REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 5, Pages 404–436 (October 26, 2001)
S 1088-4165(01)00139-X

BRANCHING THEOREMS FOR COMPACT SYMMETRIC SPACES

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ABSTRACT. A compact symmetric space, for purposes of this article, is a quotient G/K, where G is a compact connected Lie group and K is the identity component of the subgroup of fixed points of an involution. A branching theorem describes how an irreducible representation decomposes upon restriction to a subgroup. The article deals with branching theorems for the passage from G to $K_2 \times K_1$, where $G/(K_2 \times K_1)$ is any of $U(n+m)/(U(n) \times U(m))$, $SO(n+m)/(SO(n) \times SO(m))$, or $Sp(n+m)/(Sp(n) \times Sp(m))$, with $n \leq m$. For each of these compact symmetric spaces, one associates another compact symmetric space G'/K_2 with the following property: To each irreducible representation (σ, V) of G whose space V^{K_1} of K_1-fixed vectors is nonzero, there corresponds a canonical irreducible representation (σ', V') of G' such that the representations $(\sigma|_{K_2}, V^{K_1})$ and (σ', V') are equivalent. For the situations under study, G'/K_2 is equal respectively to $(U(n) \times U(n))/\mathrm{diag}(U(n))$, $U(n)/SO(n)$, and $U(2n)/Sp(n)$, independently of m. Hints of the kind of "duality" that is suggested by this result date back to a 1974 paper by S. Gelbart.

1. Branching Theorems

Branching theorems tell how an irreducible representation of a group decomposes when restricted to a subgroup. The first such theorem historically for a compact connected Lie group is due to Hermann Weyl. It already appeared in the 1931 book [W] and described how a representation of the unitary group $U(n)$ decomposes when restricted to the subgroup $U(n-1)$ embedded in the upper left $n-1$ entries. With respect to standard choices, the highest weight of the given representation may be written in the modern form $a_1 e_1 + \cdots + a_n e_n$, where $a_1 \geq \cdots \geq a_n$ are integers, or in the more traditional form $(a_1, \ldots, a_n)$. Weyl's theorem is that the representation of $U(n)$ with highest weight $(a_1, \ldots, a_n)$ decomposes with multiplicity one under $U(n-1)$, and the representations of $U(n-1)$ that appear are exactly those with highest weights $(c_1, \ldots, c_{n-1})$ such that

$$a_1 \geq c_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq c_{n-1} \geq a_n. \tag{1.1}$$

Similar results for rotation groups are due to Murnaghan and appeared in his 1938 book [Mu]; they deal with the passage from $SO(2n+1)$ to $SO(2n)$ and with the passage from $SO(2n)$ to $SO(2n-1)$, and their precise statements appear in §3 below. A corresponding result for the quaternion unitary groups $Sp(n)$ came in 1962, is due to Zhelobenko [Z], and was subsequently rediscovered by Hegerfeldt

Received by the editors March 20, 2001 and, in revised form, September 10, 2001.
2000 *Mathematics Subject Classification.* Primary 20G20, 22E45; Secondary 05E15.
Key words and phrases. Branching rule, branching theorem, representation.

[Heg]; it deals with the passage from $Sp(n)$ to $Sp(n-1)$, and its precise statement is in §4 below.

The present paper deals with branching theorems for passing in certain other cases from a compact connected Lie group G to a closed connected subgroup K. The original interest in such theorems seems to have been in analyzing the effect of the breaking of symmetry in quantum mechanics, and such theorems subsequently found other applications in mathematical physics. In mathematics nowadays the theorems tend to be studied as tools for decomposing induced representations via Frobenius reciprocity.

An unpublished theorem of B. Kostant from the 1960s, recited in a special case by J. Lepowsky [Lep] and in the general case by D. A. Vogan [V], provides one description of branching in this setting. Following Lepowsky's formulation, suppose that a regular element of K is regular in G; equivalently suppose that the centralizer in G of a maximal torus S of K is abelian and is therefore a maximal torus T of G. Let us denote complexified Lie algebras of G, K, T, $\ldots$ by $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{t}^{\mathbb{C}}$, $\ldots$. Let Δ_G be the set of roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, let Δ_K be the set of roots of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{s}^{\mathbb{C}})$, and let W_G be the Weyl group of Δ_G. Introduce compatible positive systems Δ_G^+ and Δ_K^+ by defining positivity relative to a K-regular element of $\mathfrak{s}^{\mathbb{C}}$, let bar denote restriction from the dual $(\mathfrak{t}^{\mathbb{C}})^*$ to the dual $(\mathfrak{s}^{\mathbb{C}})^*$, and let δ_G be half the sum of the members of Δ_G^+. The restrictions to $\mathfrak{s}^{\mathbb{C}}$ of the members of Δ_G^+, repeated according to their multiplicities, are the nonzero positive weights of $\mathfrak{s}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$; deleting the members of Δ_K^+, each with multiplicity one, from this set, we obtain the set Σ of positive weights of $\mathfrak{s}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}/\mathfrak{k}^{\mathbb{C}}$, repeated according to multiplicities. The associated *Kostant partition function* is defined as follows: $\mathcal{P}(\nu)$ is the number of ways that a member of $(\mathfrak{s}^{\mathbb{C}})^*$ can be written as a sum of members of Σ, with the multiple versions of a member of Σ being regarded as distinct.

Kostant's Branching Theorem. *Let G be a compact connected Lie group, let K be a closed connected subgroup, let $\lambda \in (\mathfrak{t}^{\mathbb{C}})^*$ be the highest weight of an irreducible representation σ of G, and let $\mu \in (\mathfrak{s}^{\mathbb{C}})^*$ be the highest weight of an irreducible representation τ of K. Then the multiplicity of τ in the restriction of σ to K is given by*

$$m_\lambda(\mu) = \sum_{w \in W_G} (\operatorname{sgn} w) \mathcal{P}(\overline{w(\lambda + \delta_G)} - (\mu + \overline{\delta_G})).$$

Kostant obtained this theorem as a generalization of his formula for the multiplicity of a weight [Ko]; this is the case that K is the maximal torus T and that $S = T$. A simple proof of the main result of [Ko] was found by P. Cartier [C] and is reproduced in [Kn] in an appropriate framework that first appeared in [BGG]. It is a straightforward matter to adapt this proof to prove the above branching theorem. More discussion of Kostant's theorem may be found in the book [GoW].

The hypothesis on regular elements in the Kostant branching theorem is satisfied when $\operatorname{rank} G = \operatorname{rank} K$ and also when K is the identity component of the group of fixed points of an involution (cf. Proposition 6.60 of [Kn]). The latter situation is the one that will concern us in this paper, and we shall refer to it as the situation of a *compact symmetric space*. Unfortunately the alternating sum in the Kostant theorem involves a great deal of cancellation that, in practice, is usually too hard to sort out.

A variant of Kostant's theorem, without the hypothesis on regular elements, was published by van Daele [Da] in 1970. It uses the multiplicity formula of [Ko] for both G and K and puts together the results. The formula is different from the one above but still involves an alternating sum over the Weyl group. Other authors, particularly with applications to physics in mind, have looked for algorithms that compute the branching recursively in any desired case, preferably with minimal effort. The paper of Patera and Sharp [PaS] is notable in this direction. Branching theorems that supply information for use via Frobenius reciprocity tend not to benefit from this kind of effort, however, and we shall not pursue them here.

In fact, practical formulas for complete branching from G to K that are helpful in applying Frobenius reciprocity in the setting of a compact symmetric space are available in only limited circumstances. We have already mentioned the classical branching theorems for unitary groups, rotation groups, and quaternion unitary groups. The results for rotation groups extend readily to spin groups [Mu]. One relatively easy branching formula is the case of passing from $G \times G$ to $\operatorname{diag} G$; the restriction of (σ, σ') to $\operatorname{diag} G$ is nothing more than the tensor product $\sigma \otimes \sigma'$, for which a well-known decomposition formula of Steinberg [St] is more useful than Kostant's Branching Theorem if the weights of σ or σ' are known. For some specific groups, there are combinatorial formulas for decomposing tensor products $\sigma \otimes \sigma'$. The best known of these is the Littlewood-Richardson rule [LiR] for $U(n)$. Some other such formulas may be found in D. E. Littlewood's book [Liw]. A cancellation-free formula for decomposing tensor products for any compact semisimple Lie group has been given more recently by P. Littlemann [Lim].

Another complete branching formula, which is much more complicated, is for the passage from $Sp(n+1)$ to $Sp(n) \times Sp(1)$ ([Lep], [Lee]). Littlewood [Liw], working under the assumption that tensor products for unitary groups are understood, built on ideas in [Mu] and obtained branching formulas for the passage from $U(n)$ to $O(n)$ (p. 240) and from $U(2n)$ to $Sp(n)$ (p. 295) under a condition on the highest weight, namely that it end in 0's and have only a limited number of nonzero entries—at most $[n/2]$ in the case of $O(n)$ and at most n in the case of $Sp(n)$. Newell [N] showed how Littlewood's result could be modified to remove the limitation on the number of nonzero entries. Statements of Littlewood's results for $O(n)$ and $Sp(n)$ with all the hypotheses in place appear in [DQ] and [Ma], respectively, and references to modern proofs may be found in [Ma]. Deenen and Quesne ([DQ] and [Q]) worked with $Sp(n)/U(n)$ and the theory of dual reductive pairs in doing a deeper study of $U(n)/O(n)$.

Instead of a complete analysis of branching from some groups G to their subgroups K, the main objective of the present paper is to produce some partial branching formulas for G that help decompose those induced representations arising most often in practice. One class of such induced representations consists of left regular representations of the form $L^2(G/K)$, which is nothing more than the result of inducing to G the trivial representation of K. By Frobenius reciprocity the multiplicity of an irreducible representation σ of G in this L^2 space equals the multiplicity of 1 in the restriction of σ to K. When G/K is a compact symmetric space, this multiplicity is given by a theorem of S. Helgason in §I.3 of [Hel] (see Theorem 8.49 of [Kn]). Our main interest is in the case that G/K is a fibration of one compact symmetric space by another, i.e., that there exists a closed connected subgroup K' such that $G \supseteq K' \supseteq K$ and such that G/K' and K'/K are compact symmetric spaces.

One way in which this kind of double fibration arises was pointed out by M. W. Baldoni Silva [Ba] and is in the analysis of a maximal parabolic subgroup of a noncompact real semisimple Lie group G with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, and let θ be the corresponding Cartan involution of $\mathfrak{g}$. In this situation one is led to a decomposition

$$\mathfrak{g} = (\mathfrak{a} \oplus \mathfrak{m}) \oplus \sum_{n=-N}^{N} \mathfrak{g}_{n\alpha}, \tag{1.2}$$

where $\mathfrak{a}$ is a 1-dimensional subspace of $\mathfrak{p}$, α is a nonzero linear functional on $\mathfrak{a}$, and $\mathfrak{g}_{n\alpha}$ is the simultaneous eigenspace for eigenvalue $n\alpha$ under the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$. The 0 eigenspace is the direct sum of $\mathfrak{a}$ and a θ-stable subalgebra $\mathfrak{m}$. Let K and M be the analytic subgroups of G with Lie algebras $\mathfrak{k}$ and $\mathfrak{m}$. The interest is in $L^2(K/(K \cap M))$. When the integer N in (1.2) is 1, $K/(K \cap M)$ is a compact symmetric space, and Helgason's theorem answers our question. Situations with $N = 1$ arise infrequently, however, and we are more interested in the cases $N = 2$ and $N = 3$, which are the normal thing. (In classical groups, N is at most 2, but N can be as large as 6 in exceptional groups.) In this case let

$$\mathfrak{g}' = (\mathfrak{a} \oplus \mathfrak{m}) \oplus \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{2\alpha},$$

let $\mathfrak{k}' = \mathfrak{g}' \cap \mathfrak{k}$, and let K' be the analytic subgroup of G with Lie algebra $\mathfrak{k}'$. Then K/K' and $K'/(K \cap M)$ are compact symmetric spaces, and $K/(K \cap M)$ is exhibited as a fibration of one compact symmetric space by another.

Related fibrations of one compact symmetric space by another occur in the work of W. Schmid [Sc] and S. Greenleaf [Gr].

The case that was of most interest to Baldoni Silva in [Ba] had $M \subset K$ with

$$K = Sp(n) \times Sp(1), \quad K' = Sp(n-1) \times Sp(1) \times Sp(1),$$
$$K \cap M = Sp(n-1) \times \operatorname{diag} Sp(1).$$

Induction from $K \cap M$ to K of the trivial representation of $K \cap M$ can be done in stages, and the result at the stage of K' is the sum of all representations $(1, \tau^c, \tau)$, where τ is an irreducible representation of $Sp(1)$ and $(\,\cdot\,)^c$ denotes contragredient. The important thing is that all the intermediate representations $(1, \tau^c, \tau)$ are trivial on the complicated factor $Sp(n-1)$ of K'. Consequently, the only branching theorem from K to K' that is needed to study $L^2(K/(K \cap M))$ is a branching theorem that looks for constituents that are trivial on the factor $Sp(n-1)$ of K'. Not every double fibration arising from (1.2) involves a product decomposition as in this Baldoni Silva example, but enough of them do to make their systematic study to be of interest.

We undertake such a study in this paper. Thus we are interested in branching for compact symmetric spaces $G/(K_2 \times K_1)$. We regard K_1 as the larger of K_1 and K_2. For an irreducible representation (σ, V) of G, let V^{K_1} be the subspace of vectors fixed by K_1. We seek the decomposition of this space under K_2.

