-1}F(r)(d\tau + \cos\theta d\phi)^2 + (r^2-1)g_{S^2(\frac{1}{2})}\}, \tag{7.13}$$

where the parameters $r$, $s$ satisfy $r \geq s$ and $s > 2$, the constant $E_s$ is given by

$$E_s = \frac{2}{3}\frac{s-2}{s^2-1}, \tag{7.14}$$

and the function $F(r) = F_s(r)$ is

$$F_s(r) = Er^4 + (4-6E)r^2 + \{-Es^3 + (6E-4)s + \frac{1}{s}(3E-4)\}r + (4-3E). \tag{7.15}$$

The parameter $\tau$ runs over the vertical $S^1$, and $\tau \in [0, \beta)$, with

$$\beta = 2\pi. \tag{7.16}$$

The bolt $S^2$ $\{r = s\}$ is a round, totally geodesic 2-sphere, of area $A_s = \frac{2}{3}\pi(s-2)$. These metrics are AH, with conformal infinity given by a Berger (or squashed) $S^3$, with base $S^2(\frac{1}{2})$, and Hopf fiber $S^1 = S^1(L)$ of length $L = 2\pi\sqrt{E}$.

In analogy to (7.4)-(7.5), notice that $E \to 0$ as $s \to 2$ or $s \to \infty$, and has a maximal value $E_{max} = (2-\sqrt{3})/3$ at $s = s_0 = 2+\sqrt{3}$. Note in particular that since

$$E_{max} < 1,$$

the round metric $S^3(1)$ is not in Im $\Pi(g_s)$ for any $s$.

We see that on the curve $g_s$, $s \in (2,\infty)$, the boundary map $\Pi$ has exactly the same 2-1 fold behavior as for the AdS Schwarzschild metric. The use of Theorem 2.5 as in Proposition 7.2 implies that $\Pi$ is not surjective, and in fact the Berger spheres with Hopf fiber length $L$, for $L > 2\pi\sqrt{E_{max}}$ are not in Im$\Pi$.

In contrast, one has the following behavior on more twisted disc bundles over $S^2$.

**Proposition 7.6.** *Let $M = M_k$ be the disc bundle of degree $k$ over $S^2$, $k \geq 10$, so that $\partial M = S^3/\mathbb{Z}_k$. Then*

$$deg\Pi^o = 1, \tag{7.17}$$

*and $\Pi^o$ is surjective.*

**Proof:** As seed metrics, choose again the AdS-Taub bolt metrics on $M_k$, cf. [24], [31]. These have exactly the same form as (7.13)-(7.16), except that the parameter $s$ satisfies $s > 1$, $E_s$ in (7.14) is replaced by $E_{s,k}$ of the form

$$E_{s,k} = \frac{2ks-4}{3(s^2-1)}, \tag{7.18}$$

and the period $\beta$ for $\tau$ is given by $\beta = 2\pi/k$. For each $k$, these metrics are AH, and the conformal infinity on $S^3/\mathbb{Z}_k$ is given by the Berger metric, with, as before, Hopf circle fibers of length $L = 2\pi\sqrt{E_{s,k}}/k$. When $E = 1$, conformal infinity is given by the round metric on $S^3/\mathbb{Z}_k$.

Note however from (7.18) that now the function $E_{s,k}$ is a monotone decreasing function of $s$, as $s$ increases from 1 to $\infty$. Hence, $\Pi$ is 1-1 on this curve, and in particular, the metric $g_{s_0}$, $s_0 = \frac{1}{3}(k+\sqrt{k^2-3})$ has conformal infinity the constant curvature metric on $S^3/\mathbb{Z}_k$. The symmetry argument using Theorem 2.5 as before, together with [31], implies these metrics are the unique metrics with these boundary values. The metrics $g_s$ are regular points for $\Pi$, and so (7.17) follows, since by Remark 7.4, orbifold singularities cannot occur when $k \geq 10$. ∎

**Remark 7.7.** We expect Proposition 7.6 holds for all $k \geq 3$; the proof above holds for such $k$, provided there are no orbifold degenerations, cf. Remark 7.4.

However, Proposition 7.6 does not hold when $k = 2$. In this case, (7.18) becomes

$$E_{s,2} = \frac{4}{3}\frac{1}{s+1}. \tag{7.19}$$

This is of course monotone decreasing in $s \in (1,\infty)$, but it has a finite value at $s = 1$ with $E_{1,2} = \frac{2}{3}$. As $s \to 1$, the area of the bolt $S^2$ at $\{r = s\}$ tends to 0, and vanishes at $\{r = 1\}$ when $s = 1$. Thus, the Taub-Bolt metric is an orbifold singular metric on $C(\mathbb{RP}^3)$ when $s = 1$. Note this shows that the $k = 2$ discussion in Remark 7.4 is sharp, in that as $s \to 1$ on the Taub-Bolt curve, the Ricci-flat ALE space associated to the orbifold singularity is necessarily the Eguchi-Hanson metric. In particular, only the values $E \in (0, \frac{2}{3})$ are achieved on the Taub-Bolt curve $g_s$. I am grateful to Michael Singer for pointing out this behavior of the Taub-NUT curve.

