Proceedings of Symposia in Pure Mathematics
Volume **61** (1997), pp. 1–27

Structure Theory of Semisimple Lie Groups

A. W. Knapp

This article provides a review of the elementary theory of semisimple Lie algebras and Lie groups. It is essentially a summary of much of [K3]. The four sections treat complex semisimple Lie algebras, finite-dimensional representations of complex semisimple Lie algebras, compact Lie groups and real forms of complex Lie algebras, and structure theory of noncompact semisimple groups.

1. Complex Semisimple Lie Algebras

This section deals with the structure theory of complex semisimple Lie algebras. Some references for this material are [He], [Hu], [J], [K1], [K3], and [V].

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. For the moment we shall allow the underlying field to be $\mathbb{R}$ or $\mathbb{C}$, but shortly we shall restrict to Lie algebras over $\mathbb{C}$.

Semisimple Lie algebras are defined as follows. Let $\operatorname{rad}\mathfrak{g}$ be the sum of all the solvable ideals in $\mathfrak{g}$. The sum of two solvable ideals is a solvable ideal [K3, §I.2], and the finite-dimensionality of $\mathfrak{g}$ makes $\operatorname{rad}\mathfrak{g}$ a solvable ideal. We say that $\mathfrak{g}$ is **semisimple** if $\operatorname{rad}\mathfrak{g} = 0$.

Within $\mathfrak{g}$, let $\operatorname{ad} X$ be the linear transformation given by $(\operatorname{ad} X)Z = [X, Z]$. The **Killing form** is the symmetric bilinear form on $\mathfrak{g}$ defined by $B(X, Y) = \operatorname{Tr}(\operatorname{ad} X \operatorname{ad} Y)$. It is **invariant** in the sense that $B([X, Y], Z) = B(X, [Y, Z])$ for all X, Y, Z in $\mathfrak{g}$.

Theorem 1.1 (Cartan's criterion for semisimplicity)**.** *The Lie algebra $\mathfrak{g}$ is semisimple if and only if B is nondegenerate.*

Reference. [K3, Theorem 1.42].

The Lie algebra $\mathfrak{g}$ is said to be **simple** if $\mathfrak{g}$ is nonabelian and $\mathfrak{g}$ has no proper nonzero ideals. In this case, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Semisimple Lie algebras and simple Lie algebras are related as in the following theorem.

Theorem 1.2. *The Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}$ is the direct sum of simple ideals. In this case there are no other simple ideals, the direct sum decomposition is unique up to the order of the summands, and every ideal is the sum of some subset of the simple ideals. Also in this case, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

1991 *Mathematics Subject Classification.* Primary 17B20, 20G05, 22E15.

REFERENCE. [K3, Theorem 1.51].

For the remainder of this section, $\mathfrak{g}$ will always denote a semisimple Lie algebra, and the underlying field will be $\mathbb{C}$. The dual of a vector space V will be denoted V^*.

We discuss root-space decompositions. For our semisimple Lie algebra $\mathfrak{g}$, these are decompositions of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Here $\mathfrak{h}$ is a **Cartan subalgebra**, defined in any of three equivalent ways [K3, §§II.2–3] as

(a) (usual definition) a nilpotent subalgebra $\mathfrak{h}$ whose normalizer satisfies $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$,
(b) (constructive definition) the generalized eigenspace for 0 eigenvalue for $\operatorname{ad} X$ with X regular (i.e., characteristic polynomial $\det(\lambda 1 - \operatorname{ad} X)$ is such that the lowest-order nonzero coefficient is nonzero on X),
(c) (special definition for $\mathfrak{g}$ semisimple) a maximal abelian subspace of $\mathfrak{g}$ in which every $\operatorname{ad} H$, $H \in \mathfrak{h}$, is diagonable.

The elements $\alpha \in \mathfrak{h}^*$ are **roots**, and the $\mathfrak{g}_\alpha$'s are **root spaces**, the α's being defined as the nonzero elements of $\mathfrak{h}^*$ such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$$

is nonzero.

Let Δ be the set of all roots. This is a finite set. We recall the the classical examples of root-space decompositions [K3, §II.1].