Main Theorem. *For the three types of symmetric space G/K given in Table 1 and having K of the form $K = K_2 \times K_1$ with K_1 larger than K_2, there is another compact symmetric space G'/K_2 with the following property: To each irreducible representation (σ, V) of G whose space V^{K_1} of K_1-fixed vectors is nonzero, there corresponds a canonical irreducible representation (σ', V') of G' such that the representations $(\sigma|_{K_2}, V^{K_1})$ and $(\sigma'|_{K_2}, V')$ are equivalent.*

TABLE 1. Situations to which the Main Theorem applies

G	$K_2 \times K_1$	G'/K_2	Theorem
$U(n+m),\ n \le m$	$U(n) \times U(m)$	$(U(n) \times U(n))/\operatorname{diag} U(n)$	2.1
$SO(n+m),\ n \le m$	$SO(n) \times SO(m)$	$U(n)/SO(n)$	3.1
$Sp(n+m),\ n \le m$	$Sp(n) \times Sp(m)$	$U(2n)/Sp(n)$	4.1

Remarks. 1. In the case of $U(n+m)$, an irreducible representation σ' of $U(n)\times U(n)$ is of the form $(k', k'') \mapsto \sigma'(k') \otimes \sigma'(k'')$, and the restriction to the diagonal is of the form $k \mapsto \sigma'(k) \otimes \sigma'(k)$. In other words, the theorem is that $(\sigma|_{K_2}, V^{K_1})$ is the tensor product of two irreducible representations of $K_2 \cong U(n)$.

2. The theorem does not describe the decomposition of $(\sigma'|_{K_2}, V')$ into irreducible representations, but the information that the theorem gives is in some ways better. For example, in the case of $U(n+m)$, the tensor product of two irreducible representations can always be decomposed into irreducibles by the Littlewood-Richardson rule [LiR], but there seems to be no easy prescription for saying when a sum of certain irreducible representations is actually a tensor product. Similarly Littlewood's theorems mentioned above allow for the decomposition of the representation $(\sigma'|_{K_2}, V')$ in the SO and Sp cases, but the information that the restriction is coming from an irreducible representation of G' does not seem to be encoded in the restriction in an easy way.

3. In notation that will be explained at the beginnings of §§2–4, the condition on the highest weight of σ for V^{K_1} to be nonzero turns out to be that the highest weight is of the form

$$\begin{array}{ll} (a_1, \dots, a_n, 0, \dots, 0, a_1', \dots, a_n') & \text{in the case of } U(m+n), \\ (a_1, \dots, a_n, 0, \dots, 0) & \text{in the case of } SO(m+n), \\ (a_1, \dots, a_{2n}, 0, \dots, 0) & \text{in the case of } Sp(m+n), \end{array}$$

and the highest weight of σ' in the respective cases is taken to be

$$\begin{array}{ll} (a_1, \dots, a_n)(a_1', \dots, a_n') & \text{in the case of } U(n) \times U(n), \\ (a_1, \dots, a_{n-1}, |a_n|) & \text{in the case of } U(n), \\ (a_1, \dots, a_{2n}) & \text{in the case of } U(2n). \end{array}$$

4. From Remark 3 it is apparent, in each case of the Main Theorem other than for $SO(n+m)$ with $n = m$, that the function $\sigma \mapsto \sigma'$ is *one-one* on the set of irreducible representations of G with nonzero K_1-fixed vectors, and in every case $\sigma \mapsto \sigma'|_{K_2}$ is *onto* the set of all restrictions of irreducible representations of G'.

5. Because of the absolute value signs in $|a_n|$ in Remark 3 and the exception to the one-oneness of $\sigma \mapsto \sigma'$ in Remark 4, it is tempting to rephrase the rotation-group case of the Main Theorem in terms of orthogonal groups. This rephrasing solves some expository problems while creating others, and we shall not pursue it.

6. One way of viewing the Main Theorem is as a generalization of Helgason's theorem in §I.3 of [Hel] that gives, in the case of a compact symmetric space G/K, the multiplicity of the trivial representation 1 in the restriction to K of an irreducible representation of G. The above Main Theorem gives, for any of the listed compact symmetric spaces, the multiplicity of a representation σ in the restriction to K of an irreducible representation of G under the assumption that σ is of the form $\tau \otimes 1$.

7. In each of the cases of the Main Theorem the positions of the blocks K_1 and K_2 can be reversed because $K_2 \times K_1$ is conjugate to $K_1 \times K_2$ within G.

8. For all three cases the rank of the symmetric space $G/(K_2 \times K_1)$ equals the rank of G'/K_2. This fact seems to play only a minor role in the proof, however.

The proof of the Main Theorem will occupy most of the remainder of the paper. Most of the ideas for the proof are present for the unitary case, and that case will be handled in §2. The statements of the results in the rotation and quaternion unitary cases, together with the necessary modifications in the proofs, are in §3 and §4.

A clue to the situation established by the Main Theorem appears in the paper [Ge] of S. S. Gelbart. For the case of $SO(n+m)$ with $n \leq m$, Gelbart observed for any representation (σ, V) of $SO(n+m)$ that the dimension of $V^{SO(m)}$ equals the dimension of a certain representation of $U(n)$ that he associated to the highest weight of σ. He demonstrated this equality of dimensions by a direct argument that did not involve calculating the dimensions in the respective cases, and he wondered whether his equality was an indication of some undiscovered duality. In fact, Gelbart's representation of $U(n)$ is the representation σ' in the SO case of the Main Theorem. His argument generalizes to all three cases of the Main Theorem, and it can be regarded as the main step of the proof.

The proof that we give constructs a certain equivariant linear mapping and shows that this mapping is one-one onto. At least for the unitary case, a combinatorial proof is possible that ignores the linear mapping and instead shows the equality of two versions of Kostant's branching theorem. However, the combinatorial proof is longer, taking approximately 30 pages to handle just the unitary case. So far, it has not been possible to push the combinatorial proof through in the rotation case except when n is small.

I am indebted to Roe Goodman and to David Vogan—to Goodman for making me aware of the extensive history in the subject of branching theorems, especially of the work of D. E. Littlewood, and to Vogan for suggesting ways to streamline the exposition.

2. Main Theorem for Unitary Groups

In this section we shall state and prove the Main Theorem corresponding to $U(n+m)$ in the left column of Table 1. Concerning the representation theory of unitary groups, we use the following notation: The roots for $U(N)$ are all nonzero linear functionals $e_r - e_s$ in the dual $\mathfrak{h}^*$ of the diagonal subalgebra with $1 \leq r, s \leq N$. We take the positive ones to be those with $r < s$. Dominant integral forms for $U(N)$ are expressions $a_1 e_1 + \cdots + a_N e_N$ with all a_r in $\mathbb{Z}$ and with $a_1 \geq \cdots \geq a_N$. We write such an expression as an N-tuple $(a_1, \ldots, a_N)$. We shall make use of Weyl's branching theorem (1.1) for restriction from $U(N)$ to $U(N-1)$.

Theorem 2.1. *Let $1 \leq n \leq m$, and regard $U(n)$ and $U(m)$ as embedded as block diagonal subgroups of $U(n+m)$ in the standard way with $U(n)$ in the upper left diagonal block and with $U(m)$ in the lower right diagonal block.*

(a) *If $(a_1, \ldots, a_{n+m})$ is the highest weight of an irreducible representation (σ, V) of $U(n+m)$, then a necessary and sufficient condition for the subspace $V^{U(m)}$ of $U(m)$ invariants to be nonzero is that $a_{n+1} = \cdots = a_m = 0$ and that (in case $m = n$) also $a_n \geq 0$ and $a_{m+1} \leq 0$.*

(b) *Let* $(a_1,\dots,a_n,0,\dots,0,a'_1,\dots,a'_n)$ *be the highest weight of an irreducible representation* (σ,V) *of* $U(n+m)$ *with a nonzero subspace of* $U(m)$ *invariants, and let* τ_1 *and* τ_2 *be irreducible representations of* $U(n)$ *with highest weights* $(a_1,\dots,a_n)$ *and* $(a'_1,\dots,a'_n)$. *Then the representations* $(\sigma|_{U(n)},V^{U(m)})$ *and* $\tau_1\otimes\tau_2$ *of* $U(n)$ *are equivalent, i.e.,* $(\sigma|_{U(n)},V^{U(m)})$ *is equivalent with the restriction to* $\operatorname{diag}U(n)$ *of the representation* $\sigma'=(\tau_1,\tau_2)$ *of* $U(n)\times U(n)$.

Proof of (a). To restrict σ from $U(n+m)$ to $U(m)$, we shall iterate Weyl's branching theorem (1.1) for unitary groups. Write $(a_1^{(0)},\dots,a_{n+m}^{(0)})$ for $(a_1,\dots,a_{n+m})$, and let $(a_1^{(l)},\dots,a_{n+m-l}^{(l)})$ be specified inductively so that

$$a_1^{(l-1)}\ge a_1^{(l)}\ge a_2^{(l-1)}\ge\cdots\ge a_{n+m-l}^{(l-1)}\ge a_{n+m-l}^{(l)}\ge a_{n+m-l+1}^{(l-1)}.$$

According to the branching formula, the restriction of σ contains all irreducible representations of $U(m)$ with highest weights $(a_1^{(n)},\dots,a_m^{(n)})$ and no others. Thus we seek a necessary and sufficient condition for the m-tuple $0=(0,\dots,0)$ to arise.

Examining the formulas, we see that $a_r^{(l)}\ge a_{r+s}^{(l-s)}$ whenever the indices are in bounds; taking $l=s=n$ and $r=1$, we see that the condition $0\ge a_{n+1}$ is necessary for the m-tuple 0 to arise. Also $a_r^{(l)}\ge a_r^{(l+s)}$ whenever the indices are in bounds; taking $l=0$ and $r=m$ and $s=n$, we see that $a_m\ge 0$ is necessary for the m-tuple 0 to arise. The necessity of the condition in (a) follows from the assumed dominance of the given highest weight.

For the sufficiency, suppose that $a_n\ge 0$, $a_{n+1}=\cdots=a_m=0$, and $a_{m+1}\le 0$. Define

$$a_r^{(l)}=\begin{cases} a_{l+r} & \text{for } 1\le r\le n-l\\ 0 & \text{for } n-l<r\le m\\ a_r & \text{for } m<r\le n+m-l.\end{cases}$$

Then the $a_r^{(l)}$ have the right interleaving property, and $a_r^{(n)}=0$. Thus σ has a nonzero subspace of $U(m)$ invariants.

We turn to the proof of Theorem 2.1b. Actually we shall cast most of the argument in a form in which it will apply with G equal to $SO(n+m)$ or $Sp(n+m)$, as well as $U(n+m)$. We begin with an outline of that general argument, and then we fill in the details that apply to all three classes of groups. In supplying the details, we shall sometimes prove facts that are not strictly needed for the proof but that give insight into the overall structure. After giving the details that apply to all three classes of groups, we shall finish the details for $U(n+m)$, returning to $SO(n+m)$ and $Sp(n+m)$ in §§3 and 4.

First we give the outline of the general argument. We introduce a "dual" group G^d, which will be $U(n,m)$, $SO(n,m)_0$, and $Sp(n,m)$ in the respective cases; these are the identity components of isometry groups with respect to a standard indefinite Hermitian form over $\mathbb{C}$, $\mathbb{R}$, and the quaternions $\mathbb{H}$. We pass by Weyl's unitary trick from σ as a representation of G on V to σ as a representation of G^d on V. The highest weight of σ relative to G^d is expressed in terms of a maximally compact Cartan subgroup of G^d; this group is compact except in the case of $SO(n,m)_0$ with n and m both odd. We introduce a maximally noncompact Cartan subgroup of G^d and an appropriate ordering relative to it. Examining the restricted-root spaces, we pick out a general linear group L sitting as a subgroup of G^d; this will be $GL(n,\mathbb{C})$,

$GL(n,\mathbb{R})_0$, and $GL(n,\mathbb{H})$ in the respective cases. Let K_L be the standard maximal compact subgroup of this general linear group L; the subgroup K_2, which is one of $U(n)$, $SO(n)$, and $Sp(n)$, is canonically isomorphic to K_L by a map ι. We take v_0 to be a highest weight vector of σ in this new ordering. The cyclic span of v_0 under L is denoted V', and the restriction of $\sigma|_L$ to V' is denoted σ'. The representation (σ', V') of L is irreducible. Let E be the projection of V onto V^{K_1} given by integrating $v \mapsto \sigma(k)v$ over K_1. If we take the isomorphism $K_L \cong K_2$ into account, the map E is equivariant with respect to K_2. An argument that uses the formula $K = K_1 K_L$ and the Iwasawa decomposition in G^d shows that E carries the subspace V' onto V^{K_1}.

The group L and the representation (σ', V') are transferred from G^d back to G, and the result is a strangely embedded subgroup G' of G isomorphic to $U(n)\times U(n)$, $U(n)$, or $U(2n)$ in the respective cases, together with an irreducible representation of G' that we still write as (σ', V'). The group K_L, which is also a subgroup of G, does not move in this process and hence may be regarded as a subgroup of G', embedded in the standard way that $U(n)$, $SO(n)$, and $Sp(n)$ are embedded in $U(n)\times U(n)$, $U(n)$, and $U(2n)$, respectively. However, some care is needed in working with this inclusion: the identification of G' as isomorphic to $U(n)\times U(n)$, $U(n)$, or $U(2n)$ has to allow for outer automorphisms of $U(n)\times U(n)$, $U(n)$, or $U(2n)$. For example, in embedding $U(n)$ diagonally in $U(n)\times U(n)$, we must distinguish between $U(n)$ and $\overline{U(n)}$ in the second factor in order to distinguish a tensor product $\sigma_1' \otimes \sigma_2'$ from $\sigma_1' \otimes {\sigma_2'}^c$, which has a contragredient in the second factor.

Unwinding the highest weights in question and using the indicated amount of care, we see that the highest weights match those in the statement of the theorem. Finally we use Gelbart's observation, adapted from the SO case to all of our original groups G, to show that $\dim V' = \dim V^{K_1}$; hence E is an equivalence on the level of representations of $K_L \cong K_2$. This completes the outline of the general argument.

Now we come to the details. In relating G and G^d, we shall be using Riemannian duality. Usually this duality refers to two semisimple (or perhaps reductive) groups G and G^d with G compact and G^d noncompact such that the Lie algebra $\mathfrak{g}^d$ of G^d has a Cartan decomposition $\mathfrak{g}^d = \mathfrak{k}\oplus\mathfrak{p}$ and the Lie algebra of G is given by $\mathfrak{g} = \mathfrak{k}+i\mathfrak{p}$. However, we shall impose in addition a global condition on the pair (G, G^d) so that we do not err by a covering map in the construction of the subgroups L and G'. The global condition will be that G and G^d are realized as matrix groups with isomorphic complexifications, and we insist that an isomorphism be fixed between their complexifications.