Hence the round metric $\gamma_0$ on $\mathbb{RP}^3$ is not in $Im\Pi$ on the curve $g_s$. Again, Theorem 2.5 and the classification in [31] show that $\gamma_0 \notin \Pi(\mathcal{E})$, where $\mathcal{E}$ is the moduli space of AH Einstein metrics on the disc bundle of degree 2 over $S^2$.

**Remark 7.8.** We close the paper with some observations on whether the full boundary map

$$\Pi : \mathcal{E}_{AH} \to \mathcal{C}$$

might be surjective, or almost surjective, at least when $deg\Pi^o \neq 0$.

First, recall from Corollaries 5.5 and 5.6 that both the extended map $\bar{\Pi} : \bar{\mathcal{E}}_{AH} \to \mathcal{C}$ and the restricted map $\hat{\Pi} : \hat{\mathcal{E}}_{AH} \to \hat{\mathcal{C}}$ are proper. In this regard, it would be very interesting to know if the set of boundary values of AH Einstein metrics with cusps disconnects $\mathcal{C}$ or not, that is whether

$$\bar{\Pi}(\partial \bar{\mathcal{E}}_{AH}) \subset \mathcal{C}$$

disconnects $\mathcal{C}$, or whether $\hat{\mathcal{C}} = \mathcal{C} \setminus \bar{\Pi}(\partial \bar{\mathcal{E}}_{AH})$ is path connected, (for instance if $\bar{\Pi}(\partial \bar{\mathcal{E}}_{AH})$ is of codimension at least 2). If $\hat{\mathcal{C}}$ is path connected, then the degree theory arguments of §6 and §7 hold without any change and give a well-defined degree $deg\Pi$ on each component of $\mathcal{E}_{AH}$. In particular, if this holds and $deg\Pi \neq 0$, then $\Pi$ is almost surjective, in that $\Pi$ surjects onto $\hat{\mathcal{C}}$.

On the other hand, if $\bar{\Pi}(\partial \bar{\mathcal{E}}_{AH})$ disconnects $\mathcal{C}$, then $\bar{\Pi}(\partial \bar{\mathcal{E}}_{AH})$ represents a "wall", past which it may not be possible to fill in boundary metrics with AH Einstein metrics. This would be the case for instance if $\bar{\mathcal{E}}_{AH}$ is a Banach manifold with boundary $\partial \bar{\mathcal{E}}_{AH}$ and $\bar{\Pi}$ maps $\partial \bar{\mathcal{E}}_{AH}$ onto a set of codimension 1 in $\mathcal{C}$.

The same issue arises if the condition (1.4) does not hold, so that orbifold singular metrics might arise, as discussed in Remark 7.7.

Finally, it would also be interesting to know if there are topological obstructions to the possible formation of cusps, as is the case with the formation of orbifold singularities as in (1.4). Thus, with respect to the decomposition (5.13), one would like to know for instance if some of the homology of one of the two factors injects into that of the union $M$.

## Appendix

In this appendix, we collect several formulas for the curvature of conformal compactifications of AH Einstein metrics.

Let $\bar{g} = \rho^2 \cdot g$, where $\rho$ is a defining function with respect to $\bar{g}$. The curvatures of the metrics $g$ and $\bar{g}$ are related by the following formulas:

$$\bar{K}_{ab} = \frac{K_{ab} + |\bar{\nabla}\rho|^2}{\rho^2} - \frac{1}{\rho}\{\bar{D}^2\rho(\bar{e}_a, \bar{e}_a) + \bar{D}^2\rho(\bar{e}_b, \bar{e}_b)\}. \tag{A.1}$$

$$\bar{Ric} = -(n-1)\frac{\bar{D}^2\rho}{\rho} + (n\rho^{-2}(|\bar{\nabla}\rho|^2 - 1) - \frac{\bar{\Delta}\rho}{\rho})\bar{g}, \tag{A.2}$$

$$\bar{s} = -2n\frac{\bar{\Delta}\rho}{\rho} + n(n+1)\rho^{-2}(|\bar{\nabla}\rho|^2 - 1). \tag{A.3}$$

The equation (A.2) is equivalent to the Einstein equation (1.2). Observe that (A.2) implies that if the compactification $\bar{g}$ is $C^2$, then

$$|\bar{\nabla}\rho| \to 1 \ \text{ at } \ \partial M, \tag{A.4}$$

and so by (A.1), $|K_{ab} + 1| = O(\rho^2)$. For $r = -\log\rho$, a simple calculation gives

$$|\bar{\nabla}\rho| = |\nabla r|, \tag{A.5}$$

where the norm and gradient on the left are with respect to $\bar{g}$, and on the right are with respect to $g$.