Example 1. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) = \{n\text{-by-}n \text{ complex matrices of trace } 0\}$.
The Cartan subalgebra is

$$\mathfrak{h} = \{\text{diagonal matrices in } \mathfrak{g}\}.$$

Let

$$E_{ij} = \begin{cases} 1 & \text{in } (i,j)^{\text{th}} \text{ place} \\ 0 & \text{elsewhere.} \end{cases}$$

Let $e_i \in \mathfrak{h}^*$ be defined by

$$e_i \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} = h_i.$$

Then each $H \in \mathfrak{h}$ satisfies

$$(\operatorname{ad} H)E_{ij} = [H, E_{ij}] = (e_i(H) - e_j(H))E_{ij}.$$

So E_{ij} is a simultaneous eigenvector for all $\operatorname{ad} H$, with eigenvalue $e_i(H) - e_j(H)$. We conclude that

(a) $\mathfrak{h}$ is a Cartan subalgebra,
(b) the roots are the $(e_i - e_j)$'s for $i \neq j$,
(c) $\mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{ij}$.

Example 2. $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C}) = \{n\text{-by-}n \text{ skew-symmetric complex matrices}\}$. For this example one proceeds similarly. Let

$$\mathfrak{h} = \{H \in \mathfrak{so}(2n+1,\mathbb{C}) \mid H = \text{matrix below}\}.$$

Here

H is block diagonal with n 2-by-2 blocks and one 1-by-1 block,

the j^{th} 2-by-2 block is $\begin{pmatrix} 0 & ih_j \\ -ih_j & 0 \end{pmatrix}$,

the 1-by-1 block is just (0).

Let $e_j(\text{above matrix } H) = h_j$ for $1 \le j \le n$. Then

$$\Delta = \{\pm e_i \pm e_j \text{ with } i \neq j\} \cup \{\pm e_k\}.$$

Formulas for the root vectors E_α may be found in [K3, §II.1].

Example 3. $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C})$.
This is the Lie algebra of all $2n$-by-$2n$ complex matrices X such that

$$X^t J + JX = 0, \qquad \text{where } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

For this example the Cartan subalgebra $\mathfrak{h}$ is the set of all matrices H of the form

$$H = \begin{pmatrix} h_1 & & & & & \\ & \ddots & & & & \\ & & h_n & & & \\ & & & -h_1 & & \\ & & & & \ddots & \\ & & & & & -h_n \end{pmatrix}$$

Let $e_j(\text{above matrix } H) = h_j$ for $1 \le j \le n$. Then

$$\Delta = \{\pm e_i \pm e_j \text{ with } i \neq j\} \cup \{\pm 2e_k\}.$$

Formulas for the root vectors E_α may again be found in [K3, §II.1].

Example 4. $\mathfrak{g} = \mathfrak{so}(2n,\mathbb{C})$.
This example is similar to $\mathfrak{so}(2n+1,\mathbb{C})$ but without the $(2n+1)^{\text{st}}$ entry. The set of roots is

$$\Delta = \{\pm e_i \pm e_j \text{ with } i \neq j\}.$$

We return to the discussion of general semisimple Lie algebras $\mathfrak{g}$. The following are some elementary properties of root-space decompositions:

(a) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
(b) If α and β are in $\Delta \cup \{0\}$ and $\alpha + \beta \neq 0$, then $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.
(c) If α is in Δ, then B is nonsingular on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$.
(d) If α is in Δ, then so is $-\alpha$.
(e) $B|_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate. Define H_α to be the element of $\mathfrak{h}$ paired with α.
(f) Δ spans $\mathfrak{h}^*$.

See [K3, §II.4]. We isolate some deeper properties of root-space decompositions as a theorem.

Theorem 1.3. *Root-space decompositions have the following properties:*

(a) *If α is in Δ, then* $\dim \mathfrak{g}_\alpha = 1$.
(b) *If α is in Δ, then $n\alpha$ is not in Δ for any integer $n \geq 2$.*
(c) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ *if $\alpha + \beta \neq 0$.*
(d) *The real subspace $\mathfrak{h}_0$ of $\mathfrak{h}$ on which all roots are real is a real form of $\mathfrak{h}$, and $B|_{\mathfrak{h}_0 \times \mathfrak{h}_0}$ is an inner product. Transfer $B|_{\mathfrak{h}_0 \times \mathfrak{h}_0}$ to the real span $\mathfrak{h}_0^*$ of the roots, obtaining $\langle \cdot , \cdot \rangle$ and $| \cdot |^2$.*

Reference. [K3, §II.4].