The groups G and their respective subgroups $K = K_2 \times K_1$ are as in Table 1, and we write $\mathfrak{k}$ for the Lie algebra of K. The respective noncompact groups G^d corresponding to G are, as we said above, the indefinite isometry groups $U(m,n)$, $SO(n,m)_0$, and $Sp(n,m)$; here we regard $Sp(n,m)$ as a group of square matrices of size $n+m$ over the quaternions $\mathbb{H}$. The quaternions are taken to have the usual $\mathbb{R}$ basis $\{1, i, j, k\}$.

The respective groups K are subgroups of G^d as well as of G. We write $\mathfrak{g} = \mathfrak{k}\oplus i\mathfrak{p}$ and $\mathfrak{g}^d = \mathfrak{k}\oplus\mathfrak{p}^d$ with $i\mathfrak{p}$ and $\mathfrak{p}^d$ given by the sets of matrices $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$ in $\mathfrak{g}$ and $\mathfrak{g}^d$. The involutions of $\mathfrak{g}$ and $\mathfrak{g}^d$ fixing $\mathfrak{k}$ and acting by -1 on $\mathfrak{p}$ and $\mathfrak{p}^d$, respectively, are denoted θ and θ^d. We select and fix realizations of $\mathfrak{g}^{\mathbb{C}}$ and $(\mathfrak{g}^d)^{\mathbb{C}}$ as complex Lie algebras of complex matrices, together with a $\mathbb{C}$ isomorphism φ of $\mathfrak{g}^{\mathbb{C}}$ onto $(\mathfrak{g}^d)^{\mathbb{C}}$, such that

(i) φ lifts to an isomorphism, also called φ, of the corresponding analytic groups $G^{\mathbb{C}}$ and $(G^d)^{\mathbb{C}}$ of complex matrices,
(ii) G and G^d map one-one into their complexifications $G^{\mathbb{C}}$ and $(G^d)^{\mathbb{C}}$,
(iii) the pull-back to $\mathfrak{k} \subset \mathfrak{g}$ of the Lie-algebra isomorphism φ is the identity mapping from $\mathfrak{k} \subset \mathfrak{g}$ into $\mathfrak{k} \subset \mathfrak{g}^d$, i.e., the diagram

$$\begin{array}{ccc} \mathfrak{g}^{\mathbb{C}} & \xrightarrow{\varphi} & (\mathfrak{g}^d)^{\mathbb{C}} \\ {\scriptstyle \mathrm{inc}}\uparrow & & {\scriptstyle \mathrm{inc}}\uparrow \\ \mathfrak{k} & \xrightarrow{1} & \mathfrak{k} \end{array}, \tag{2.1}$$

in which the maps "inc" are the natural inclusions, commutes, and
(iv) the pull-back to $i\mathfrak{p} \subset \mathfrak{g}$ of the Lie-algebra isomorphism φ carries $\mathfrak{p}$ to $\mathfrak{p}^d$, i.e., the diagram

$$\begin{array}{ccc} \mathfrak{g}^{\mathbb{C}} & \xrightarrow{\varphi} & (\mathfrak{g}^d)^{\mathbb{C}} \\ {\scriptstyle \mathrm{inc}}\uparrow & & {\scriptstyle \mathrm{inc}}\uparrow \\ \mathfrak{p} & \longrightarrow & \mathfrak{p}^d \end{array} \tag{2.2}$$

commutes.

For $G = U(n+m)$ and $SO(n+m)$, we can let $\mathfrak{g}^{\mathbb{C}}$ and $(\mathfrak{g}^d)^{\mathbb{C}}$ be the natural matrix complexifications of $\mathfrak{g}$ and $\mathfrak{g}^d$, and we can let φ be conjugation by the block-diagonal matrix $\left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the respective diagonal blocks being of sizes n-by-n and m-by-m. (Another possible choice with $G = U(n+m)$ is to let φ be the identity map, but we do not use this choice.) For $Sp(n+m)$, the mapping φ is more complicated to set up; first, one has to embed the quaternion matrices into complex matrices of twice the size. We omit the details in this case.

The given representation (σ, V), initially defined on G, extends holomorphically to $G^{\mathbb{C}}$. Using φ to pass to $(G^d)^{\mathbb{C}}$ and then restricting to G^d, we obtain an interpretation for (σ, V) as a representation of G^d.

Any θ stable Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ has a counterpart in $\mathfrak{g}^d$, and vice versa. This correspondence is achieved on a theoretical level by using the same $\mathfrak{k}$ part of $\mathfrak{s}$ in both $\mathfrak{g}$ and $\mathfrak{g}^d$ and by dropping the i in the $i\mathfrak{p}$ part and mapping the $\mathfrak{p}$ part to the $\mathfrak{p}^d$ part via the bottom row of (2.2). Moreover, this correspondence extends to a correspondence for the associated analytic subgroups of G and G^d. On a practical level the correspondence in the case of our particular groups is easy to write down in one realization. If the matrices in question are broken into blocks of sizes m and n and if

$$\mathfrak{s} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \right\} + \left\{ \begin{pmatrix} 0 & Y \\ -Y^* & 0 \end{pmatrix} \right\} \tag{2.3a}$$

is given, then

$$\mathfrak{s}^d = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \right\} + \left\{ \begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix} \right\}; \tag{2.3b}$$

here $(\,\cdot\,)^*$ denotes the ordinary adjoint. In the reverse direction if $\mathfrak{s}^d$ is given by (2.3b), then the corresponding $\mathfrak{s}$ is given by (2.3a).

The given highest weight is defined on a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ consisting, in the cases of $U(n+m)$ and $Sp(n+m)$, of diagonal matrices whose diagonal entries are real multiples of i (with i complex or quaternion in the two cases). In the case of $SO(n+m)$, $\mathfrak{h}$ consists of certain 2-by-2 blocks that will be described more precisely in §3. The subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ lies in $\mathfrak{k}$ in the cases of $U(n+m)$ and $Sp(n+m)$, and we shall arrange that it is θ stable in the case of $SO(n+m)$. Therefore $\mathfrak{g}^d$ in every case contains a corresponding Cartan subalgebra, which we denote $\mathfrak{h}^d$. Among all Cartan subalgebras of $\mathfrak{g}^d$, $\mathfrak{h}^d$ is maximally compact; it is actually compact except for $SO(n,m)_0$ with n and m both odd.

Let us introduce a maximally noncompact θ^d stable Cartan subalgebra $\mathfrak{a}\oplus\mathfrak{t}$ of $\mathfrak{g}^d$. The ingredients $\mathfrak{a}$ and $\mathfrak{t}$ are given in blocks of sizes $n, m-n, n$ by

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & 0 & \begin{matrix} 0 & \cdots & 0 & x_1 \\ 0 & \cdots & x_2 & 0 \\ \vdots & & & \vdots \\ x_n & \cdots & 0 & 0 \end{matrix} \\ 0 & 0 & 0 \\ \begin{matrix} 0 & 0 & \cdots & x_n \\ \vdots & & & \vdots \\ 0 & x_2 & \cdots & 0 \\ x_1 & 0 & \cdots & 0 \end{matrix} & 0 & 0 \end{pmatrix} \right\} \tag{2.4}$$

and

$$\mathfrak{t} = \left\{ \begin{pmatrix} \begin{matrix} iy_1 & 0 & \cdots & 0 \\ 0 & iy_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & iy_n \end{matrix} & 0 & 0 \\ 0 & \begin{matrix}\text{compact} \\ \text{Cartan}\end{matrix} & 0 \\ 0 & 0 & \begin{matrix} iy_n & \cdots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & iy_2 & 0 \\ 0 & \cdots & 0 & iy_1 \end{matrix} \end{pmatrix} \right\}. \tag{2.5}$$

Here the entries of $\mathfrak{a}$ are real, and the entries iy_r of $\mathfrak{t}$ are purely imaginary in the case of $U(n+m)$, are 0 in the case of $SO(n+m)$, and are real multiples of the quaternion i in the case of $Sp(n+m)$. Define f_r of the matrix in (2.4) to be x_r. In the cases of $U(n+m)$ and $Sp(n+m)$, define f_r' of the matrix in (2.5) to be iy_r, this i being the one in $\mathbb{C}$.

The Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{a}\oplus\mathfrak{t}$ of $\mathfrak{g}^d$ are conjugate via $\mathrm{Ad}((G^d)^{\mathbb{C}})$, and we shall need to fix a particular member of $\mathrm{Ad}((G^d)^{\mathbb{C}})$ achieving this conjugation in order to carry weights from $\mathfrak{h}$ to $\mathfrak{a}\oplus\mathfrak{t}$. This transport of weights requires a little care as we do not want to err by an outer automorphism. Cayley transforms are handy for achieving the conjugation, and we return to this point when we consider our three cases separately.

We shall make use also of the Lie subalgebra

$$\mathfrak{b} = \left\{ \begin{pmatrix} \begin{matrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & q_n \end{matrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \begin{matrix} q_n & \cdots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & q_2 & 0 \\ 0 & \cdots & 0 & q_1 \end{matrix} \end{pmatrix} \right\}. \tag{2.6}$$

Here the entries q_r of $\mathfrak{b}$ are 1-by-1 skew Hermitian, i.e., they are imaginary numbers in the case of $U(n+m)$, 0 in the case of $SO(n+m)$, and linear combinations of i, j, k in the case of $Sp(n+m)$.

We introduce a lexicographic ordering on $\mathfrak{a}^*$, the dual of $\mathfrak{a}$, so that

$$f_1 > f_2 > \cdots > f_n.$$

The restricted roots in the cases of $U(m+n)$ and $Sp(n+m)$ are

$$\begin{array}{lll} C_n: & \{\pm f_r \pm f_s,\ r < s\} \cup \{\pm 2f_r\} & \text{if } n = m, \\ (BC)_n: & \{\pm f_r \pm f_s,\ r < s\} \cup \{\pm 2f_r\} \cup \{\pm f_r\} & \text{if } n < m; \end{array}$$

in the case of $SO(n+m)$ they are

$$\begin{array}{lll} D_n: & \{\pm f_r \pm f_s,\ r < s\} & \text{if } n = m, \\ B_n: & \{\pm f_r \pm f_s,\ r < s\} \cup \{\pm f_r\} & \text{if } n < m. \end{array}$$

In each case the positive restricted roots are the $f_r \pm f_s$ with $r < s$, together with any f_{2r} and f_r that exist. Put $A = \exp \mathfrak{a}$, and let N be the exponential of the sum of the restricted root spaces for the positive restricted roots. Then we have an Iwasawa decomposition

$$G^d = KAN. \tag{2.7}$$

We shall be interested in the details of the restricted-root spaces only for the restricted roots $\pm(f_r - f_s)$, $r < s$. These are of multiplicity

$$\begin{array}{ll} 2 & \text{in the case of } U(n+m), \\ 1 & \text{in the case of } SO(n+m), \\ 4 & \text{in the case of } Sp(n+m). \end{array}$$

For $r < s$, the corresponding restricted-root spaces $\mathfrak{g}_{f_r - f_s}$ and $\mathfrak{g}_{-f_r + f_s}$ within $\mathfrak{g}^d$ have nonzero entries only in rows and columns numbered r, s, $n+m+1-s$,

$n+m+1-r$. In those rows and columns the entries are given by

$$\mathfrak{g}_{f_r-f_s} = \left\{ \begin{pmatrix} 0 & z & z & 0 \\ -\bar{z} & 0 & 0 & \bar{z} \\ \bar{z} & 0 & 0 & -\bar{z} \\ 0 & z & z & 0 \end{pmatrix} \right\}, \tag{2.8a}$$

$$\mathfrak{g}_{-f_r+f_s} = \left\{ \begin{pmatrix} 0 & z & -z & 0 \\ -\bar{z} & 0 & 0 & -\bar{z} \\ -\bar{z} & 0 & 0 & -\bar{z} \\ 0 & -z & z & 0 \end{pmatrix} \right\}. \tag{2.8b}$$

Here the bar denotes conjugation in $\mathbb{C}$ or $\mathbb{H}$, and the bar is to be ignored in $\mathbb{R}$. Define $\mathfrak{l}$ to be the Lie subalgebra of $\mathfrak{g}^d$ given by

$$\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{b} \oplus \sum_{r \neq s} \mathfrak{g}_{f_r - f_s}. \tag{2.9}$$

This is isomorphic with $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{gl}(n, \mathbb{R})$, and $\mathfrak{gl}(n, \mathbb{H})$ in our three cases. Let L be the analytic subgroup of G^d with Lie algebra $\mathfrak{l}$. Although it is not logically necessary to do so, we shall show that L is globally isomorphic with $GL(n, \mathbb{C})$, $GL(n, \mathbb{R})_0$, and $GL(n, \mathbb{H})$ in our three cases.

First let us observe that $\mathfrak{l}$ is stable under θ^d. In fact, we have $\mathfrak{a} \subseteq \mathfrak{p}^d$ and $\mathfrak{b} \subseteq \mathfrak{k}$. Also if we take sums and differences of (2.8a) and (2.8b), we see that the complementary part of $\mathfrak{l}$ consists of all real linear combinations of matrices of the two forms

$$\begin{pmatrix} 0 & z & 0 & 0 \\ -\bar{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{z} \\ 0 & 0 & z & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & z & 0 \\ 0 & 0 & 0 & \bar{z} \\ \bar{z} & 0 & 0 & 0 \\ 0 & z & 0 & 0 \end{pmatrix}. \tag{2.10}$$

These two matrices are in $\mathfrak{k}$ and $\mathfrak{p}^d$, respectively. Hence $\mathfrak{l}$ is stable under θ^d.

