1. Some answers to problems from §1.5

1) We have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{x(2\pi-x)}{4}\,dx=\frac{\pi^2}{6}\,,$$

and so

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\,.$$

(In case you haven't thought about it, this is a rather remarkable result: π is a number defined as half the ratio of the length of the circumference and its radius. That its square could possibly be related to the series whose terms are the inverses of the squares of the whole numbers is, at first glance, impressive. And it is still impressive after a while.)

3) The Fourier sine series of f is the Fourier series of the 2π-periodic extension of x defined on $-\pi < x < \pi$ (correct?). That periodic function is sectionally smooth but *it is not* continuous. So the theorem that permits term-by-term differentiation does not apply.

On the other hand, the Fourier cosine series of f is the Fourier series of the 2π-periodic extension of $|x|$ defined on $-\pi < x < \pi$ (correct?). This periodic function is continuous and sectionally smooth. So the answer in this case is yes: we can indeed differentiate the cosine series of f term-by-term.

We can actually see these two results by direct analysis of the explicit expressions for these sine and cosine series, that are given by

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\,,\qquad \frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi-1}{n^2}\cos nx\,,$$

respectively. The fundamental difference is that the coefficients of the first series decrease as $1/n$, while those of the second do so as $1/n^2$. As you can verify directly, the term-by-term differentiation of these series lead to results that behave quite different, one converges for all x that are not multiples of π to the "square wave function," while the other is quite hard to analyse, and it certainly does not converge to the constant function 1.

8) For the term-by-term integration of the Fourier series of a function to be valid, it is only required that the function be periodic and sectionally continuous. The given sine series is the Fourier series of the 2π-periodic extension of the function x, and such a function satisfies these conditions. So we may integrate the identity

$$x=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\,,\quad 0<x<\pi\,, \tag{1}$$

term-by-term. For convenience, we compute the integral of both sides of this identity from 0 to x (since we call x the upper limit of our integration, we shall use s for the argument of the integrands, and carry the integration with respect to s):

$$\begin{aligned}\int_0^x s\,ds &= 2\sum_{n=1}^{\infty}\int_0^x \frac{(-1)^{n+1}}{n}\sin ns\,ds\\ \frac{x^2}{2} &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}(1-\cos nx)\,,\end{aligned}$$

and so we obtain that

$$x^2 = 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx\,.$$

Multiplying the series (1) by π, and subtracting from the result the series above for x^2, we obtain that

$$\pi x - x^2 = -4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx + 2\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\,, \quad 0<x<\pi\,.$$

(Could you guess from this identity the value of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$?)

9) Since f is an odd periodic function that is sectionally smooth, by the theorem on convergence of Fourier series we know that

$$\sum_{n=1}^{\infty} b_n \sin nx$$

converges at every x to the average $(f(x^+)+f(x^-))/2$. So the function

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin nx\,, \quad t\geq 0\,, \tag{2}$$

restricts at $t=0$ to

$$u(x,0) = \sum_{n=1}^{\infty} b_n e^{-n^2 0}\sin nx = \sum_{n=1}^{\infty} b_n \sin nx = \frac{f(x^+)+f(x^-)}{2}\,.$$

Furthermore, since $\sin 0 = \sin n\pi = 0$, we have that $u(0,t)=u(\pi,t)=0$ for any positive t.

Finally, term by term differentiation with respect to x twice yields the formal expression

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} n^2 b_n e^{-n^2 t}\sin nx\,. \tag{3}$$

The whole issue to go beyond these formal manipulations is to see why the series (2) and (3), that define u and $\partial_x^2 u$, respectively, converge. I will leave the details of that argument to you.

2. Some answers to problems from §2.1

1) This is actually a rather simple exercise provided we assume *Fourier's law of heat conduction* and *Newton's law of cooling* as valid. The first one says that

$$q = -\kappa \frac{\partial u}{\partial x}\,,$$

where $q = q(x,t)$ is the rate of heat flow at a point x and time t (we used this law in the derivation of the heat equation satisfied by u). The second law says that at a point a of the rod exposed to a fluid at temperature $T = T(t)$, the heat is transfered from the rod to the fluid by convection at a rate proportional to the difference in temperature between the rod and the fluid, and so

$$q(a,t) = c(u(a,t) - T(t))\,,$$

where c is the constant of proportionality.

So let us suppose that the point $a = 0$ is exposed to convection. We can take the value at $x = 0$ of the equation expressing Fourier's law and obtain that

$$q(0,t) = -\kappa \frac{\partial u}{\partial x}(0,t)\,.$$

On the other hand, the equation expressing Newton's cooling law will read

$$q(0,t) = c(u(0,t) - T(t))\,.$$

But heat flows in the direction of lower temperature, so if $u(0,t) > T(t)$ the point $x = 0$ looses heat, that is to say, $q(0,t) < 0$, while the opposite is true if $u(0,t) < T(t)$. This property implies that the constant c must be negative, and we write it as $c = -h$ for some positive h. We thus obtain

$$-\kappa \frac{\partial u}{\partial x}(0,t) = -h(u(0,t) - T(t))\,,$$

that simplifies to the condition of third kind, or Robin, given by

$$\kappa \frac{\partial u}{\partial x}(0,t) = hu(0,t) - hT(t)\,.$$

2) Again, this amounts to interpreting the assertion about *Boltzmann law of radiation*: the rate of heat radiation q is proportional to the difference of the fourth power of the temperature u of the rod and the temperature T of the ambient body. So we obtain

$$q(a,t) = \sigma(u^4(a,t) - T^4(t))\,,$$

at a point $x = a$ of the rod exposed to radiation. Notice that in combination with Fourier's law, this yields

$$-\kappa \frac{\partial u}{\partial x}(a,t) = \sigma(u^4(a,t) - T^4(t))$$

at any such point.

8) The function $u(x,t) = e^{-\lambda^2 kt}\cos \lambda x$ has the following partial derivatives:

$$\frac{\partial}{\partial t}u(x,t) = -\lambda^2 k e^{-\lambda^2 kt}\cos \lambda x = -\lambda^2 k u(x,t)\,, \tag{4}$$

$$\frac{\partial}{\partial x}u(x,t) = -\lambda e^{-\lambda^2 kt}\sin\lambda x\,, \tag{5}$$

$$\frac{\partial^2}{\partial x^2}u(x,t) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u\right) = -\lambda^2 e^{-\lambda^2 kt}\cos\lambda x = -\lambda^2 u(x,t)\,. \tag{6}$$

Therefor, if we divide (4) by k and subtract (6) from the result, we obtain

$$\frac{1}{k}\frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = -\lambda^2 u(x,t) - (-\lambda^2 u(x,t)) = 0\,,$$

as desired.

A similar argument applies to the function defined by $e^{-\lambda^2 kt}\sin\lambda x$ instead.

These two functions show that the heat equation alone may have more than one solution. Uniqueness, if at all, must come from the imposition of further conditions at the boundary of the (x,t)-region where this equation is solved. You should think of this as somewhat analogous to solutions of ordinary differential equations, that are unique if one imposes an initial condition.