Let us now consider root strings. By definition the α **string containing** β (for $\alpha \in \Delta$, $\beta \in \Delta \cup \{0\}$) consists of all members of $\Delta \cup \{0\}$ of the form $\beta + n\alpha$ with $n \in \mathbb{Z}$. The n's in question form an interval with $-p \leq n \leq q$ and $p - q = \dfrac{2\langle \beta, \alpha \rangle}{|\alpha|^2}$. Here $p - q$ is a measure of how centered β is in the root string. When $p - q$ is 0, β is exactly in the center. When $p - q$ is large and positive, β is close to the end $\beta + q\alpha$ of the root string. In any event, it follows that $\dfrac{2\langle \beta, \alpha \rangle}{|\alpha|^2}$ is always an integer. A consequence of the form of root strings is that if α is in Δ, then the orthogonal transformation of $\mathfrak{h}_0^*$ given by

$$s_\alpha(\varphi) = \varphi - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|^2}\, \alpha$$

carries Δ into itself. The linear transformation s_α is called the **root reflection** in α.

An **abstract root system** is a finite set Δ of nonzero elements in a real inner product space V such that

(a) Δ spans V,
(b) all s_α for $\alpha \in \Delta$ carry Δ to itself,
(c) $\dfrac{2\langle \beta, \alpha \rangle}{|\alpha|^2}$ is an integer whenever α and β are in Δ.

We say that an abstract root system is **reduced** if $\alpha \in \Delta$ implies $2\alpha \notin \Delta$.

The relevance of these notions to semisimple Lie algebras is that the root system of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ forms a reduced abstract root system in $\mathfrak{h}_0^*$. See [K3, Theorem 2.42].

There are four kinds of classical reduced root systems:

A_n has $V = \left\{ \sum_{i=1}^{n+1} e_i \right\}^\perp$ in $\mathbb{R}^{n+1}$ and $\Delta = \{e_i - e_j \mid i \neq j\}$. The system A_n arises from $\mathfrak{sl}(n+1, \mathbb{C})$.
B_n has $V = \mathbb{R}^n$ and $\Delta = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm e_k\}$. The system B_n arises from $\mathfrak{so}(2n+1, \mathbb{C})$.
C_n has $V = \mathbb{R}^n$ and $\Delta = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_k\}$. The system C_n arises from $\mathfrak{sp}(n, \mathbb{C})$.
D_n has $V = \mathbb{R}^n$ and $\Delta = \{\pm e_i \pm e_j \mid i \neq j\}$. . The system D_n arises from $\mathfrak{so}(2n, \mathbb{C})$.

We say that an abstract root system Δ is **reducible** if $\Delta = \Delta' \cup \Delta''$ with $\Delta' \perp \Delta''$. Otherwise Δ is **irreducible**.

Theorem 1.4. *A semisimple Lie algebra $\mathfrak{g}$ is simple if and only if the corresponding root system Δ is irreducible.*

Reference. [K3, Proposition 2.44].

Now we introduce the notions of lexicographic ordering and positive roots for an abstract root system. The construction is as follows. Let $\varphi_1, \ldots, \varphi_m$ be a spanning set for V. Define φ to be **positive** (written $\varphi > 0$) if there exists an index k such that $\langle \varphi, \varphi_i \rangle = 0$ for $1 \leq i \leq k-1$ and $\langle \varphi, \varphi_k \rangle > 0$. The corresponding **lexicographic ordering** has $\varphi > \psi$ if $\varphi - \psi$ is positive. Fix such an ordering. Call the root α **simple** if $\alpha > 0$ and if α does not decompose as $\alpha = \beta_1 + \beta_2$ with β_1 and β_2 both positive roots.

Theorem 1.5. *If $l = \dim V$, then there are l simple roots $\alpha_1, \ldots, \alpha_l$, and they are linearly independent. If β is a root and is written as*

$$\beta = x_1 \alpha_1 + \cdots + x_l \alpha_l,$$

then all the x_j have the same sign (if 0 is allowed to be positive or negative), and all the x_j are integers.