In fact, we see that $\mathfrak{k} \cap \mathfrak{l}$ is spanned by $\mathfrak{b}$ and all the matrices of the first kind in (2.10). To describe $\mathfrak{k} \cap \mathfrak{l}$ more explicitly, it is helpful to introduce a tool from the theory of automorphic forms—the notion of transpose about the opposite diagonal from usual. It is a kind of backwards transpose. For a square matrix C of size N, the *backwards transpose* ${}^tC$ of C is defined by

$$({}^tC)_{rs} = C_{N+1-s, N+1-r}. \tag{2.11a}$$

The mapping $C \mapsto {}^tC$ respects addition and scalar multiplication, reverses order under multiplication, and maps the identity matrix to itself. It follows that it commutes with complex or quaternion conjugation, powers, inversion, and the exponential map. We define a *backwards adjoint* by

$${}^*C = {}^t(\overline{C}). \tag{2.11b}$$

The upper left block of the first matrix in (2.10), when combined with the corresponding entries from $\mathfrak{b}$, yields a copy of $\mathfrak{u}(2)$, $\mathfrak{so}(2)$, and $\mathfrak{sp}(2)$ in our three cases, and the lower right block is obtained as minus the backwards adjoint. Thus

$$\mathfrak{k} \cap \mathfrak{l} = \left\{ \begin{pmatrix} Z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -{}^*Z \end{pmatrix} \;\middle|\; Z \in \mathfrak{k}_2 \right\}.$$

The corresponding analytic subgroup K_L of L is thus given by

$$K_L = \left\{ \begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^*k^{-1} \end{pmatrix} \;\middle|\; k \in K_2 \right\}. \tag{2.12}$$

We let $\iota : K_2 \to K_L$ be the isomorphism indicated by (2.12). From general theory it follows that

$$L \cong K_L \exp(\mathfrak{l} \cap \mathfrak{p}^d), \tag{2.13}$$

and therefore L is globally isomorphic to the identity component of the appropriate general linear group.

There is also a direct way to see the isomorphism (2.13), and this direct approach gives further insight into the structure. Let $\mathbb{F}$ be $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{H}$, and regard the space $\mathbb{F}^{n+m}$ of $(n+m)$-component column vectors with entries in $\mathbb{F}$ as a right vector space over $\mathbb{F}$. Write $G(n,m)$ for $U(n,m)$, $O(n,m)$, or $Sp(n,m)$ in the respective cases; we may identify $G(n,m)$ with the group of $\mathbb{F}$-linear transformations of $\mathbb{F}^{n+m}$ preserving the standard indefinite Hermitian form $\langle\,\cdot\,,\,\cdot\,\rangle_{n,m}$ of signature (n,m). Let $\{u_i\}$ be the standard basis of $\mathbb{F}^{n+m}$. Fix $p \le n$, and define

$$v_i = \frac{1}{\sqrt{2}}(u_i + u_{n+m+1-i}) \quad \text{and} \quad w_i = \frac{1}{\sqrt{2}}(u_i - u_{n+m+1-i}) \quad \text{for } 1 \le i \le p.$$

Let V_p be the span of the v_i, and let W_p be the span of the w_i. The form $\langle\,\cdot\,,\,\cdot\,\rangle_{n,m}$ is 0 on V_p and W_p, and it exhibits W_p as the Hermitian dual of V_p. If we write $\mathbb{F}^{n+m-2p}$ for the span of $u_{p+1}, \dots, u_{n+m-p}$, then we have

$$\mathbb{F}^{n+m} = V_p \oplus \mathbb{F}^{n+m-2p} \oplus W_p. \tag{2.14}$$

Regard g in $GL(p,\mathbb{F})$ as acting on V_p and denote the Hermitian dual action on W_p by $\widetilde{g}$. If h is in $G(n-p, m-p)$, then (g,h) acts on $\mathbb{F}^{n+m}$ by $(g,h)(v,u,w) = (gv, hu, \widetilde{g}w)$, preserving the decomposition (2.14) and respecting the form $\langle\,\cdot\,,\,\cdot\,\rangle_{n,m}$. Consequently we see that

$$GL(p,\mathbb{F}) \times G(n-p, m-p) \quad \text{embeds in} \quad G(n,m). \tag{2.15a}$$

For $p = n$, the result is that $GL(n,\mathbb{F}) \times G(0, m-n)$ embeds in $G(n,m)$. The subgroup L is the identity component of the factor $GL(n,\mathbb{F})$, and in matrices written in terms of the basis $\{v_1, \dots, v_p, u_{p+1}, \dots, u_{n+m-p}, w_p, \dots, w_1\}$, the set of matrices in L is given by

$$\left\{ \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^*g^{-1} \end{pmatrix} \;\middle|\; g \in GL(n,\mathbb{F})_0 \right\}. \tag{2.15b}$$

We order the roots of $\mathfrak{g}^d$ with respect to the Cartan subalgebra $\mathfrak{a} \oplus \mathfrak{t}$ in a fashion that takes $\mathfrak{a}$ first, takes $i(\mathfrak{b} \cap \mathfrak{t})$ next, and ends with the part of $i\mathfrak{t}$ that goes with matrix indices $n+1$ through m; we require also that the ordering be compatible with the ordering on the restricted roots. We obtain an ordering for the roots of $\mathfrak{l}$ with respect to its Cartan subalgebra $\mathfrak{a} \oplus (\mathfrak{b} \cap \mathfrak{t})$ by restriction.

Let v_0 be a nonzero highest weight vector for G^d in the representation space V. Then v_0 is also a highest weight vector for L, and hence the vector subspace

$$V' = U(\mathfrak{l}^{\mathbb{C}})v_0$$

is irreducible under the action of L; here $U(\mathfrak{l}^{\mathbb{C}})$ is the universal enveloping algebra of the complexification of $\mathfrak{l}$. We denote the representation of L on V' by σ'.

It may be helpful to see this irreducibility in a wider context. In (2.15a) we saw that the group $GL(n,\mathbb{F})\times G(0,m-n)$, which we call $\widetilde{L}$ for the moment, is a subgroup of $G(n,m)$. In fact, $\widetilde{L}\cap G(n,m)_0$ is the Levi subgroup of the maximal parabolic subgroup of $G(n,m)_0$ built from the simple restricted roots $f_1-f_2,\ \ldots,f_{n-1}-f_n$. The unipotent radical $\widetilde{N}$ of this parabolic subgroup is generated by the positive restricted roots other than the f_i-f_j with $i<j$. The subspace of $\widetilde{N}$ invariants in (σ,V) is stable under $\widetilde{L}_0$, and $\widetilde{L}_0=L\times G(0,m-n)_0$ acts irreducibly on it, with v_0 as highest weight vector, as a consequence of the general theory. The representation of $\widetilde{L}_0$ is therefore an outer tensor product of an irreducible representation of L and an irreducible representation of $G(0,m-n)_0$. It will turn out for our σ that the latter representation is 1-dimensional. Consequently V' is the entire space of $\widetilde{N}$ invariants in V, and L acts irreducibly in it.

Returning to the main line of the proof, let E be the projection of V to V^{K_1} given by

$$E(v)=\int_{U(m)}\sigma\begin{pmatrix}1&0\\0&k\end{pmatrix}v\,dk. \tag{2.16}$$

A change of variables shows that

$$E(\sigma(k_1)v)=E(v)\qquad\text{for all } k_1\in K_1. \tag{2.17}$$

Lemma 2.2. *E carries V' onto V^{K_1}, and it is equivariant with respect to K_2 in the sense that $\sigma(k_2)E(v)=E(\sigma(\iota(k_2))(v))$ for $k_2\in K_2$, where $\iota:K_2\to K_L$ is the canonical isomorphism indicated in* (2.12).

Proof. Since σ is irreducible for G^d, there is a finite set of elements $g_i\in G^d$ such that $\{\sigma(g_i)v_0\}$ spans V. Then the vectors $E(\sigma(g_i)v_0)$ span V^{K_1}. Write $g_i=k_ia_in_i$ according to the Iwasawa decomposition (2.7). Since $\sigma(n_i)$ fixes v_0 and $\sigma(a_i)$ multiplies v_0 by a positive scalar, the vectors $E(\sigma(k_i)v_0)$ span V^{K_1}. We can decompose k_i as $k_i=k_i^{(1)}k_i^L$ with $k_i^{(1)}\in K_1$ and $k_i^L\in K_L$ by writing

$$\begin{pmatrix}k_2&0\\0&k_1\end{pmatrix}=\begin{pmatrix}1&0\\0&k_1\,{}^*k_2\end{pmatrix}\begin{pmatrix}k_2&0\\0&{}^*k_2^{-1}\end{pmatrix},$$

and then it follows from (2.17) that the vectors $E(\sigma(k_i^L)v_0)$ span V^{K_1}. Since $\sigma(k_i^L)v_0$ is in V', we see that $E(V')=V^{K_1}$.

For the equivariance we write $k_2=\left(\begin{smallmatrix}k'&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)$. Then we have

$$\begin{aligned}E(\sigma(\iota(k_2))(v))&=\int_{U(m)}\sigma\begin{pmatrix}1&0&0\\0&&\\0&&k\end{pmatrix}\sigma\begin{pmatrix}k'&0&0\\0&1&0\\0&0&{}^*k'^{-1}\end{pmatrix}(v)\,dk\\&=\sigma\begin{pmatrix}k'&0&0\\0&1&0\\0&0&1\end{pmatrix}\int_{U(m)}\sigma\begin{pmatrix}1&0\\0&k\begin{pmatrix}1&0\\0&{}^*k'^{-1}\end{pmatrix}\end{pmatrix}(v)\,dk\\&=\sigma(k_2)\int_{U(m)}\sigma\begin{pmatrix}1&0\\0&k\end{pmatrix}(v)\,dk\\&=\sigma(k_2)E(v).\end{aligned}$$

This proves the lemma.

The next step is to transfer the group L and the representation (σ', V') from G^d back to G, obtaining a group G' and regarding (σ', V') as a representation of G'. The procedure for obtaining the Lie algebra $\mathfrak{g}'$ of G' is given in (2.3). The Lie algebra $\mathfrak{l}$ consists of all real linear combinations of matrices as in (2.4), matrices as in (2.6), and matrices indicated by (2.10). Therefore $\mathfrak{g}'$ consists of all real linear combinations of

$$\begin{pmatrix}
\begin{matrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & q_n \end{matrix} & 0 & \begin{matrix} 0 & \cdots & 0 & x_1 \\ 0 & \cdots & x_2 & 0 \\ \vdots & & & \vdots \\ x_n & \cdots & 0 & 0 \end{matrix} \\
0 & 0 & 0 \\
\begin{matrix} 0 & 0 & \cdots & -x_n \\ \vdots & & & \vdots \\ 0 & -x_2 & \cdots & 0 \\ -x_1 & 0 & \cdots & 0 \end{matrix} & 0 & \begin{matrix} q_n & \cdots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & q_2 & 0 \\ 0 & \cdots & 0 & q_1 \end{matrix}
\end{pmatrix}, \tag{2.18}$$

$$\begin{pmatrix} 0 & z & 0 & 0 \\ -\bar{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{z} \\ 0 & 0 & z & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & z & 0 \\ 0 & 0 & 0 & \bar{z} \\ -\bar{z} & 0 & 0 & 0 \\ 0 & -z & 0 & 0 \end{pmatrix}. \tag{2.19}$$

As usual, the last two of these are to be interpreted as indicating only rows and columns numbered $r, s, n+m+1-s, n+m+1-r$. Let G' be the analytic subgroup of G with Lie algebra $\mathfrak{g}'$.

The Lie algebra $\mathfrak{g}'$ is θ stable, and the $+1$ eigenspace under θ is $\mathfrak{k}_L$. Since a compact form of a complex semisimple Lie algebra is unique up to isomorphism, it follows that $\mathfrak{g}'$ is isomorphic to $\mathfrak{u}(n) \oplus \mathfrak{u}(n)$, $\mathfrak{u}(n)$, and $\mathfrak{u}(2n)$ in our three cases and that $\mathfrak{k}_L$ is embedded in the standard way in each case. As we noted above, we shall need in each case to take into account any possible effects of outer automorphisms on the highest weights that occur. In doing so, we shall have to consider each of our three cases separately and we shall make use of the following lemma.

Lemma 2.3. *For $U(N)$ with $N \geq 1$, there are exactly two outer automorphisms modulo inner automorphisms, namely complex conjugation and the identity map.*

Proof. The group $U(N)$ is the commuting product of $SU(N)$ and the subgroup Z of scalar matrices in $U(N)$, and any automorphism of $U(N)$ must preserve $SU(N)$ and Z and must agree on their intersection. In the case of Z, there are two automorphisms, namely complex conjugation and the identity.

For the Lie algebra $\mathfrak{su}(N)$ of the special unitary group and then also for the simply connected group $SU(N)$ itself, the outer automorphisms modulo inner automorphisms are given by automorphisms of the Dynkin diagram. For $N \geq 3$ the group in question has order 2, and for N equal to 1 or 2 it has order 1. In each case the inner automorphisms fix each member of $SU(N) \cap Z$. Complex conjugation of $SU(N)$ is a representative of the nontrivial class of automorphisms for $N \geq 3$ because it does not fix the members of $SU(N) \cap Z$ when $N \geq 3$.

Thus for all N, we see from the effect on Z that there are at least two classes of outer automorphisms modulo inner automorphisms for $U(N)$. For $N \geq 3$ it appears at first that there may be four classes for $U(N)$. However, the need for the restrictions to $SU(N)$ and Z to coincide on $SU(N) \cap Z$ eliminates two of the classes. Thus the number of classes is exactly two for all $N \geq 1$.

Partly because the statement of the Main Theorem requires it and partly because we shall want to limit the number of outer automorphisms by means of Lemma 2.3, we shall want to see that G' is globally isomorphic to $U(n) \times U(n)$, $U(n)$, and $U(2n)$ in the three cases, not merely locally isomorphic. This step will be carried out for each of our three cases separately. The argument above that L is globally isomorphic to a general linear group gives a clue how to prove this result, but we still need to consider the cases separately to handle G'.

Then we shall unwind the highest weights to see that they are as asserted, taking into account any information about outer automorphisms that is relevant. This step too will be carried out for each of our three cases separately. This concludes the discussion of the details of the proof of Theorem 2.1b that apply to all three cases of the Main Theorem.

For the remainder of this section, we specialize to $G = U(n+m)$ and $G^d = U(n,m)$. The first unproved detail that needs to be addressed is the construction of a particular member of $\mathrm{Ad}((G^d)^{\mathbb{C}})$ that transforms $\mathfrak{h}^{\mathbb{C}}$ into $(\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}}$. We shall identify this element by using Cayley transforms. However, since we need only to know the mapping of weights to weights, we shall not need to write down the effect of any Cayley transform on a particular matrix, and there will be no need to refer directly to the complexifications $(\mathfrak{g}^d)^{\mathbb{C}}$ and $(G^d)^{\mathbb{C}}$.

