1. Some answers to problems from §4.2

1) The functions $u_1(y) = \sinh \lambda y$ and $u_2(y) = \sinh \lambda(b-y)$ each solve the equation. We verify this for the latter. We have

$$\frac{d}{dy} u_2(y) = -\lambda \cosh \lambda(b-y)\,,$$

and therefore,

$$\frac{d^2}{dy^2} u_2(y) = \lambda^2 \sinh \lambda(b-y) = \lambda^2 u_2(y)\,,$$

This last identity shows what we wanted to verify.

These two functions are linearly independent. This means that if we choose coefficients c_1 and c_2, the only combination of the form $c_1 u_1(y) + c_2 u_2(y)$ that results into the zero function is the combination corresponding to $c_1 = c_2 = 0$. Let's see if that is the case. If

$$c_1 u_1(y) + c_2 u_2(y) = 0\,,$$

then taking the value at $y = 0$ we obtain

$$0 = c_1 u_1(0) + c_2 u_2(0) = c_2 \sinh \lambda b\,.$$

But $\sinh \lambda b \neq 0$, so this implies that $c_2 = 0$. In turn this means that

$$c_1 u_1(y) = 0$$

for all y. So in particular, this is so at $y = 1$. But since $u_1(1) \neq 0$, we conclude that the coefficient c_1 must be zero.

The work above shows that $\{\sinh \lambda y, \sinh \lambda(b-y)\}$ is a linearly independent set of functions each one of which satisfies the second order differential equation $\ddot{Y} - \lambda^2 Y = 0$. So this set is a basis of the space of solutions to this equation. That is to say, any solution Y can be written as a linear combination

$$Y(y) = \alpha_1 \sinh \lambda y + \alpha_2 \sinh \lambda(b-y)\,,$$

for some coefficients α_1, α_2.

7a) Let us consider the constant function $v = v(x,y) = 1$. We attempt to find our solution $u = u(x,y)$ as $u = v + w$ for some function w. Since u and v are harmonic inside the rectangle and the Laplace operator is linear, the function w must be harmonic inside the rectangle as well. On the other hand, since u and v are equal to 1 on the bottom, left, and upper side of the rectangle, we must have that w is zero on those sides. Finally, $\partial_x w(0,y)$ must be zero since the same is truce of $\partial_x u(0,y)$ and $\partial_x v(0,y)$. Thus, w solves the boundary value problem

$$\begin{aligned}
&\Delta w = 0\,, \quad 0 < x < a\,,\ 0 < y < b\,,\\
&\partial_x w(0,y) = 0\,, \quad w = 0 \text{ on the remainder of the boundary}\,.
\end{aligned}$$

We show below that the only solution to this problem is the function $w = 0$. Hence, the solution to problem we started with is the function $u(x,y) = v(x,y) = 1$.

In order to solve this problem for w, we apply the method of separation of variables. There several ways of proceeding here. We describe one below.

If we try to find solutions of $\Delta w = 0$ of the form $w(x,y) = \varphi(x)\psi(y)$ such that $\partial_x w(0,y) = 0$ and $w(a,y) = 0$, then we end up with the eigenvalue problem

$$\begin{aligned} \ddot{\varphi} + \lambda^2\varphi &= 0\,, \\ \dot{\varphi}(0) &= 0\,, \\ \varphi(a) &= 0\,, \end{aligned}$$

and the differential equation

$$\ddot{\psi} - \lambda^2\psi = 0\,.$$

The eigenvalues at $\lambda_0 = 0$ with eigenfunction $\varphi_0 = \cos\lambda_0 x = 1$, and $\lambda_n = (2n+1)\pi/(2a)$, with eigenfunction $\varphi_n = \cos\lambda_n x$. Solving the equation for ψ corresponding to the various eigenvalues, we obtain the family of harmonic functions

$$\cos\lambda_n x(A_n\cosh\lambda_n y + B_n\sinh\lambda_n y)$$

whose normal derivative at $x = 0$ and value at $x = a$ are both zero. We seek for a solution to the boundary value problem that w satisfies of the form

$$w(x,y) = \sum_{n=1}^{\infty}\cos\lambda_n x(A_n\cosh\lambda_n y + B_n\sinh\lambda_n y\,,$$

and fix the coefficients A_n and B_n so that the conditions at $y = 0$ and $y = b$ hold. But from the condition at $y = 0$ we obtain

$$w(x,0) = 0 = \sum_{n=1}^{\infty} A_n\cos\lambda_n x\,,$$

and this forces A_n to be zero for all n. Once we know this, the expression for the solution w we seek only involves sinh terms. Then from the condition at $y = b$ we obtain

$$w(x,b) = 0 = \sum_{n=1}^{\infty} B_n\sinh\lambda b\cos\lambda_n x\,,$$

and this implies that $B_n\sinh\lambda b = 0$ for all n, that in turn imples that $B_n = 0$. So all coefficients are zero and $w(x,y) \equiv 0$, as claimed earlier.