When standard choices are made, the following are the positive roots and simple roots for the classical reduced root systems:

A_n. The positive roots are the $e_i - e_j$ with $i < j$. The simple roots are all $e_i - e_{i+1}$ with $1 \leq i \leq n$.

B_n. The positive roots are the $e_i \pm e_j$ with $i < j$ and all e_k. The simple roots are e_n and all $e_i - e_{i+1}$ with $1 \leq i \leq n-1$.

C_n. The positive roots are the $e_i \pm e_j$ with $i < j$ and all $2e_k$. The simple roots are $2e_n$ and all $e_i - e_{i+1}$ with $1 \leq i \leq n-1$.

D_n. The positive roots are the $e_i \pm e_j$ with $i < j$. The simple roots are $e_{n-1} + e_n$ and all $e_i - e_{i+1}$ with $1 \leq i \leq n-1$.

A root α is called **reduced** if $\frac{1}{2}\alpha$ is not a root. Every simple root is reduced. By a **simple system** for Δ, we mean the set of simple roots for some ordering. By Theorem 1.5, a simple system $\{\alpha_1, \ldots, \alpha_l\}$ has the property that any root α, when expressed as $\sum_i x_i \alpha_i$, has all x_i of the same sign. Conversely any subset $\{\alpha_1, \ldots, \alpha_l\}$ of reduced roots with the property that any root α, when expressed as $\sum_i x_i \alpha_i$, has all x_i of the same sign is a simple system.

Let l be the dimension of the underlying space V of an abstract root system Δ. The number l is called the **rank**. If Δ is the root system of a semisimple Lie algebra $\mathfrak{g}$, we also refer to $l = \dim \mathfrak{h}$ as the **rank** of $\mathfrak{g}$. Relative to a given simple system $\{\alpha_1, \ldots, \alpha_l\}$, the **Cartan matrix** is the l-by-l matrix with entries

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2}.$$

It has the following properties:

(a) A_{ij} is in $\mathbb{Z}$ for all i and j,
(b) $A_{ii} = 2$ for all i,
(c) $A_{ij} \leq 0$ for $i \neq j$,
(d) $A_{ij} = 0$ if and only if $A_{ji} = 0$,
(e) there exists a diagonal matrix D with positive diagonal entries such that DAD^{-1} is symmetric positive definite.

An **abstract Cartan matrix** is a square matrix satisfying properties (a) through (e) as above. To such a matrix we can associate a **Dynkin diagram** in the standard way. See [K3, §II.5].

We come to the first principal result.

Theorem 1.6 (Isomorphism Theorem)**.** *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be complex semisimple Lie algebras with respective Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ and respective root systems Δ and Δ'. Suppose that a vector space isomorphism $\varphi : \mathfrak{h} \to \mathfrak{h}'$ is given with the property that φ carries Δ one-one onto Δ'. Let the mapping of Δ to Δ' be denoted $\alpha \mapsto \alpha'$. Fix a simple system Π for Δ. For each α in Π, select nonzero root vectors $E_\alpha \in \mathfrak{g}$ for α and $E_{\alpha'} \in \mathfrak{g}'$ for α'. Then there exists one and only one Lie algebra isomorphism $\tilde{\varphi} : \mathfrak{g} \to \mathfrak{g}'$ such that $\tilde{\varphi}|_{\mathfrak{h}} = \varphi$ and $\tilde{\varphi}(E_\alpha) = E_{\alpha'}$ for all $\alpha \in \Pi$.*

Reference. [K3, Theorem 2.108].

Examples.

1) An automorphism of the Dynkin diagram yields an automorphism of the Lie algebra.

2) Let $\varphi = -1$ on $\mathfrak{h}$. This extends to $\tilde{\varphi} : \mathfrak{g} \to \mathfrak{g}$ and is used in constructing real forms of $\mathfrak{g}$. See Theorem 3.5 and the discussion that follows it.

The **Weyl group** $W(\Delta)$ of an abstract root system Δ is defined to be the finite group generated by all root reflections s_α for $\alpha \in \Delta$.

