1. Some answers to problems from §2.2

1) The steady-state solution of the given problem is the solution $\nu(x)$ of the following boundary value problem:

$$\frac{d^2\nu}{dx^2} - \gamma^2(\nu - T) = 0\,, \tag{1}$$

$$\nu(0) = T_0\,, \quad \nu(a) = T_1\,. \tag{2}$$

The most general solution to (1) is of the form

$$\nu(x) = \nu_h(x) + \nu_{nh}(x)\,,$$

where ν_h is a solution of the homogeneous equation

$$\frac{d^2\nu}{dx^2} - \gamma^2\nu = 0\,,$$

and $\nu_{nh}(x)$ is any particular solution of (1). It is obvious that one such particular solution is given by the constant function

$$\nu_{nh}(x) = T\,,$$

while the general solution to the homogeneous equation is given by

$$\nu_h(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x}\,.$$

Therefore,

$$\nu(x) = T + c_1 e^{\gamma x} + c_2 e^{-\gamma x}\,.$$

But $\nu(x)$ must satisfy the boundary conditions (2). So

$$\nu(0) = T + c_1 + c_2 = T_0\,, \tag{3}$$

while

$$\nu(a) = T + c_1 e^{\gamma a} + c_2 e^{-\gamma a} = T_1\,. \tag{4}$$

Solving the system of equations (3), (4) for c_1 and c_2, we obtain that

$$c_1 = \frac{T_1 - (T_0 - T)e^{-\gamma a} - T}{e^{\gamma a} - e^{-\gamma a}}\,, \quad c_2 = \frac{T_1 - (T_0 - T)e^{\gamma a} - T}{e^{-\gamma a} - e^{\gamma a}}\,.$$

So the steady-state solution that we search for is given by

$$\nu(x) = T + \frac{T_1 - (T_0 - T)e^{-\gamma a} - T}{e^{\gamma a} - e^{-\gamma a}} e^{\gamma x} + \frac{T_1 - (T_0 - T)e^{\gamma a} - T}{e^{-\gamma a} - e^{\gamma a}} e^{-\gamma x}\,.$$

6a) The steady-state solution solves the boundary value problem

$$\frac{d^2u}{dx^2} = 0\,, \quad 0 < x < a\,,$$

$$\frac{du}{dx}(0) = 0\,, \quad u(a) = T_0\,.$$

The most general solution to the ordinary differential equation is

$$u(x) = c_1 + c_2 x\,.$$

We find the constants c_1 and c_2 by imposing the boundary conditions. By the vanishing of the derivative at $x = 0$, we conclude that $c_2 = 0$. On the other hand, $u(a) = c_1 = T_0$. So the desired steady-state solution is given by the function

$$u(x) = T_0\,,$$

a constant.

2. Some answers to problems from §2.3

1 Let us call $w_n(x,t)$ the terms in the series:

$$w_n(x,t) = -\frac{2}{\pi}\frac{T_0 - (-1)^n T_1}{n} e^{-\lambda_n^2 kt} \sin(\lambda_n kx)\,, \qquad \lambda_n = n\pi/a\,.$$

Then the first four terms in the series are:

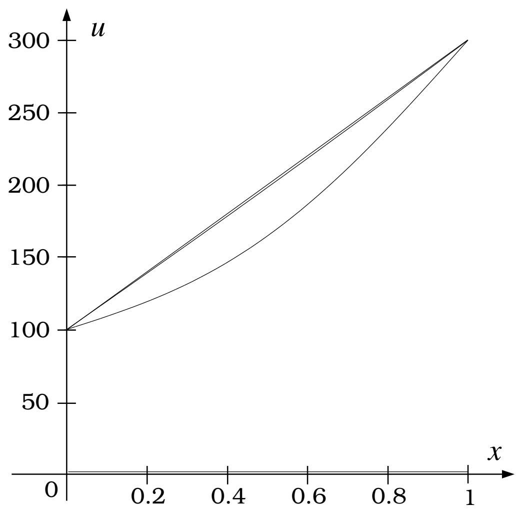
$$w_1(x,t) = -\frac{2}{\pi}(T_0 + T_1) e^{-\frac{\pi^2 kt}{a^2}} \sin\left(\frac{\pi kx}{a}\right),$$

$$w_2(x,t) = -\frac{2}{\pi}\frac{T_0 - T_1}{2} e^{-\frac{4\pi^2 kt}{a^2}} \sin\left(\frac{2\pi kx}{a}\right),$$

$$w_3(x,t) = -\frac{2}{\pi}\frac{T_0 + T_1}{3} e^{-\frac{9\pi^2 kt}{a^2}} \sin\left(\frac{3\pi kx}{a}\right),$$

$$w_4(x,t) = -\frac{2}{\pi}\frac{T_0 - T_1}{4} e^{-\frac{16\pi^2 kt}{a^2}} \sin\left(\frac{4\pi kx}{a}\right).$$

2) Initially, that is to say, at $t = 0$, the function u is identically zero: $u(x,0) = 0$. On the other hand, as $t \to \infty$, the function u approaches the steady-state solution to the problem, that is $\lim_{t\to\infty} u(x,t) = \nu(x) = 100 + 200x$. The graphs below depict the solution at $t = 0$, $t = 0.2$, $t = 0.5$ and $t = \infty$ (since there are very few differences between the graphs at $t = 0.5$, $t = 1.0$ and $t = \infty$, I chose to depict these graphs instead of those asked for in the exercise; these graphs were obtained using Maple):



Notice that there is very little distinction between the graphs of $u(x, 0.5)$ and $u(x,\infty)$. So in just 0.5 sec, the distribution of temperature in the rod is almost equal to the steady-state temperature.

6) Since the boundary conditions are homogeneous, the steady-state solution of this problem is the function identically zero. If we apply the method of separation of variables, we find the family of functions $w_n(x,t) = e^{-\lambda_n^2 kt} \sin \lambda_n x$, $\lambda_n = n\pi/a$, $n = 1, 2, 3, \ldots$, solutions to the heat equation and the given boundary conditions. We then apply the principle of superposition to obtain a solution that satisfies the initial value as well. If

$$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{a}\right)^2 kt} \sin\left(\frac{n\pi x}{a}\right),$$

then

$$w(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) = \beta x\,, \quad 0 < x < a\,,$$

if, and only if, the b_n's are the coefficients of the sine Fourier series of the function βx. So we must choose

$$b_n = \frac{2}{a}\int_0^a \beta x \sin\left(\frac{n\pi x}{a}\right) dx = \frac{2\beta a}{n\pi}\cos n\pi = \frac{2\beta a}{n\pi}(-1)^n\,,$$

and the desired solution to our initial boundary value problem is

$$u(x,t) = \frac{2\beta a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{-\left(\frac{n\pi}{a}\right)^2 kt} \sin\left(\frac{n\pi x}{a}\right).$$