A defining function $\rho = t$ is a geodesic defining function if

$$|\bar{\nabla}t| = 1, \tag{A.6}$$

in a collar neighborhood $U$ of $\partial M$. Clearly, the formulas (A.1)-(A.3) simplify considerably in this situation. The function $t$ is the distance function to $\partial M$ on $(M, \bar{g})$, and similarly by (A.5), the function $r$ is a (signed) distance function on $(M, g)$. The integral curves of $\bar{\nabla} t$ and $\nabla r$ are geodesics in $(M, \bar{g})$ and $(M, g)$ respectively.

The $2^{\text{nd}}$ fundamental form $\bar{A}$ of the level sets $S(t)$ of $t$ in $(M, \bar{g})$ is given by $\bar{A} = D^2 t$, with $\bar{H} = \bar{\Delta} t$ giving the mean curvature of $S(t)$. Along $t$-geodesics of $(M, \bar{g})$, one has the standard Riccati equation

$$\bar{H}' + |\bar{A}|^2 + \bar{Ric}(\bar{\nabla} t, \bar{\nabla} t) = 0, \tag{A.7}$$

where $\bar{H}' = d\bar{H}/dt$.

The following formulas relating the curvatures of $(M, \bar{g})$ at $\partial M$ to the intrinsic curvatures of $(\partial M, \gamma)$ may be found in [6, §1].

Let $\bar{g}$ be a $C^2$ geodesic compactification of $(M, g)$, with $C^2$ boundary metric $\gamma$. Then at $\partial M$,

$$\bar{s} = 2n\bar{Ric}(N, N) = \frac{n}{n-1} s_\gamma, \tag{A.8}$$

where $N = \bar{\nabla} t$ is the unit normal to $\partial M$ with respect to $\bar{g}$. If $X$ is tangent to $\partial M$, then

$$\bar{Ric}(N, X) = 0, \tag{A.9}$$

while if $T$ denotes the projection onto $T(\partial M)$, then

$$(\bar{Ric})^T = \frac{1}{n-2}((n-1)Ric_\gamma - \frac{s_\gamma}{2(n-1)}\gamma). \tag{A.10}$$

In particular, the full curvature of ambient metric $\bar{g}$ at $\partial M$ is determined by the curvature of the boundary metric $\gamma$.

Finally, we have the following formula for $\bar{s}' = d\bar{s}/dt$:

$$\bar{s}' = 2nt^{-1}|\bar{D}^2 t|^2 \geq \frac{1}{2n^2} t\bar{s}^2 \geq 0. \tag{A.11}$$

Hence, if $\bar{s}(0) > 0$, then

$$t^2 < 4n^2/\bar{s}(0). \tag{A.12}$$

## References


[1] M. Anderson, Ricci curvature bounds and Einstein metrics on compact manifolds, Jour. Amer. Math. Soc., **2**, (1989), 455-490

[2] M. Anderson, Convergence and rigidity for manifolds under Ricci curvature bounds, Inventiones Math. **102**, (1990), 429-445.

[3] M. Anderson, The $L^2$ structure of moduli spaces of Einstein metrics on 4-manifolds, Geom. & Funct. Analysis, **2**, (1992), 29-89.

[4] M. Anderson, Extrema of curvature functionals on the space of metrics on 3-manifolds, Calc. Var. & P.D.E., **5**, (1997), 199-269.

[5] M. Anderson, $L^2$ curvature and volume renormalization for AHE metrics on 4-manifolds, Math. Research Lett., **8**, (2001), 171-188, math.DG/0011051.

[6] M. Anderson, Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds, Advances in Math. **179**, (2003), 205-249, math.DG/0104171.

[7] M. Anderson, On the structure of conformally compact Einstein metrics, (preprint), math.DG/0402198.

[8] M. Anderson and M. Herzlich, Unique continuation results for Ricci curvature and applications, Jour. Geom. & Physics, **58**, (2008), 179-207, arXiv:0710.1305 [math.DG].

[9] L. Andersson and M. Dahl, Scalar curvature rigidity for asymptotically locally hyperbolic manifolds, Ann. Glob. Anal. Geom., **16**, (1998), 1-27.

[10] A. Besse, Einstein Manifolds, Ergebnisse Series, vol 3:10, Springer Verlag, New York, (1987).

[11] O. Biquard, Métriques d'Einstein asymptotiquement symétriques, Astérisque, **265**, (2000).