Theorem 1.7. *The Weyl group $W(\Delta)$ of the abstract root system Δ has the following properties:*

(a) *Fix a simple system $\Pi = \{\alpha_1, \dots, \alpha_l\}$ for Δ. Then $W(\Delta)$ is generated by all s_{α_i}, $\alpha_i \in \Pi$. If α is any reduced root, then there exist $\alpha_j \in \Pi$ and $s \in W(\Delta)$ such that $s\alpha_j = \alpha$.*

(b) *If Π and Π' are two simple systems for Δ, then there exists one and only one element $s \in W(\Delta)$ such that $s\Pi = \Pi'$.*

Reference. [K3, Proposition 2.62 and Theorem 2.63].

Briefly conclusion (b) says that $W(\Delta)$ acts simply transitively on the set of all simple systems. There is a geometric way of formulating this property. Regard V as the dual of its dual V^*, so that each root has a kernel in V^*. A **Weyl chamber** of V^* is a connected component of the subset of V^* on which every root is nonzero. Each Weyl chamber is an open convex cone, and each root has constant sign on each Weyl chamber. To each simple system corresponds exactly one Weyl chamber, namely the set where each simple root is positive. Conversely each Weyl chamber determines a simple system by this procedure. If the action of $W(\Delta)$ on V is transferred to an action on V^*, then (b) says that $W(\Delta)$ acts simply transitively on the set of Weyl chambers.

Dominance is a notion that plays a role with finite-dimensional representations and will be discussed in detail in §2. We call $\lambda \in V$ **dominant** if $\langle \lambda, \alpha \rangle \geq 0$ for all positive roots α. Equivalently $\langle \lambda, \alpha \rangle \geq 0$ is to hold for all simple roots α.

Theorem 1.8. *Fix an abstract root system Δ.*

(a) *If λ is in V, then there exists a simple system Π for which λ is dominant.*

(b) *If λ is in V and if a simple system is specified, then there is some element w of the Weyl group such that $w\lambda$ is dominant.*

Reference. [K3, Proposition 2.67 and Corollary 2.68].

Here is a handy result that uses dominance in its proof.

Theorem 1.9 (Chevalley's Lemma). *Fix v in V, and let W_0 be the subgroup of $W(\Delta)$ fixing v. Then W_0 is generated by the root reflections s_α such that $\langle v, \alpha\rangle = 0$.*

Reference. [K3, Proposition 2.72].

Examples.
1) The only reflections s_φ in $W(\Delta)$ are the root reflections.
2) If an element v of V is fixed by a nontrivial element of $W(\Delta)$, then some root is orthogonal to v.
3) Any element of order 2 in $W(\Delta)$ is the product of commuting root reflections.

The main correspondence involving complex semisimple Lie algebras relates three classes of objects and isomorphisms, identifying each one with the other two:

(1) complex semisimple Lie algebras and isomorphisms of Lie algebras,
(2) abstract reduced root systems and invertible linear maps carrying Δ to Δ' and respecting the integers $2\langle\beta,\alpha\rangle/|\alpha|^2$,
(3) abstract Cartan matrices and equality up to permutation of indices.

The passage from (1) to (2) is well defined because any two Cartan subalgebras of $\mathfrak{g}$ are conjugate via $\operatorname{Int}\mathfrak{g}$ (see [K3, Theorem 2.15]); here $\operatorname{Int}\mathfrak{g}$ is the analytic subgroup of $GL(\mathfrak{g})$ with Lie algebra $\operatorname{ad}\mathfrak{g}$. The passage from (1) to (2) is one-one by the Isomorphism Theorem (Theorem 1.6 above), and it is onto by a result known as the Existence Theorem (see [K3, Theorem 2.111]).

The passage (2) to (3) is well defined because any two simple systems are conjugate via the Weyl group (Theorem 1.7b above). It is one-one by Theorem 1.7a above, and it is onto by a case-by-case construction.

2. Finite-Dimensional Representations of Complex Semisimple Lie Algebras

This section deals with finite-dimensional representations of complex semisimple Lie algebras and with the tools needed in their study. Some references for this material are [Hu], [J], [K1], [K2], [K3], and [V].

