1. Some answers to problems from §2.4

1) Since $a_n \to 0$ as $n \to \infty$, the sequence $|a_n|$ must be bounded. Hence, there exists a constant M such

$$|a_n| < M \text{ for all } n\,.$$

Since the functions $\cos \lambda_n x$ are bounded by 1, in order to prove that the series defining $\partial_x^k u(x,t)$ converges, it is sufficient to prove the series

$$\sum_{n=1}^{\infty} A_n(t_1) n^k$$

converges uniformly (why is that?), that is to say, that

$$\sum_{n=1}^{\infty} |A_n(t_1) n^k|$$

converges. We use comparison and the ratio test.

Indeed,

$$|A_n(t_1)n^k| = |a_n| n^k e^{-\frac{n^2\pi^2}{a^2}kt_1} \leq M n^k e^{-\frac{n^2\pi^2}{a^2}kt_1} := b_n\,.$$

Now the series $\sum_{n=1}^{\infty} b_n$ converges as can be proven using the ration test. For

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^k e^{-\frac{(n+1)^2\pi^2}{a^2}kt_1}}{n^k e^{-\frac{n^2\pi^2}{a^2}kt_1}} = \left(\frac{n+1}{n}\right)^k e^{\frac{\pi^2}{a^2}kt_1(-(n+1)^2+n^2)} = \left(\frac{n+1}{n}\right)^k e^{-\frac{\pi^2}{a^2}kt_1(2n+1)}\,,$$

and since k and t_1 are positive, this quotient goes to zero as $n \to \infty$.

3) The solution to this problem is given by

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{a^2}kt} \cos\frac{n\pi}{a}x\,,$$

where the coefficients are chosen so that $u(x,0)$ is the given initial condition. In our case, the initial condition is the function $T_1 x/a$, and so

$$a_0 = \frac{1}{a}\int_0^a T_1 \frac{x}{a}\,dx = \frac{T_1}{a^2}\int_0^a x\,dx = \frac{T_1}{2}\,,$$

while

$$a_n = \frac{2T_1}{a^2}\int_0^a x\cos\frac{n\pi}{a}x\,dx = \frac{2T_1}{n^2\pi^2}(\cos n\pi - 1)\,,\ n = 1, 2, \ldots$$

Thus, we have

$$u(x,t) = \frac{T_1}{2} + \frac{2T_1}{\pi^2}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)}{n^2} e^{-\frac{n^2\pi^2}{a^2}kt}\cos\frac{n\pi}{a}x\,.$$

2. Some answers to problems from §2.5

1) The boundary value-initial value problem in question is

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \quad &\text{on} \quad 0 < x < a\,,\ 0 < t\,, \\ u(0,t) = T_0 \quad &\text{for} \quad 0 < t\,, \\ \frac{\partial u}{\partial x}(a,t) = 0 \quad &\text{for} \quad 0 < t\,, \\ u(x,0) = f(x) \quad &\text{for} \quad 0 < x < a\,. \end{aligned}$$

The steady-state solution is a function $\nu(x)$ that satisfies the equation and boundary conditions. So we must have

$$\frac{d^2\nu}{dx^2} = 0\,,$$

subject to the conditions

$$\nu(0) = T_0\,, \quad \frac{d\nu}{dx}(a) = 0\,.$$

Any solutions of the differential equation above is given by $\nu(x) = c_1 x + c_2$ for constants c_1 and c_2. If we want $\nu(0) = T_0$, we must choose a solution with $c_2 = T_0$. If we want $\dot{\nu}(a) = 0$, we must choose the solution with $c_1 = 0$. Therefore, the steady-state solution is given by function

$$\nu(x) = T_0\,,$$

a constant.

2) The number $\lambda = 0$ will be an eigenvalue if we can find a non-trivial solution (that is to say, a solution that is not identically zero) of the boundary value problem

$$\ddot{\phi} = 0\,, \quad 0 < x < a\,,$$

$$\phi(0) = 0\,, \quad \phi^{'}(a) = 0\,.$$

The differential equation is solved by $\phi(x) = c_1 x + c_2$, and the choices of constants that make this function satisfy the boundary conditions are $c_1 = c_2 = 0$. So the only solution to the problem above is the trivial function $\phi(x) = 0$. Thus, $\lambda = 0$ is not an eigenvalue.

11) The solution to this problem can be obtained by the procedure discussed in class, or by using the solution to the problem studied in §2.5, subject to the transformation $(x,t) \mapsto (a-x,t)$. We adopt the latter method here.

In the textbook, the problem in §2.5 is solved by

$$u(x,t) = T_0 + (T_1 - T_0)\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}e^{-\frac{(2n-1)^2\pi^2 kt}{4a^2}}\sin\left(\frac{(2n-1)\pi}{2a}x\right).$$

This function satisfies the heat equation, the boundary conditions $u(0,t) = T_0$ and $\partial_x u(a,t) = 0$, and has initial value $u(x,0) = T_1$. We define a new function in terms of this u:

$$v(x,t) = u(a-x,t) = T_0 + (T_1 - T_0)\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}e^{-\frac{(2n-1)^2\pi^2 kt}{4a^2}}\sin\left(\frac{(2n-1)\pi}{2a}(a-x)\right). \tag{1}$$

This is defined for $0 < x < a$ and $t > 0$. Observe that the value of $v(x,t)$ at $x = 0$ corresponds to that of $u(x,t)$ at $x = a$, and vice versa.

Notice that

$$\frac{\partial v}{\partial x}(x,t) = -\frac{\partial u}{\partial x}(a-x,t)\,.$$

Using this twice, we conclude that

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{k}\frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{1}{k}\frac{\partial u}{\partial t} = 0\,.$$

Thus, v solves the heat equation as well.

Now,

$$\frac{\partial v}{\partial x}(0,t) = -\frac{\partial u}{\partial x}(a,t) = 0\,,$$

while

$$v(a,t) = u(0,t) = T_0\,.$$

Thus, the function v satisfies the desired boundary conditions.

Finally, at $t = 0$ we have $v(x,0) = u(a-x,0)$. But at $t = 0$ the function u is the constant T_1. So $u(a-x,0) = T_1 = v(x,0)$.

Therefore, the function $v(x,t)$ defined in (1) is the desired solution to our initial value boundary value problem.