[12] R. Böhme and A.J. Tromba, The index theorem for classical minimal surfaces, Ann. of Math., **113**, (1981), 447-499.

[13] J. Cheeger and T. Colding, On the structure of spaces with Ricci curvature bounded below, Jour. Diff. Geom., **46**, (1997), 406-470.
[14] T. Colding, Ricci curvature and volume convergence, Annals of Math., **145**, (1997), 477-501.
[15] J. Cheeger and M. Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded, I, II, Jour. Diff. Geom., **23**, (1986), 309-346 and **32**, (1990), 269-298.
[16] P. Chruściel, E. Delay, J. Lee and D. Skinner, Boundary regularity of conformally compact Einstein metrics, Jour. Diff. Geom., **69**, (2005), 111-136, math.DG/0401386.
[17] G. Craig, Dehn filling and asymptotically hyperbolic Einstein metrics, Comm. Anal. Geom., 14, (2006), 725-764, math.DG/0502491.
[18] S. de Haro, K. Skenderis and S.N. Solodukhin, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, Comm. Math. Phys., **217**, (2001), 595-622, hep-th/0002230.
[19] C. Fefferman and C.R. Graham, Conformal invariants, in Élie Cartan et les Mathématiques d'Aujourd'hui, Astérisque, (1985), 95-116.
[20] C.R. Graham, Volume and area normalization for conformally compact Einstein metrics, Rend. Circ. Math. Palermo, (2) Suppl. **63**, (2000), 31-42, math.DG/0009042
[21] C.R. Graham and J.M. Lee, Einstein metrics with prescribed conformal infinity on the ball, Advances in Math., **87**, (1991), 186-225.
[22] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Prog. in Math. Series, **152**, Birkhauser Verlag, Boston, (1999).
[23] S.W. Hawking and D.N. Page, Thermodynamics of black holes in Anti-de Sitter space, Comm. Math. Phys., **87**, (1983), 577-588.
[24] S.W. Hawking, C.J. Hunter and D.N. Page, Nut charge, Anti-de Sitter space and entropy, Phys. Rev., **D59**, (1999), 044033, hep-th/9809035.
[25] D. Helliwell, Boundary regularity for conformally compact Einstein metrics in even dimensions, (preprint), arXiv:0705.2625, [math.DG].
[26] N. Hitchin, Twistor spaces, Einstein metrics and isomonodromic deformations, Jour. Diff. Geom., **42**, (1995), 30-112.
[27] N. Hitchin, Einstein metrics and the eta-invariant, Boll. Un. Mat. Italia, **llB**, Suppl. fasc. 2, (1997), 95-105.
[28] S. Kichenassamy, On a conjecture of Fefferman and Graham, Adv. in Math., **184**, (2004), 268-288.
[29] P. Kronheimer, A Torelli-type theorem for gravitational instantons, Jour. Diff. Geom, **29**, (1989), 685-697.
[30] J. M. Lee, Fredholm operators and Einstein metrics on conformally compact manifolds, Mem. Amer. Math. Soc, **183**, (2006, math.DG/0105046.
[31] D.N. Page and C.N. Pope, Inhomogeneous Einstein metrics on complex line bundles, Class. Quantum Grav., **4**, (1987), 213-225.
[32] H. Pedersen, Einstein metrics, spinning top motions and monopoles, Math. Annalen, **274**, (1986), 35-59.
[33] K. Skenderis, Asymptotically anti-de-Sitter spacetimes and their stress-energy tensor, Int. Jour. Mod. Physics **A16**, (2001), 740-749, hep-th/0010138.
[34] S. Smale, An infinite dimensional version of Sard's theorem, Amer. Jour. Math., **87**, (1965), 861-866.
[35] A.J. Tromba, Degree theory on oriented infinite dimensional varieties and Morse number of minimal surfaces spanning a curve in $\mathbb{R}^n$, Trans. Amer. Math. Soc., **290**, (1985), 385-413.
[36] B. White, The space of $m$-dimensional surfaces that are stationary for a parametric elliptic functional, Ind. Univ. Math. Jour., **36**, (1987), 567-603.
[37] B. White, The space of minimal submanifolds for varying Riemannian metrics, Ind. Univ. Math. Jour., **40**, (1991), 161-200.
[38] E. Witten, Anti De Sitter space and holography, Adv. Theor. Math. Phys., **2**, (1998), 253-291, hep-th/9802150.
[39] E. Witten and S.-T. Yau, Connectedness of the boundary in the AdS/CFT correspondence, Adv. Theor. Math. Phys., **3**, (1999), 1635-1655, hep-th/9910245.

March, 2008

Department of Mathematics
S.U.N.Y. at Stony Brook
Stony Brook, N.Y.11794-3651
anderson@math.sunysb.edu