Except for one segment about the universal enveloping algebra where $\mathfrak{g}$ will be allowed to be more general, the notation in this section will be as follows:

$$\begin{aligned}
\mathfrak{g} &= \text{complex semisimple Lie algebra}\\
\mathfrak{h} &= \text{Cartan subalgebra}\\
\Delta &= \Delta(\mathfrak{g},\mathfrak{h}) = \text{set of roots}\\
\mathfrak{h}_0 &= \text{real form of } \mathfrak{h} \text{ where roots are real-valued}\\
B &= \text{nondegenerate symmetric invariant bilinear form}\\
&\quad\ \text{on } \mathfrak{g} \text{ that is positive definite on } \mathfrak{h}_0\\
H_\lambda &= \text{member of } \mathfrak{h}_0 \text{ corresponding to } \lambda \in \mathfrak{h}_0^*
\end{aligned}$$

Here B can be the Killing form, but it does not need to be. In the definition of H_λ, it is understood that $(\,\cdot\,)^*$ refers to the vector space dual; the correspondence of λ to H_λ is the one induced by B.

A **representation** φ on a complex vector space V is a linear map $\varphi : \mathfrak{g} \to \operatorname{End} V$ with

$$\varphi[X, Y] = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X)$$

for all X and Y in $\mathfrak{g}$. Isomorphism of representations is called **equivalence**. An **irreducible representation** is a representation φ on a nonzero space V such that $\varphi(\mathfrak{g})U \subseteq U$ fails for all proper nonzero subspaces U.

Fix such a φ. For $\lambda \in \mathfrak{h}^*$, let V_λ be the set of all $v \in V$ with $(\varphi(H)-\lambda(H)1)^n v = 0$ for all $H \in \mathfrak{h}$ and some $n = n(H, V)$. If V_λ is nonzero, V_λ is called a **generalized weight space**, and λ is called a **weight**. If $\dim V$ is finite-dimensional, V is the direct sum of its generalized weight spaces. This is a generalization of the fact from linear algebra about a linear transformation L on a finite-dimensional V that V is the direct sum of the generalized eigenspaces of L. If λ is a weight, then the subspace

$$\{v \in V \mid \varphi(H)v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}$$

is nonzero and is called the **weight space** corresponding to λ.

A source of finite-dimensional representations of $\mathfrak{g}$ is group representations. Suppose that G is a compact connected Lie group whose Lie algebra $\mathfrak{g}_0$ has complexification $\mathfrak{g}$. A **representation** Φ of G on a complex vector space V is a continuous group homomorphism $\Phi : G \to GL(V)$. If V is finite-dimensional, then Φ is automatically smooth. We can differentiate to get a representation φ of $\mathfrak{g}_0$ on V, and then we can complexify, writing

$$\varphi(X + iY) = \varphi(X) + i\varphi(Y),$$

to obtain a representation φ of $\mathfrak{g}$ on V.

We can obtain some initial examples of this sort with $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$. We start with $G = SU(n)$ and $G = SO(n)$ in the two cases. Each of these has a standard representation on $\mathbb{C}^n$, given by the multiplication of matrices and column vectors. For each we can form a contragredient representation on the dual space $(\mathbb{C}^n)^*$. Then we can form tensor products of copies of the standard representation and its dual. Finally we can pass to skew-symmetric tensors, symmetric tensors, and similar such subspaces. Representations in polynomials arise as symmetric tensors in the tensor product of copies of $(\mathbb{C}^n)^*$.

More examples come by starting with the compact connected Lie group $G = U(2n) \cap Sp(n, \mathbb{C})$, whose complexified Lie algebra is $\mathfrak{sp}(n, \mathbb{C})$. In this case the standard representation has dimension $2n$.

In the examples below, we list some representations obtained in this way from $G = SU(n)$ and $G = SO(2n + 1)$. In each case the weights are identified. Also the highest weight, i.e., the largest weight, is identified relative to the lexicographic ordering. The Cartan subalgebras and sets of positive roots for $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{so}(2n + 1, \mathbb{C})$ are the ones in §1.

Examples. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Here the Cartan subalgebra is the diagonal subalgebra.

1) Let V be the space of polynomials in $z_1, \ldots, z_n$ and their conjugates homogeneous of degree N. The action is

$$(\Phi(g)P)(z, \bar{z}) = P(g^{-1}z, \bar{g}^{-1}\bar{z}).$$

The weights are all expressions $\sum_{j=1}^n (l_j - k_j)e_j$ with all $k_j \geq 0$ and $l_j \geq 0$ and with $\sum_{j=1}^n (k_j + l_j) = N$. The highest weight is Ne_1.