We do, however, need to use enough care to take into account the outer automorphisms of $G' \cong U(n) \times U(n)$. Lemma 2.3 shows that the group of outer automorphisms modulo inner automorphisms has order at least 8. This is too large to dismiss immediately. Instead of accounting for the effect of each class of automorphisms, we shall ultimately verify directly that the restriction of σ' is the correct tensor product, not involving any contragredients for example. In that way we will have seen that the outer automorphisms did not cause a problem.

We have taken the diagonal subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ as a compact Cartan subalgebra of $\mathfrak{g}^d$, and we have written $e_1, \ldots, e_{n+m}$ for the evaluation functionals on the diagonal entries. We introduce the usual ordering that makes $e_1 \geq \cdots \geq e_{n+m}$. Relative to $U(n,m)$, the roots

$$e_1 - e_{n+m},\ e_2 - e_{n+m-1},\ \ldots, e_n - e_{m+1} \tag{2.20}$$

form as large as possible a strongly orthogonal sequence of noncompact positive roots, and we form the product of the Cayley transforms relative to these roots, as in §§VI.7 and VI.11 of [Kn]. Each Cayley transform factor involves some limited choices, and it is assumed that these choices are made in the same way for each of the roots (2.20).

The resulting product of Cayley transforms matches the complexifications of $\mathfrak{h}$ and $\mathfrak{a} \oplus \mathfrak{t}$. The Cayley transformed roots (2.20) are denoted $2f_1, \ldots, 2f_n$, so that f_r agrees with the linear functional on $(\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}}$ whose value on the matrix in (2.4) is

x_r and whose value on $\mathfrak{t}$ is 0. Let f'_r be the linear functional on $(\mathfrak{a}\oplus\mathfrak{t})^{\mathbb{C}}$ whose value on the matrix in (2.5) is iy_r and whose value on $\mathfrak{a}$ is 0; this definition is consistent with our earlier definition of f'_r for all cases of the Main Theorem. Up to Cayley transforms, we therefore have

$$f_r = \tfrac{1}{2}(e_r - e_{n+m+1-r}) \quad \text{and} \quad f'_r = \tfrac{1}{2}(e_r + e_{n+m+1-r}).$$

In the passage from the complexification of $\mathfrak{h}$ to the complexification of $\mathfrak{a}\oplus\mathfrak{t}$, the highest weight

$$\begin{aligned} a_1e_1 + \cdots + a_ne_n + a'_1e_{m+1} + \cdots + a'_ne_{n+m} \\ &= \tfrac{1}{2}(a_1 - a'_n)(e_1 - e_{n+m}) + \tfrac{1}{2}(a_2 - a'_{n-1})(e_2 - e_{n+m-1}) \\ &\quad + \cdots + \tfrac{1}{2}(a_n - a'_1)(e_n - e_{m+1}) \\ &\quad + \tfrac{1}{2}(a_1 + a'_n)(e_1 + e_{n+m}) + \tfrac{1}{2}(a_2 + a'_{n-1})(e_2 + e_{n+m-1}) \\ &\quad + \cdots + \tfrac{1}{2}(a_n + a'_1)(e_n + e_{m+1}) \end{aligned}$$

of (σ, V) gets transformed into

$$\begin{aligned} (a_1 - a'_n)f_1 + (a_2 - a'_{n-1})f_2 + \cdots + (a_n - a'_1)f_n \\ &+ (a_1 + a'_n)f'_1 + (a_2 + a'_{n-1})f'_2 + \cdots + (a_n + a'_1)f'_n \\ &= a_1(f'_1 + f_1) + a_2(f'_2 + f_2) + \cdots + a_n(f'_n + f_n) \\ &\quad + a'_1(f'_n - f_n) + \cdots + a'_{n-1}(f'_2 - f_2) + a'_n(f'_1 - f_1). \end{aligned} \tag{2.21}$$

Since the ordering has changed, this expression is not *a priori* the highest weight of σ, but it is at least an extreme weight, still characterizing σ up to equivalence.

But in fact it is highest. The reason lies in the structure of the roots of $\mathfrak{g}^d$. The roots relative to $\mathfrak{h}$ are all of the form $e_r - e_s$, and it follows that an expression for a root relative to $\mathfrak{a}\oplus\mathfrak{t}$ involves f_r if and only if it involves f'_r. If we let ψ stand for a nonzero expression carried on the part of $\mathfrak{t}$ involving indices $n+1$ through m, then it follows that the positive roots are all necessarily of the form

$$\begin{array}{ll} 2f_r, & \\ (f_r - f_s) + (\pm f'_r \pm f'_s) & \text{with } r < s, \\ (f_r) + (\pm f'_r) + \psi & \text{if } n < m, \\ \psi & \text{if } n+1 < m. \end{array} \tag{2.22}$$

Each of these has inner product ≥ 0 with the right side of (2.21), and it follows from the fact that (2.21) is extreme that (2.21) is then highest. Therefore (2.21) is the highest weight of (σ', V').

The next step is to identify G' globally. We know that $\mathfrak{g}'$ is isomorphic to $\mathfrak{u}(n)\oplus\mathfrak{u}(n)$, and we want to see that G' is isomorphic to $U(n)\times U(n)$.

The Lie algebra $\mathfrak{g}'$ consists of all real linear combinations of the appropriate matrices (2.18) and of embedded versions of the matrices in (2.19). We introduce the matrix

$$M' = \begin{pmatrix} \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix} & 0 & \begin{matrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & \cdots & 0 & 0 \end{matrix} \\ 0 & \text{identity} & 0 \\ \begin{matrix} 0 & 0 & \cdots & i \\ \vdots & & & \vdots \\ 0 & i & \cdots & 0 \\ i & 0 & \cdots & 0 \end{matrix} & 0 & \begin{matrix} -i & \cdots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & -i & 0 \\ 0 & \cdots & 0 & -i \end{matrix} \end{pmatrix}. \tag{2.23}$$

Then we have

$$M'^{-1}G'M' = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^*v^{-1} \end{pmatrix} \;\middle|\; u \in U(n),\ v \in U(n) \right\}, \tag{2.24}$$

and furthermore conjugation by M'^{-1} leaves the members of K_L elementwise fixed. Thus K_L is embedded as the subgroup of (2.24) in which $u = v$. In more detail the relevant square submatrix of M'^{-1} conjugates rows and columns r and $n+m+1-r$ of the matrix in (2.18), in which we set $q_r = iy_r$, and the real linear combinations

$$\begin{pmatrix} 0 & z & -iz & 0 \\ -\bar{z} & 0 & 0 & i\bar{z} \\ -i\bar{z} & 0 & 0 & -\bar{z} \\ 0 & iz & z & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & z & iz & 0 \\ -\bar{z} & 0 & 0 & -i\bar{z} \\ i\bar{z} & 0 & 0 & -\bar{z} \\ 0 & -iz & z & 0 \end{pmatrix}$$

of cases of the two matrices in (2.19) respectively into

$$\begin{pmatrix} i(y_r + x_r) & 0 \\ 0 & i(y_r - x_r) \end{pmatrix}, \quad \begin{pmatrix} 0 & 2z & 0 & 0 \\ -2\bar{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\bar{z} \\ 0 & 0 & 2z & 0 \end{pmatrix}; \tag{2.25}$$

then (2.24) and the nature of the embedding of K_L follow.

Let us unwind the roots and weights, passing from L to G'. For this G' it is easier to analyze the weights fully than it is to make use of Lemma 2.3 to handle outer automorphisms.

The roots of L are given by (2.22), and the highest weight of (σ', V') is given by (2.21). In passing from L to G', we have changed the part of the Cartan subalgebra down the backwards diagonal of (2.18). On the matrix (2.18), we can still think of f'_r as taking the value iy_r. For f_r, we have a choice of ix_r or $-ix_r$ as value, and we need to make a consistent choice. Let us take ix_r as value for definiteness.

The expression $f'_r + f_r$ came via Cayley transform from e_r while $f'_r - f_r$ came via Cayley transform from $e_{n+m+1-r}$. It is apparent that $f'_r + f_r$ vanishes on the matrices (2.18) with all $iy_s = -ix_s$ while $f'_r - f_r$ vanishes on the matrices (2.18) with all $iy_s = ix_s$. From (2.25) and (2.24) we see that $f'_r + f_r$ is carried on one of the factors $\mathfrak{u}(n)$ and $f'_r - f_r$ is carried on the other one. Thus

$$a_1(f'_1 + f_1) + a_2(f'_2 + f_2) + \cdots + a_n(f'_n + f_n) \tag{2.26a}$$

is carried on the Cartan subalgebra of one of the ideals $\mathfrak{u}(n)$ while

$$a'_1(f'_n - f_n) + \cdots + a'_{n-1}(f'_2 - f_2) + a'_n(f'_1 - f_1) \tag{2.26b}$$

is carried on the Cartan subalgebra of the other. If τ_1 and τ_2 are irreducible representations of the two ideals with respective extreme weights (2.26a) and (2.26b), then we see that σ', as a representation of $\mathfrak{g}'$, is equivalent with (τ_1, τ_2).

The last thing to check is that the restriction of σ' to the subgroup K_L is the tensor product $\tau_1 \otimes \tau_2$. It is enough to check that the weights are in agreement. Let

$$c_1(f'_1 + f_1) + c_2(f'_2 + f_2) + \cdots + c_n(f'_n + f_n) + c'_1(f'_n - f_n) + \dots \\ + c'_{n-1}(f'_2 - f_2) + c'_n(f'_1 - f_1)$$

be a weight of σ'; we can rewrite it as

$$(c_1, c_2, \dots, c_n)(c'_1, \dots, c'_{n-1}, c'_n).$$

The restriction of this weight to $\mathfrak{b}$ is obtained by setting all the f_j equal to 0 and is therefore equal to $(c_1 + c'_n)f'_1 + (c_2 + c'_{n-1})f'_2 + \cdots + (c_n + c'_1)f'_n$, which we can rewrite as

$$(c_1 + c'_n,\, c_2 + c'_{n-1},\, \dots,\, c_n + c'_1).$$

This is the sum of the two expressions $(c_1, c_2, \dots, c_n)$ and $(c'_n, c'_{n-1}, \dots, c'_1)$. The first of these is a weight by inspection, and the second of these is a weight because it is a permutation of $(c'_1, \dots, c'_{n-1}, c'_n)$. Thus the restriction of σ' to K_L is exhibited as having for its weights all sums of a weight of τ_1 and a weight of τ_2, and it follows that the restriction of σ' to K_L is equivalent with $\tau_1 \otimes \tau_2$.

To complete the proof of Theorem 2.1, it suffices to show that the mapping $E : V' \to V^{K_1}$ is one-one. Since Lemma 2.2 shows E to be onto, it is enough to prove that $\dim V' = \dim V^{K_1}$. This equality of dimensions will be proved in Lemma 2.6 below. The circle of ideas that form the basis of the proof is due to Gelbart [Ge]. The tools, in one form or another, date back to Gelfand and Cetlin [GeC]. For more discussion of the tools, see [Pr].

We regard the sequence $U(1) \subset U(2) \subset \cdots \subset U(N)$ of unitary groups to be nested in a standard way, such as with each one embedded as the lower right block of the next one. A *system* for $U(N)$ of *level* r coming from a dominant integral N-tuple $(c_1, \dots, c_N)$ is a collection $\{(c_1^{(k)}, \dots, c_{N-k}^{(k)}) \mid 0 \le k \le r\}$ consisting of one $(N-k)$-tuple for each k with $0 \le k \le r$ such that

$$(c_1^{(0)}, \dots, c_N^{(0)}) = (c_1, \dots, c_N);$$

the successive tuples are dominant integral for $U(N)$, $U(N-1)$, $\dots$, $U(N-r)$; and

$$c_1^{(k-1)} \ge c_1^{(k)} \ge c_2^{(k-1)} \ge \cdots \ge c_{N-k}^{(k-1)} \ge c_{N-k}^{(k)} \ge c_{N-k+1}^{(k-1)} \quad \text{for } 1 \le k \le r.$$

The *end* of the system is the $(N-r)$-tuple $(c_1^{(r)}, \dots, c_{N-r}^{(r)})$.

Lemma 2.4 (Gelfand-Cetlin). *Let τ be an irreducible representation of $U(N)$ with highest weight $(c_1, \dots, c_N)$, let $1 \le r < N$, and let τ' be an irreducible representation of $U(N-r)$ with highest weight $(d_1, \dots, d_{N-r})$. Then the number of systems for $U(N)$ of level r coming from $(c_1, \dots, c_N)$ and having end $(d_1, \dots, d_{N-r})$ equals the multiplicity of τ' in $\tau|_{U(N-r)}$.*

Proof. For $r = 1$, the number of such systems is 1 or 0, and the Weyl branching theorem (1.1) says that this number matches the asserted multiplicity. Inductively assume the lemma to be true for $r - 1$. If τ'' is an irreducible representation of $U(N - r + 1)$ with highest weight $(x_1, \dots, x_{N-r+1})$, then the inductive hypothesis implies that the number of systems for $U(N)$ of level $N - r + 1$ coming from $(c_1, \dots, c_N)$ and having end $(x_1, \dots, x_{N-r+1})$ equals the multiplicity of τ'' in $\tau|_{U(N-r+1)}$. By Weyl's branching theorem the multiplicity of τ' in $\tau''|_{U(N-r)}$ is 1 or 0 according as the system of level $r - 1$ ending with the highest weight τ'' continues to a system of level r ending with the highest weight of τ' or does not so continue. Summing over all τ'', we obtain the result for r.

Corollary 2.5 (Gelfand-Cetlin). *Let (τ, V) be an irreducible representation of $U(N)$ with highest weight $(c_1, \dots, c_N)$. Then the number of systems for $U(N)$ of level $N - 1$ coming from $(c_1, \dots, c_N)$ equals the dimension of V.*

Remark. In essence the corollary says that Lemma 2.4 remains valid for $r = N$.

Proof. Since irreducible representations of $U(1)$ are one-dimensional, the dimension of V equals the sum of the multiplicities of all the irreducible representations of $U(1)$ in $\tau|_{U(1)}$. Then the corollary follows from the case $r = N-1$ of Lemma 2.4.

