1. Some answers to problems from §1.1

1b) The function $f(x) = |x|$, $-\pi < x < \pi$ is even. Therefore, its Fourier series expansion only involves the cosine terms, as all the coefficients of the sine terms vanish for even functions.

We have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\, dx = \frac{1}{2\pi} 2\int_{0}^{\pi} |x|\, dx = \frac{1}{2\pi}\int_{0}^{\pi} 2x\, dx = \frac{\pi}{2}\,.$$

On the other hand,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\, dx = \frac{2}{\pi}\int_{0}^{\pi} x \cos(nx)\, dx\,,$$

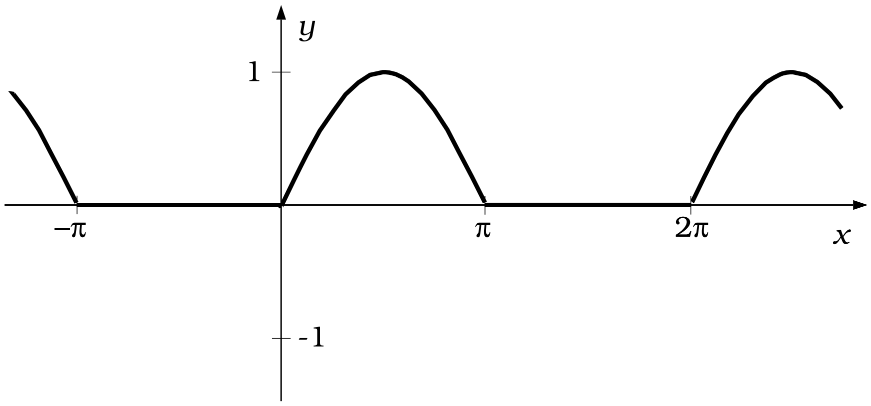
and integration by parts yields

$$a_n = \frac{2}{\pi}\frac{\cos(n\pi) - 1}{n^2} = \begin{cases} -\dfrac{4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

The desired Fourier series is given by

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}\,.$$

2d) The graph over a suitable region is depicted below:



3) By definition, a function $f(x)$ is periodic of period $p > 0$ if two conditions are satisfied:
   a) $f(x)$ is defined for all $x \in \mathbb{R}$.
   b) $f(x + p) = f(x)$ for all x.

The constant function $f(x) = 1$ is defined everywhere, so condition (a) holds. Now, let us choose an arbitrary $p > 0$. We show that (b) holds no matter what p is chosen. For $f(x + p) = 1 = f(x)$ for all x, because $f(x)$ is the constant 1 for any value of x.

Thus, $f(x) = 1$ is periodic of period p, for any $p > 0$.

2. Some answers to problems from §1.2

1b) This is a rescaling of the interval of periodicity of an example discussed in class, the example given over the interval $[-\pi, \pi]$ while here we are asked to work over the interval $[-2, 2]$. We'll recompute the coefficients all over (repeating the example), though the desired result follows from what we did in class. Do you see how?

Since the function is odd, only the sine terms survive. And we have

$$b_n = \frac{1}{a}\int_{-a}^{a} f(x)\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{a}\int_{0}^{a} f(x)\sin\left(\frac{n\pi x}{a}\right)dx\,,$$

where $a = 2$. Thus,

$$b_n = \int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx = \frac{2}{\pi}\frac{1-\cos(n\pi)}{n} = \begin{cases} \dfrac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}\,,$$

and the desired Fourier series is

$$f(x) \sim \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin\left(\frac{(2k+1)\pi x}{2}\right)}{2k+1}\,.$$

2) We use the identity

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

with $\alpha = n\pi x/a$, $\beta = m\pi x/a$. Thus,

$$\int_{-a}^{a}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)dx = \int_{-a}^{a}\left(\sin\left(\frac{(n+m)\pi x)}{a}\right) + \sin\left(\frac{(n-m)\pi x)}{a}\right)\right)dx\,,$$

which is zero because the integrands are odd functions and the interval of integration is symmetric about the origin (if you don't see this, you should carry out the integrations explicitly).

4) For any function $f(x)$, its decomposition into the sum of an even and an odd function is just

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = f_{even}(x) + f_{odd}(x)\,.$$

If $f(x) = e^x$, we have

$$\begin{aligned} f_{even}(x) &= \frac{e^x + e^{-x}}{2} = \cosh x\,, \\ f_{odd}(x) &= \frac{e^x - e^{-x}}{2} = \sinh x\,, \end{aligned}$$

and the result follows.

5) (b) and (c) are even; (a), (d) and (e) are odd; (f) is neither even nor odd.

11d) The half-range expansion of the even extension of the function $f(x) = \sin x$, $0 < x < \pi$ was given in class. Since the even extension is given by $|\sin x|$, by the result in class, we have

$$|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}\,.$$

On the other hand, the odd extension of this function is just the function $\sin x$ over the interval $-\pi < x < \pi$. Its sine series consists of a single term, the sine function itself.