2) Let V be the subspace of holomorphic polynomials in the preceding example. The action is $\Phi(g)(z) = P(g^{-1}z)$. The weights are all expressions $-\sum_{j=1}^n k_j e_j$ with all $k_j \geq 0$ and with $\sum_{j=1}^n k_j = N$. The highest weight is $-Ne_n$.

3) Let $V = \bigwedge^l \mathbb{C}^n$ with action

$$\Phi(g)(v_1 \wedge \cdots \wedge v_l) = gv_1 \wedge \cdots \wedge gv_l.$$

The weights are all expressions $\sum_{k=1}^l e_{j_k}$, and the highest weight is $\sum_{k=1}^l e_k$.

Examples. Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$. Here the Cartan subalgebra is block diagonal, containing n 2-by-2 skew-symmetric blocks and one 1-by-1 block whose entry is 0.

1) Let V be the space of all polynomials in $x_1, \dots, x_{2n+1}$ that are homogeneous of degree N, the action being $\Phi(g)(x) = P(g^{-1}x)$. The weights are all expressions $\sum_{j=1}^n (l_j - k_j)e_j$ with all $k_j \geq 0$ and $l_j \geq 0$ and with $k_0 + \sum_{j=1}^n (k_j + l_j) = N$. The highest weight is Ne_1.

2) Let $V = \bigwedge^l \mathbb{C}^{2n+1}$ with $l \leq n$ and with action as in Example 3 for $\mathfrak{sl}(n, \mathbb{C})$. The weights are all expressions $\pm e_{j_1} \pm \cdots \pm e_{j_r}$ with $j_1 < \cdots < j_r$ and $r \leq l$. The highest weight is $\sum_{k=1}^l e_k$. When $V = \bigwedge^m \mathbb{C}^{2n+1}$ with $m > n$, we again get a representation, and it can be shown to be equivalent with the representation on $\bigwedge^{2n+1-m} \mathbb{C}^{2n+1}$.

A member λ of $\mathfrak{h}^*$ is said to be **algebraically integral** if $2\langle \lambda, \alpha \rangle / |\alpha|^2$ is in $\mathbb{Z}$ for each $\alpha \in \Delta$.

Some elementary properties of a finite-dimensional representation φ on a vector space V are as follows:

(a) $\varphi(\mathfrak{h})$ acts diagonably on V, so that every generalized weight vector is a weight vector and V is the direct sum of all the weight spaces,
(b) every weight is real-valued on $\mathfrak{h}_0$ and is algebraically integral,
(c) roots and weights are related by $\varphi(\mathfrak{g}_\alpha)V_\lambda \subseteq V_{\lambda+\alpha}$.

Properties (a) and (b) follow by restricting φ to copies of $\mathfrak{sl}(2, \mathbb{C})$ lying in $\mathfrak{g}$ and then using the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, which we do not review. See [K3, §I.9].

Fix a lexicographic ordering, and let Δ^+ be the set of positive roots. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the corresponding simple system. There are three main theorems on representation theory in this section, and we come now to the first of the three.

Theorem 2.1 (Theorem of the Highest Weight)**.** *Apart from equivalence the irreducible finite-dimensional representations φ of $\mathfrak{g}$ stand in one-one correspondence with the algebraically integral dominant linear functionals λ on $\mathfrak{h}$, the correspondence being that λ is the highest weight of φ_λ. The highest weight λ of φ_λ has these additional properties:*

(a) *λ depends only on the simple system Π and not on the ordering used to define Π.*
(b) *the weight space V_λ for λ is 1-dimensional.*
(c) *each root vector E_α for arbitrary $\alpha \in \Delta^+$ annihilates the members of V_λ, and the members of V_λ are the only vectors with this property.*
(d) *every weight of φ_λ is of the form $\lambda - \sum_{i=1}^l n_i \alpha_i$ with the integers ≥ 0 and the α_i in Π.*

(e) *each weight space* V_μ *for* φ_λ *has* $\dim V_{w\mu} = \dim V_\mu$ *for all* w *in the Weyl group* $W(\Delta)$*, and each weight* μ *has* $|\mu| \le |\lambda|$ *with equality only if* μ *is in the orbit* $W(\Delta)\lambda$*.*

REFERENCE. [K3, Theorem 5.5]. Later in this section we discuss tools used in the proof.