Now we return to the notation of Theorem 2.1b. The given irreducible representation (σ, V) of $U(n + m)$ has highest weight

$$(a_1, \dots, a_n, 0, \dots, 0, a'_1, \dots, a'_n), \tag{2.27}$$

and it is understood that $a_n \geq 0 \geq a'_1$ even if $n = m$. The constructed irreducible representation (σ', V') of $U(n) \times U(n)$ has highest weight

$$(a_1, \dots, a_n)(a'_1, \dots, a'_n).$$

Let (τ_1, V'_1) and (τ_2, V'_2) be irreducible representations of $U(n)$ with respective highest weights $(a_1, \dots, a_n)$ and $(a'_1, \dots, a'_n)$.

Lemma 2.6. $\dim V' = \dim V^{K_1}$.

Proof. The right side is the multiplicity of the trivial representation of $K_1 = U(m)$ in $\sigma|_{U(m)}$, and Lemma 2.4 shows that this multiplicity equals the number of systems for $U(n + m)$ of level n coming from (2.27) and having end the m-tuple $(0, \dots, 0)$.

We shall compute this number of systems in a second way and obtain the answer $\dim V'$. Suppose that

$$\{(c_1^{(k)}, \dots, c_{n+m-k}^{(k)}) \mid 0 \leq k \leq m\} \tag{2.28}$$

is a system for $U(n+m)$ of level n coming from (2.27). As in the proof of Theorem 2.1a, we have

$$c_{l+r}^{(k-r)} \leq c_l^{(k)} \tag{2.29a}$$

whenever the indices are in bounds. If the system (2.28) has end $(0, \dots, 0)$, then $c_1^{(n)} = \cdots = c_m^{(n)} = 0$. Taking $k = n$, $r = n - s$, and $l = 1$ in (2.29a), we obtain

$$c_{n+1-s}^{(s)} \leq c_1^{(n)} = 0 \quad \text{for } 0 \leq s \leq n. \tag{2.29b}$$

Similarly (2.28) satisfies

$$c_l^{(k)} \geq c_l^{(k+r)} \tag{2.30a}$$

whenever the indices are in bounds. If (2.28) has end $(0,\dots,0)$, then we take $k=s$, $r=n-s$, and $l=m$ in (2.30a) to see that

$$c_m^{(s)} \geq c_m^{(n)} = 0 \quad \text{for } 0 \leq s \leq n. \tag{2.30b}$$

Combining (2.29b) and (2.30b) and using dominance, we see that

$$c_l^{(s)} = 0 \quad \text{for } n+1-s \leq l \leq m. \tag{2.31}$$

In view of (2.31), the initial segment

$$\{(c_1^{(s)},\dots,c_{n-s}^{(s)}) \mid 0 \leq s \leq n\}$$

of (2.28), as (2.28) ranges over all possibilities, is a completely general system for $U(n)$ of level n coming from $(a_1,\dots,a_n)$. Corollary 2.5 shows that there are $\dim V_1'$ possibilities for this initial segment as (2.28) varies. Similarly the final segment

$$\{(c_{m+1}^{(k)},\dots,c_{n+m-k}^{(k)}) \mid 0 \leq k \leq n\}$$

of (2.28), as (2.28) ranges over all possibilities, is a completely general system for $U(n)$ of level n coming from $(a_1',\dots,a_n')$. Corollary 2.5 shows that there are $\dim V_2'$ possibilities for this final segment as (2.28) varies. Since, according to (2.31), the entries in between the initial segment and the final segment are all 0, the arbitrariness of the initial segment is independent of the arbitrariness of the final segment (in the sense that the pair of segments is arbitrary) because the entries of these segments never overlap: the largest l for $c_l^{(s)}$ in the initial segment is $n-s$, and the smallest l for $c_l^{(s)}$ in the final segment is $m+1 \geq n+1$. We conclude that the number of systems (2.28) ending in $(0,\dots,0)$ is equal to $(\dim V_1')(\dim V_2') = \dim V'$. This completes the proof of Lemma 2.6 and also Theorem 2.1b.

3. Main Theorem for Rotation Groups

In this section we shall state and prove the Main Theorem corresponding to $SO(n+m)$ in the left column of Table 1. The details will depend slightly on the parity of n and m as we shall see.

A Cartan subalgebra of $SO(N)$ can be taken to consist of two-by-two diagonal blocks $\left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right)$ starting, say, from the upper left. If the jth such block is $\left(\begin{smallmatrix} 0 & it \\ -it & 0 \end{smallmatrix}\right)$, the associated evaluation functional e_j on the complexification of the Cartan subalgebra takes the value t. There are $[N/2]$ such blocks, $[\,\cdot\,]$ denoting the greatest-integer function. When N is even, say $N=2d$, the roots are the functionals $\pm e_i \pm e_j$ with $1 \leq i < j \leq d$. When N is odd, say $N=2d+1$, the roots are the functionals $\pm e_i \pm e_j$ with $1 \leq i < j \leq d$ and also the $\pm e_j$ with $1 \leq j \leq d$. We take the positive roots to be the $e_i \pm e_j$ with $i<j$ and, when N is odd, the e_j.

The dominant integral forms for $SO(N)$ are given by expressions

$$a_1e_1 + \cdots + a_de_d \longleftrightarrow (a_1,\dots,a_d)$$

$$\text{with} \quad \begin{cases} a_1 \geq \cdots \geq a_{d-1} \geq |a_d| & \text{when } N=2d, \\ a_1 \geq \cdots \geq a_d \geq 0 & \text{when } N=2d+1, \end{cases}$$

with all the a_j's understood to be integers.

The theorem for branching from $SO(2d+1)$ to $SO(2d)$ is that the representation of $SO(2d+1)$ with highest weight $(a_1,\dots,a_d)$ decomposes with multiplicity one under $SO(2d)$, and the representations of $SO(2d)$ that appear are exactly those with highest weights $(c_1, \ldots, c_d)$ such that

$$a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq a_{d-1} \geq c_{d-1} \geq a_d \geq |c_d|. \tag{3.1a}$$

The theorem for branching from $SO(2d)$ to $SO(2d-1)$ is that the representation of $SO(2d)$ with highest weight $(a_1, \ldots, a_d)$ decomposes with multiplicity one under $SO(2d-1)$, and the representations of $SO(2d-1)$ that appear are exactly those with highest weights $(c_1, \ldots, c_{d-1})$ such that

$$a_1 \geq c_1 \geq a_2 \geq c_2 \geq \cdots \geq a_{d-1} \geq c_{d-1} \geq |a_d|. \tag{3.1b}$$

Theorem 3.1. *Let $1 \leq n \leq m$, and regard $SO(n)$ and $SO(m)$ as embedded as block diagonal subgroups of $SO(n+m)$ in the standard way with $SO(n)$ in the upper left diagonal block and with $SO(m)$ in the lower right diagonal block.*

(a) *If $(a_1, \ldots, a_{[\frac{1}{2}(n+m)]})$ is the highest weight of an irreducible representation (σ, V) of $SO(n+m)$, then a necessary and sufficient condition for the subspace $V^{SO(m)}$ of $SO(m)$ invariants to be nonzero is that $a_{n+1} = \cdots = a_{[\frac{1}{2}(n+m)]} = 0$.*

(b) *Let $(a_1, \ldots, a_n, 0, \ldots, 0)$ be the highest weight of an irreducible representation (σ, V) of $SO(n+m)$ with a nonzero subspace of $SO(m)$ invariants, and let (σ', V') be an irreducible representation of $U(n)$ with highest weight $(a_1, \ldots, a_{n-1}, |a_n|)$. Then the representation $(\sigma|_{SO(n)}, V^{SO(m)})$ is equivalent with the restriction to $SO(n)$ of the representation (σ', V') of $U(n)$.*

Remarks. The need for the absolute value signs around a_n in the highest weight of σ' in (b) arises only when $n = m$. Otherwise a_n is automatically ≥ 0. When $n = m$ and $a_n \neq 0$, it follows from (b) that the two inequivalent σ's with highest weights $(a_1, \ldots, a_{n-1}, a_n)$ and $(a_1, \ldots, a_{n-1}, -a_n)$ lead to equivalent σ''s. The example of σ for $SO(4)$ with highest weight $(1, -1)$ shows that σ' cannot necessarily be taken to have highest weight$(a_1, \ldots, a_n)$ if $a_n < 0$.

The proof of Theorem 3.1a is similar to the proof of Theorem 2.1a and is given in [Ge]. Let us therefore move to Theorem 3.1b.

Most of the proof of Theorem 3.1b has been given in §2, but some details have been left for this section.

The first detail concerns constructing the maximally compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^d$. This subalgebra needs to be set up so as to allow the complexification of $\mathfrak{h}$ to be transformed into the complexification of $\mathfrak{a} \oplus \mathfrak{t}$ by Cayley transforms. The point of using Cayley transforms is to keep accurate track of how weights move from one Cartan subalgebra to another. In particular, we do not want to err by confusing two weights that differ by an outer automorphism.

However, we can relax somewhat about this matter because of Lemma 2.3: The inclusion of $K_L \cong SO(n)$ into $G' \cong U(n)$ is a version of the inclusion of $SO(n) \subset U(n)$, and Lemma 2.3 says that the only automorphism of $U(n)$ that is of concern is complex conjugation, i.e., θ. This automorphism fixes $SO(n)$. So a representation σ' of $U(n)$ and its composition $\sigma' \circ \theta$ have the same restriction to $SO(n)$, and it does not matter if we confuse σ' with $\sigma' \circ \theta$.

There are two other matters concerning automorphisms to dispose of. One is that in the case $n = m$, a highest weight $(a_1, \ldots, a_{n-1}, a_n)$ for σ on $SO(2n)$ with $a_n < 0$ leads not to the highest weight $(a_1, \ldots, a_{n-1}, a_n)$ for σ' on $U(n)$ but to

$(a_1, \ldots, a_{n-1}, |a_n|)$. This fact cries out for a simple explanation, and Lemma 3.2a below gives such an explanation.

The other matter is a symmetry relative to $SO(n)$ when n is even. For even n, $SO(n)$ has a nontrivial outer automorphism, and this extends to an automorphism of $U(n)$ that is inner. How is this fact reflected in the context of Theorem 3.1? Lemma 3.2b will give an answer.

Lemma 3.2. (a) *Let $n = m$, let the given representation (σ, V) of $G = SO(2n)$ have highest weight $(a_1, \ldots, a_{n-1}, a_n)$, and let $(\widehat{\sigma}, V)$ be the representation of $SO(2n)$ given by conjugating by the diagonal matrix $D = \operatorname{diag}(1, \ldots, 1, -1) : \widehat{\sigma}(k) = \sigma(D^{-1}kD)$. Then the subspaces of K_1 invariants are the same, and the two actions of K_2 on this space of K_1 invariants are identical.*

(b) *If n is even and a representation τ of $K_2 \cong SO(n)$ with highest weight $(c_1, \ldots, c_{n/2-1}, c_{n/2})$ occurs in V^{K_1}, then the representation $\widehat{\tau}$ with highest weight $(c_1, \ldots, c_{n/2-1}, -c_{n/2})$ occurs, and it has the same multiplicity.*

Remarks. Lemma 3.2a will allow us to assume in all cases, without loss of generality, that the integers in the highest weight are ≥ 0.

Proof. If $\sigma(K_1)$ fixes v, then $\widehat{\sigma}(K_1)$ fixes v because conjugation by D carries K_1 to itself. On all of V, we have $\sigma(k_2) = \widehat{\sigma}(k_2)$ for $k_2 \in K_2$ because conjugation by D fixes K_2. This proves (a).

Let d be the diagonal matrix of size $n + m$ that is -1 in diagonal entries n and $n + 1$ and is 1 in the other diagonal entries, and put $(d\sigma)(k) = \sigma(d^{-1}kd)$. Since d is in $SO(n + m)$, d is equivalent with $d\sigma$. The space V^{K_1} of vectors fixed by K_1 is the same for σ as for $d\sigma$ because conjugation by d carries K_1 to itself. On V^{K_1} the restrictions of σ and $d\sigma$ are related by a nontrivial outer automorphism of K_2. The lemma follows.

Now let us specify the maximally compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^d$. We distinguish cases according to the parities of n and m:

Case 1: $n = 2n'$ and $m = 2m'$ even. We use n' two-by-two diagonal blocks within $\mathfrak{so}(n)$ and m' two-by-two diagonal blocks within $\mathfrak{so}(m)$. These blocks and their corresponding e_s's are numbered consecutively from 1 to $n'+m'$. The strongly orthogonal sequence of noncompact roots to use for Cayley transforms is

$$e_1 \pm e_{n'+m'},\ e_2 \pm e_{n'+m'-1},\ \ldots,\ e_{n'} \pm e_{m'+1}. \tag{3.2}$$

With suitable consistently made choices for the Cayley transforms, these roots transform into $f_1 \pm f_2,\ f_3 \pm f_4, \ldots, f_{n-1} \pm f_n$, so that we can think of f_1 as corresponding to e_1, f_2 as corresponding to $e_{n'+m'}$, f_3 as corresponding to e_2, and so on.

Case 2: $n = 2n'$ even and $m = 2m' + 1$ odd. We use n' two-by-two diagonal blocks within $\mathfrak{so}(n)$ and m' two-by-two diagonal blocks within $\mathfrak{so}(m)$. The latter are to start with entries $(n + 2, n + 3)$, skipping entry $n + 1$. The strongly orthogonal sequence of noncompact roots to use for Cayley transforms, as well as the identification of f_r's with e_s's, is the same as in Case 1.

Case 3: $n = 2n' + 1$ odd and $m = 2m'$ even. We use n' two-by-two diagonal blocks within $\mathfrak{so}(n)$ and m' two-by-two diagonal blocks within $\mathfrak{so}(m)$. The blocks within $\mathfrak{so}(n)$ omit entry n. The strongly orthogonal sequence of noncompact roots to use for Cayley transforms consists of (3.2) and $e_{m'}$; the choices for the Cayley transform relative to $e_{m'}$ need to be made so that $\mathbb{R}(E_{n,m+1} + E_{m+1,n})$ becomes part of $\mathfrak{a}$. The identification of f_r's with e_s's begins as in Case 1 and concludes with the correspondence of f_n with $e_{m'}$.

Case 4: $n = 2n'+1$ and $m = 2m'+1$ odd. In this case $\mathfrak{g}^d$ does not have a compact Cartan subalgebra. We choose the compact part of the maximally compact Cartan subalgebra $\mathfrak{h}$ to consist of $m' + n'$ two-by-two diagonal blocks that omit entries n and $m+1$. If $E_{i,j}$ denotes the matrix that is 1 in the (i, j)th place and 0 elsewhere, then the noncompact part of the Cartan subalgebra consists of $\mathbb{R}(E_{n,m+1}+E_{m+1,n})$. The strongly orthogonal sequence of noncompact roots to use for Cayley transforms consists of (3.2) alone, and the identification of f_r's with e_s's accounts for all the f_r's except f_n, which acts on $\mathbb{R}(E_{n,m+1}+E_{m+1,n})$ and is not affected by the Cayley transforms.

If we let ψ stand for a nonzero expression carried on $\mathfrak{t}$, then the positive roots relative to $\mathfrak{a} \oplus \mathfrak{t}$ are all necessarily of the form

$$\begin{array}{ll} f_r \pm f_s & \text{with } r < s, \\ f_r & \text{if } n + m \text{ is odd,} \\ f_r + \psi & \text{if } n + 1 < m, \\ \psi & \text{if } n + 2 < m. \end{array}$$

The given highest weight $a_1e_1 + \cdots + a_ne_n$ of (σ, V) relative to $\mathfrak{h}$ transforms to an integer combination of f_r's, together possibly with a term carried on $\mathfrak{t}$. The transformed expression is an extreme weight. To make it dominant, we permute coefficients, including those corresponding to the $\mathfrak{t}$ part, and we obtain $a_1f_1 + \cdots + a_nf_n$. In the case that $n = m$, a_n may in principle be < 0. But Lemma 3.2a says that we may, without loss of generality, replace a_n by $|a_n|$. Thus we may work with the highest weight of (σ, V) relative to $\mathfrak{a} \oplus \mathfrak{t}$ as if it is

$$a_1f_1 + \cdots + a_{n-1}f_{n-1} + |a_n|f_n. \tag{3.3}$$

The expression (3.3) may then be taken as the highest weight of (σ', V') relative to $\mathfrak{a}$.

The next step is to identify G' globally. We know that $\mathfrak{g}'$ is isomorphic to $\mathfrak{u}(n)$, and we want to see that G' is isomorphic to $U(n)$. The Lie algebra $\mathfrak{g}'$ consists of all real linear combinations of the matrices (2.18) with $y_1 = \cdots = y_n = 0$ and of embedded versions of the real matrices in (2.19), i.e., of all real linear combinations of

$$\begin{pmatrix} & & & & & 0 & \cdots & 0 & x_1 \\ & 0 & & & 0 & 0 & \cdots & x_2 & 0 \\ & & & & & \vdots & & & \vdots \\ & & & & & x_n & \cdots & 0 & 0 \\ & 0 & & & 0 & & 0 & & \\ 0 & 0 & \cdots & -x_n & & & & & \\ \vdots & & & \vdots & 0 & & 0 & & \\ 0 & -x_2 & \cdots & 0 & & & & & \\ -x_1 & 0 & \cdots & 0 & & & & & \end{pmatrix} \tag{3.4}$$

and of embedded versions of the real matrices

$$\begin{pmatrix} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & -x \\ 0 & 0 & x & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Define M' as in (2.23). Then we have

$$M'^{-1}G'M' = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^tu^{-1} \end{pmatrix} \;\middle|\; u \in U(n) \right\}, \tag{3.6}$$

and furthermore conjugation by M'^{-1} leaves the members of K_L elementwise fixed. The group K_L is embedded as the subgroup of (3.6) in which u is real, i.e., $K_L \cong SO(n)$ is embedded in the standard way in $U(n)$. In more detail the relevant square submatrix of M'^{-1} conjugates rows and columns r and $n+m+1-r$ of the matrix in (3.4) and the two matrices in (3.5) respectively into

$$\begin{pmatrix} ix_r & 0 \\ 0 & -ix_r \end{pmatrix}, \quad \begin{pmatrix} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & -x \\ 0 & 0 & x & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & ix & 0 & 0 \\ ix & 0 & 0 & 0 \\ 0 & 0 & 0 & -ix \\ 0 & 0 & -ix & 0 \end{pmatrix}; \tag{3.7}$$

then (3.6) and the nature of the embedding of K_L follow.

To complete the proof of Theorem 3.1b, it suffices to show that the mapping $E : V' \to V^{K_1}$ is one-one. Since Lemma 2.2 shows E to be onto, it is enough to prove that $\dim V' = \dim V^{K_1}$. A proof of this equality is essentially in Gelbart [Ge]. We give a proof anyway so that the result can be cast in our notation.

We regard the sequence $SO(1) \subset SO(2) \subset \cdots \subset SO(N)$ of rotation groups to be nested in a standard way, such as with each one embedded as the lower right block of the next one. A *system* for $SO(N)$ of *level* r coming from a dominant integral $[N/2]$-tuple $(c_1, \ldots, c_{[N/2]})$ is a collection $\{(c_1^{(k)}, \ldots, c_{[(N-k)/2]}^{(k)}) \mid 0 \le k \le r\}$ consisting of one $(N-k)$-tuple for each k such that

$$(c_1^{(0)}, \ldots, c_{[N/2]}^{(0)}) = (c_1, \ldots, c_{[N/2]});$$

the successive tuples are dominant integral for $SO(N), SO(N-1), \ldots, SO(N-r)$; and the kth tuple, for $k \ge 1$, is obtained from the $(k-1)$st tuple by (3.1a) or (3.1b). The *end* of the system is the $[(N-r)/2]$-tuple $(c_1^{(r)}, \ldots, c_{[(N-r)/2]}^{(r)})$.

Lemma 3.3. *Let τ be an irreducible representation of $SO(N)$ with highest weight $(c_1, \ldots, c_{[N/2]})$, let $1 \le r < N$, and let τ' be an irreducible representation of $SO(N-r)$ with highest weight $(d_1, \ldots, d_{[(N-r)/2]})$. Then the number of systems for $SO(N)$ of level r coming from $(c_1, \ldots, c_{[N/2]})$ and having end $(d_1, \ldots, d_{[(N-r)/2]})$ equals the multiplicity of τ' in $\tau|_{SO(N-r)}$.*

Proof. The argument is the same as for Lemma 2.4 except that the branching theorems (3.1a) and (3.1b) are used in place of the branching theorem (1.1).

Now we return to the notation of Theorem 3.1b. The given irreducible representation (σ, V) of $SO(n+m)$ has highest weight the $[(n+m)/2]$-tuple

$$(a_1, \ldots, a_n, 0, \ldots, 0), \tag{3.8}$$

and it is understood that we can take $a_n \geq 0$ even if $n = m$. The constructed irreducible representation (σ', V') of $U(n)$ has highest weight

$$(a_1, \dots, a_n).$$

Lemma 3.4. $\dim V' = \dim V^{K_1}$.

Proof. The right side is the multiplicity of the trivial representation of $K_1 = SO(m)$ in $\sigma|_{SO(m)}$, and Lemma 3.3 shows that this multiplicity equals the number of systems for $SO(n+m)$ of level n coming from the $[(n+m)/2]$-tuple (3.8) and having end the $[m/2]$-tuple $(0, \dots, 0)$.

We shall compute this number of systems in a second way and obtain the answer $\dim V'$. More specifically, we shall show that the $SO(n+m)$ systems of level n coming from the $[(n+m)/2]$-tuple (3.8) and having end the $[m/2]$-tuple $(0, \dots, 0)$ are in one-one correspondence with the $U(n+m)$ systems of level n coming from the $(n+m)$-tuple (3.8) and having end the m-tuple $(0, \dots, 0)$. The correspondence is as follows: to pass from an $SO(n+m)$ system to a $U(n+m)$ system, we pad the right ends of the tuples with 0's; to pass from a $U(n+m)$ system to an $SO(n+m)$ system, we drop the appropriate number of entries from the right ends of the tuples. To see that this is a one-one correspondence, we need to check that

(i) the $U(n+m)$ tuples are always at least as long as the $SO(n+m)$ tuples,
(ii) any entry that gets dropped from a $U(n+m)$ tuple in carrying out the correspondence has a 0 in it, and
(iii) no negative entries can arise in the SO system.

Each kind of system consists of $n+1$ tuples numbered from 0 to n, the kth tuple being of length

$$\begin{aligned} &[(n+m-k)/2] && \text{in the } SO(n+m) \text{ case,} \\ &n+m-k && \text{in the } U(n+m) \text{ case.} \end{aligned} \tag{3.9}$$

Fact (i) above follows from the inequalities

$$[(n+m-k)/2] \leq (n+m-k)/2 \leq n+m-k.$$

To prove (ii), suppose that $n+m-s > [(n+m-s)/2]$ and that

$$\{(c_1^{(k)}, \dots, c_{n+m-k}^{(k)}) \mid 0 \leq k \leq n\}$$

is a $U(n+m)$ system of level n coming from the $(n+m)$-tuple (3.8) and having end the m-tuple $(0, \dots, 0)$. We are to show that

$$c^{(s)}_{[(n+m-s)/2]+1} = 0. \tag{3.10}$$

Since all entries of (3.8) are ≥ 0 and the branching rule (1.1) is in force, we have

$$c_l^{(s)} \geq 0 \quad \text{for } 1 \leq l \leq n+m-s. \tag{3.11}$$

On the other hand, (2.29b) shows that $c^{(s)}_{n+1-s} \leq 0$. Thus (3.11) shows that

$$c^{(s)}_{n+1-s} = 0. \tag{3.12}$$

Now

$$n+1-s \leq [(n+n-s)/2]+1 \leq [(n+m-s)/2]+1, \tag{3.13}$$

and thus (3.10) follows from (3.12), (3.13), dominance, and (3.11).

To prove (iii), let $\{(c_1^{(k)}, \ldots, c_{[(n+m-k)/2]}^{(k)}) \mid 0 \le k \le n\}$ be a system for $SO(n+m)$ of level n coming from (3.8) and having end the $[m/2]$-tuple $(0, \ldots, 0)$. Here the kth tuple for the system is dominant integral for $SO(n+m-k)$. We are to prove that its last entry is ≥ 0:

$$c_{[(n+m-k)/2]}^{(k)} \ge 0 \quad \text{for } 0 \le k \le n. \tag{3.14}$$

We are given that $c_1^{(n)} = 0$, and thus (3.1) implies that

$$0 = c_1^{(n)} \ge c_2^{(n-1)} \ge \cdots \ge |c_{n+1-k}^{(k)}|$$

as long as $n+1-k \le [(n+m-k)/2]$. Since $n+1-k \le [(n+m-k)/2]$ for $k \ge 2$, dominance gives $c_{[(n+m-k)/2]}^{(k)} \ge 0$ for $k \ge 2$. That is, (3.14) holds for $k \ge 2$. By construction (3.14) holds for $k = 0$. Thus we have only to check $k = 1$. If (3.14) fails for $k = 1$, then $n+m-1$ must be even, say equal to $2d$. So $[(n+m-2)/2] = d-1$, and (3.1b) and (3.14) for $k = 2$ give

$$0 = c_{[(n+m-2)/2]}^{(2)} = c_{[(n+m-1)/2]-1}^{(2)} \ge |c_{[(n+m-1)/2]}^{(1)}| > 0,$$

contradiction. We conclude that (3.14) holds for $k = 1$, and this proves (iii).

Thus the number of systems for $SO(n+m)$ of level n coming from the $[(n+m)/2]$-tuple (3.8) and having end the $[m/2]$-tuple $(0, \ldots, 0)$ equals the number of systems for $U(n+m)$ of level n coming from the $(n+m)$-tuple (3.8) and having end the m-tuple $(0, \ldots, 0)$. This latter number, by Corollary 2.5 and the argument in the proof of Lemma 2.6, equals the dimension of V'. This completes the proof of Lemma 3.4 and also Theorem 3.1b.

4. Main Theorem for Quaternion Unitary Groups

In this section we shall state and prove the Main Theorem corresponding to $Sp(n+m)$ in the left column of Table 1. We regard $Sp(n+m)$ as the group of unitary matrices over the quaternions, and we write quaternions using the customary basis $1, i, j, k$. The group $Sp(N)$ has a standard realization as a subgroup of $U(2N)$ obtained by writing each quaternion as a 2-by-2 complex matrix (cf. [Kn], §I.8).

A Cartan subalgebra of $Sp(N)$ can be taken to consist of the diagonal matrices whose entries are real multiples of i. Let e_r denote evaluation of the rth diagonal entry. The roots for $Sp(N)$ are all $\pm e_r \pm e_s$ with $r < s$ and all $\pm 2e_r$. We take the postive roots to be the $e_r \pm e_s$ with $r < s$, as well as the $2e_r$. The dominant integral forms for $Sp(N)$ are the expressions $a_1 e_1 + \cdots + a_N e_N$ with all a_i in $\mathbb{Z}$ and with $a_1 \ge \cdots \ge a_N \ge 0$. We write such an expression as an N-tuple $(a_1, \ldots, a_N)$.

Zhelobenko's branching theorem [Z] for passing from $Sp(N)$ to $Sp(N-1)$ says that the number of times the representation of $Sp(N-1)$ with highest weight $(c_1, \ldots, c_{N-1})$ occurs in the representation of $Sp(N)$ with highest weight $(a_1, \ldots, a_N)$ equals the number of integer N-tuples $(b_1, \ldots, b_N)$ such that

$$\begin{gathered} a_1 \ge b_1 \ge a_2 \ge \cdots \ge a_{N-1} \ge b_{N-1} \ge a_N \ge b_N \ge 0, \\ b_1 \ge c_1 \ge b_2 \ge \cdots \ge b_{N-1} \ge c_{N-1} \ge b_N. \end{gathered} \tag{4.1}$$

If there are no such N-tuples $(b_1, \ldots, b_N)$, then it is understood that the multiplicity is 0.

Theorem 4.1. *Let $1 \le n \le m$, and regard $Sp(n)$ and $Sp(m)$ as embedded as block diagonal subgroups of $Sp(n+m)$ in the standard way with $Sp(n)$ in the upper left diagonal block and with $Sp(m)$ in the lower right diagonal block.*

(a) *If* $(a_1, \dots, a_{n+m})$ *is the highest weight of an irreducible representation* (σ, V) *of* $Sp(n{+}m)$*, then a necessary and sufficient condition for the subspace* $V^{Sp(m)}$ *of* $Sp(m)$ *invariants to be nonzero is that* $a_{2n+1} = \cdots = a_{n+m} = 0$.

(b) *Let* $(a_1, \dots, a_{2n}, 0, \dots, 0)$ *be the highest weight of an irreducible representation* (σ, V) *of* $Sp(n+m)$ *with a nonzero subspace of* $Sp(m)$ *invariants, and let* (σ', V') *be an irreducible representation of* $U(2n)$ *with highest weight* $(a_1, \dots, a_{2n})$*. Then the representation* $(\sigma|_{Sp(n)}, V^{Sp(m)})$ *is equivalent with the restriction to* $Sp(n)$ *of the representation* (σ', V') *of* $U(2n)$.

The proof of Theorem 4.1a is similar to the proof of Theorem 2.1a. Let us therefore move to Theorem 4.1b.

Most of the proof of Theorem 4.1b has been given in §2, but some details have been left for this section. The first detail left for now is the construction of a particular member of $\mathrm{Ad}((G^d)^{\mathbb{C}})$ that transforms $\mathfrak{h}^{\mathbb{C}}$ into $(\mathfrak{a} \oplus \mathfrak{t})^{\mathbb{C}}$. This member is constructed as a product of Cayley transforms, and we need to indicate what roots are used in constructing the Cayley transforms.

There will be no difficulty with outer automorphisms in connection with Theorem 4.1. In fact, the inclusion $K_L \subset G'$ is a version of the inclusion $Sp(n) \subset U(2n)$, and Lemma 2.3 says that only one outer automorphism of $G' \cong U(2n)$ is of concern. We may take this to be θ, which fixes $Sp(n)$. A representation σ' of $U(2n)$ and its composition $\sigma' \circ \theta$ have the same restriction to $Sp(n)$, and so it does not matter if we confuse σ' with $\sigma' \circ \theta$.

In addition, the group $K_L \cong Sp(n)$ admits no nontrivial outer automorphisms, and hence no special symmetries require explanation.

Let us return to the passage from $\mathfrak{h}$ to $\mathfrak{a} \oplus \mathfrak{t}$. We begin by observing that the roots $e_r \pm e_s$ are compact if r and s are both $\le n$ or both $\ge n+1$, and they are noncompact if $r \le n$ and $s \ge n+1$. The roots $\pm 2e_r$ are compact. The roots $e_r \pm e_s$ are not strongly orthogonal, and hence the two cannot both be used in a strongly orthogonal sequence. Instead we use the strongly orthogonal sequence

$$e_1 - e_{n+m},\ e_2 - e_{n+m-1},\ \dots,\ e_n - e_{m+1}$$

to form Cayley transforms. The Cayley transforms are denoted

$$2f_1,\ 2f_2,\ \dots,\ 2f_n,$$

where f_r is the linear functional on $\mathfrak{a} \oplus \mathfrak{t}$ whose value on the matrix (2.4) is x_r and whose value on $\mathfrak{t}$ is 0; this definition consistently extends the definition in §2. Let f'_r be the linear functional on $\mathfrak{a} \oplus \mathfrak{t}$ that is 0 on $\mathfrak{a}$ and whose value on the quaternion matrix in (2.5) is iy_r, where i denotes the i in $\mathbb{C}$ rather than the i in $\mathbb{H}$. Up to Cayley transforms, we therefore have

$$f_r = \tfrac{1}{2}(e_r - e_{n+m+1-r}) \quad \text{and} \quad f'_r = \tfrac{1}{2}(e_r + e_{n+m+1-r}).$$

We may then make the following identifications, via Cayley transforms:

$$\begin{array}{rcll} 2(f'_r + f_r) & \longleftrightarrow & 2e_r & \text{if } r \le n, \\ 2(f'_r - f_r) & \longleftrightarrow & 2e_{n+m+1-r} & \text{if } r \le n, \\ (f'_r + f_r) \pm (f'_s + f_s) & \longleftrightarrow & e_r \pm e_s & \text{if } r, s \le n. \end{array}$$

The conditions on the ordering of the roots relative to $\mathfrak{a} \oplus \mathfrak{t}$ will be satisfied if we insist that

$$f_1 > \cdots > f_n > f'_1 > \cdots > f'_n > \psi$$

for all nonzero expressions ψ carried on the part of $\mathfrak{t}$ involving indices $n+1$ through m. Then the positive roots are all necessarily of the form

$$\begin{array}{lll} & 2f_r, & \\ & (f_r - f_s) + (\pm f_r' \pm f_s') & \text{with } r < s, \\ (4.2) & (f_r) + (\pm f_r') + \psi & \text{if } n < m, \\ & 2f_r', & \\ & \psi & \text{if } n < m. \end{array}$$

The given highest weight $a_1e_1 + \cdots + a_ne_n$ of (σ, V) relative to $\mathfrak{h}$ transforms to an integer combination of f_r's and f_s''s, together possibly with a term carried on $\mathfrak{t}$. The transformed expression is an extreme weight. To make it dominant, we permute coefficients of the e_r's, including those corresponding to the $\mathfrak{t}$ part, and use sign changes. Then the result, as in (2.21), is that the highest weight of (σ, V) relative to $\mathfrak{a} \oplus \mathfrak{t}$ is

$$\begin{aligned} (a_1 - a_n')f_1 + (a_2 - a_{n-1}')f_2 + \cdots + (a_n - a_1')f_n \\ + (a_1 + a_n')f_1' + (a_2 + a_{n-1}')f_2' + \cdots + (a_n + a_1')f_n' \end{aligned} \tag{4.3}$$

with no ψ term. The expression (4.3) consequently is the highest weight of (σ', V') relative to $\mathfrak{a}$.

The next step is to identify G' globally. We know that $\mathfrak{g}'$ is isomorphic to $\mathfrak{u}(2n)$, and we want to see that G' is isomorphic to $U(2n)$. The Lie algebra $\mathfrak{g}'$ consists of all real linear combinations of the matrices (2.18) with $y_1 = \cdots = y_n = 0$ and of embedded versions of the quaternion matrices in (2.19). The argument for this step involves conjugating by a matrix as in the previous two cases, but an additional complication arises in that we first have to change the quaternion matrices to complex matrices. Since all indices $1, \ldots, n$ used in the quaternion case behave in the same fashion, it will be enough to handle two such indices, i.e., to do the identification for $n = 2$. Thus we will be working with 4-by-4 quaternion matrices and 8-by-8 complex matrices. Following §I.8 of [Kn], let Q be a 4-by-4 quaternion matrix, and write Q in terms of 4-by-4 real matrices as $Q = A + Bi + Cj + Dk$. Put $Q_1 = A + Bi$ and $Q_2 = C - Di$. Then the 8-by-8 complex matrix corresponding to Q is

$$Z(Q) = \begin{pmatrix} Q_1 & -\overline{Q}_2 \\ Q_2 & \overline{Q}_1 \end{pmatrix}. \tag{4.4}$$

We apply this transformation to the part of (2.18) corresponding to indices 1 and 2, as well as to the two matrices in (2.19). Let W be the 4-by-4 complex matrix

$$W = \begin{pmatrix} 1 & 0 & 1 & 0 \\ i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & -i \end{pmatrix},$$

and let M'' be the 8-by-8 matrix that is constructed by using W in row and column indices 1, 4, 5, 8 and by using W again in row and column indices 2, 3, 6, 7. For each of the three matrices $Z(Q)$ obtained by the transformation (4.4), we form $M''^{-1}Z(Q)M''$. Then we check by inspection that the resulting three 8-by-8 matrices are block diagonal with two 4-by-4 diagonal blocks, that real linear combinations of these block diagonal matrices yield arbitary skew-Hermitian matrices for the upper left 4-by-4 block, and that the lower right 4-by-4 block is a function of the upper left 4-by-4 block. Then it follows that the group in question is $U(4)$ for the case $n = 2$ that is under study and hence is $U(2n)$ in general. We omit the details.

To complete the proof of Theorem 4.1b, it suffices to show that the mapping $E : V' \to V^{K_1}$ is one-one. Since Lemma 2.2 shows E to be onto, it is enough to prove that $\dim V' = \dim V^{K_1}$.

We regard the sequence $Sp(1) \subset Sp(2) \subset \cdots \subset Sp(N)$ of unitary quaternion groups to be nested in a standard way, such as with each one embedded as the lower right block of the next one. A *system* for $Sp(N)$ of *level* $2r$ coming from a dominant integral N-tuple $(c_1, \ldots, c_N)$ is a collection $\{(c_1^{(k)}, \ldots, c_{N-[k/2]}^{(k)}) \mid 0 \le k \le 2r\}$ consisting of one $(N - [k/2])$-tuple for each k such that

$$(c_1^{(0)}, \ldots, c_N^{(0)}) = (c_1, \ldots, c_N);$$

the kth tuple is dominant integral for $Sp(N - [k/2])$; and,

> for k even and ≥ 2, the $(k-1)$st and kth tuples are obtained from the $(k-2)$nd tuple by (4.1).

The *end* of the system is the $(N - r)$-tuple $(c_1^{(2r)}, \ldots, c_{N-r}^{(2r)})$.

Lemma 4.2. *Let τ be an irreducible representation of $Sp(N)$ with highest weight $(c_1, \ldots, c_N)$, let $1 \le r < N$, and let τ' be an irreducible representation of $Sp(N - r)$ with highest weight $(d_1, \ldots, d_{N-r})$. Then the number of systems for $Sp(N)$ of level $2r$ coming from $(c_1, \ldots, c_N)$ and having end $(d_1, \ldots, d_{N-r})$ equals the multiplicity of τ' in $\tau|_{Sp(N-r)}$.*

Proof. The argument is the same as for Lemma 2.4 except that the branching theorem (4.1) is used in place of the branching theorem (1.1).

Now we return to the notation of Theorem 4.1b. The given irreducible representation (σ, V) of $Sp(n + m)$ has highest weight the $(n + m)$-tuple

$$(a_1, \ldots, a_{2n}, 0, \ldots, 0). \tag{4.5}$$

The constructed irreducible representation (σ', V') of $U(n)$ has highest weight

$$(a_1, \ldots, a_{2n}).$$

Lemma 4.3. $\dim V' = \dim V^{K_1}$.

Proof. The right side is the multiplicity of the trivial representation of $K_1 = Sp(m)$ in $\sigma|_{Sp(m)}$, and Lemma 4.2 shows that this multiplicity equals the number of systems for $Sp(n + m)$ of level $2n$ coming from the $(n + m)$-tuple (4.5) and having end the m-tuple $(0, \ldots, 0)$.

We shall compute this number of systems in a second way and obtain the answer $\dim V'$. More specifically we shall show that the $Sp(n + m)$ systems of level $2n$ coming from the $(n + m)$-tuple (4.5) and having end the m-tuple $(0, \ldots, 0)$ are in one-one correspondence with the $U(n + m)$ systems of level $2n$ coming from the $(n + m)$-tuple (4.5) and having end the $(m - n)$-tuple $(0, \ldots, 0)$. (When $n = m$, the end tuple is to be a 0-tuple, i.e., is to be empty.) The correspondence is as follows: to pass from a $U(n + m)$ system to an $Sp(n + m)$ system, we pad the right ends of the tuples with 0's; to pass from an $Sp(n + m)$ system to a $U(n + m)$ system, we drop the appropriate number of entries from the right ends of the tuples. To see that this is a one-one correspondence, we need to check that

(i) the $Sp(n+m)$ tuples are always at least as long as the $U(n+m)$ tuples,
(ii) any entry that gets dropped from an $Sp(n+m)$ tuple in carrying out the correspondence has a 0 in it.

Each kind of system consists of $2n+1$ tuples numbered from 0 to $2n$, the kth tuple being of length

$$\begin{aligned} &n+m-[k/2] \qquad \text{in the } Sp(n+m) \text{ case,} \\ &n+m-k \qquad \text{in the } U(n+m) \text{ case.} \end{aligned} \tag{4.6}$$

Fact (i) above follows from the inequality

$$n+m-k \le n+m-[k/2].$$

To prove (ii), suppose that $n+m-[s/2] > n+m-s$ and that

$$\{(c_1^{(k)}, \ldots, c_{n+m-[k/2]}^{(k)}) \mid 0 \le k \le 2n\}$$

is an $Sp(n+m)$ system of level $2n$ coming from the $(n+m)$-tuple (4.5) and having end the m-tuple $(0,\ldots,0)$. We are to show that

$$c_{n+m-s+1}^{(s)} = 0. \tag{4.7}$$

Since all entries of (4.5) are ≥ 0 and the branching rule (4.1) is in force, we have

$$c_l^{(s)} \ge 0 \quad \text{for } 1 \le l \le n+m-[s/2]. \tag{4.8}$$

On the other hand, (2.29a) shows that $c_{l+r}^{(k-r)} \le c_l^{(k)}$ whenever the indices are in bounds. Taking $k=2n$, $r=2n-s$, and $l=1$, we obtain

$$c_{2n+1-s}^{(s)} \le c_1^{(2n)} = 0. \tag{4.9}$$

Thus (4.8) shows that

$$c_{2n+1-s}^{(s)} = 0. \tag{4.10}$$

Since

$$2n+1-s \le n+m-s+1,$$

(4.7) follows from (4.10), dominance, and (4.8).

Thus the number of systems for $Sp(n+m)$ of level $2n$ coming from the $(n+m)$-tuple (4.5) and having end the m-tuple $(0,\ldots,0)$ equals the number of systems for $U(n+m)$ of level $2n$ coming from the $(n+m)$-tuple (4.5) and having end the $(m-n)$-tuple $(0,\ldots,0)$. This latter number, by Corollary 2.5 and the argument in the proof of Lemma 2.6, equals the dimension of V'. This completes the proof of Lemma 4.4 and also Theorem 4.1b.

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