REMARK. As a consequence of (e), the Weyl group acts on the weights, preserving multiplicities. The extreme weights are those in the orbit of the highest weight.

We can immediately state the second main theorem of the section on representation theory. It concerns complete reducibility.

Theorem 2.2. *Let* φ *be a complex-linear representation of* $\mathfrak{g}$ *on a finite-dimensional complex vector space* V*. Then* V *is completely reducible in the sense that there exist invariant subspaces* $U_1, \dots, U_r$ *of* V *such that* $V = U_1 \oplus \cdots \oplus U_r$ *and such that the restriction of the representation to each* U_i *is irreducible.*

REFERENCE. [K3, Theorem 5.29].

The proofs of Theorems 2.1 and 2.2 use three tools:

(a) universal enveloping algebra,
(b) Casimir element,
(c) Verma modules.

We review each of these in turn.

First we take up the universal enveloping algebra. In the discussion, we shall allow $\mathfrak{g}$ to be any complex Lie algebra. Let $T(\mathfrak{g})$ be the tensor algebra

$$T(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots .$$

In $T(\mathfrak{g})$, let J be the two-sided ideal generated by all $X \otimes Y - Y \otimes X - [X, Y]$ with X and Y in the space $T^1(\mathfrak{g})$ of first-order tensors. The **universal enveloping algebra** of $\mathfrak{g}$ is the associative algebra (with identity) given by

$$U(\mathfrak{g}) = T(\mathfrak{g})/J.$$

Let $\iota : \mathfrak{g} \to U(\mathfrak{g})$ be the composition $\iota : \mathfrak{g} \cong T^1(\mathfrak{g}) \hookrightarrow T(\mathfrak{g}) \to U(\mathfrak{g})$, so that

$$\iota[X, Y] = \iota(X)\iota(Y) - \iota(Y)\iota(X).$$

The universal enveloping algebra is so named because of the following universal mapping property.

Theorem 2.3. *Whenever* A *is a complex associative algebra with identity and* $\pi : \mathfrak{g} \to A$ *is a linear mapping such that*

$$\pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi[X, Y]$$

for all X, Y *in* $\mathfrak{g}$*, then there exists a unique algebra homomorphism* $\tilde{\pi} : U(\mathfrak{g}) \to A$ *such that* $\tilde{\pi}(1) = 1$ *and* $\pi = \tilde{\pi} \circ \iota$*.*

REFERENCE. [K3, Proposition 3.3].

Remark. One thinks of $\tilde{\pi}$ in the theorem as an extension of π from $\mathfrak{g}$ to all of $U(\mathfrak{g})$. This attitude about $\tilde{\pi}$ implicitly assumes that ι is one-one, a fact that follows from Theorem 2.5 below.

Theorem 2.4. *Representations of $\mathfrak{g}$ on complex vector spaces stand in one-one correspondence with left $U(\mathfrak{g})$ modules in which 1 acts as 1.*

Reference. [K3, Corollary 3.6].

Remark. The one-one correspondence comes from $\pi \mapsto \tilde{\pi}$ in the notation of Theorem 2.3.

Theorem 2.5 (Poincaré-Birkhoff-Witt Theorem)**.** *Let $\{X_i\}_{i\in A}$ be a basis of $\mathfrak{g}$, and suppose that a simple ordering has been imposed on the index set A. Then the set of all monomials*

$$(\iota X_{i_1})^{j_1}\cdots(iX_{i_n})^{j_n}$$

with $i_1 < \cdots < i_n$ and with all $j_k \geq 0$, is a basis of $U(\mathfrak{g})$. In particular the canonical map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is one-one.

Reference. [K3, Theorem 3.8].

Let us now return to our assumption that $\mathfrak{g}$ is semisimple. We also return to the other notation listed at the start of this section. We shall apply the theorems about $U(\mathfrak{g})$ to a representation φ of $\mathfrak{g}$ on a finite-dimensional vector space V. We enumerate the positive roots as $\beta_1,\dots,\beta_m$, and we let $H_1,\dots,H_l$ be a basis of $\mathfrak{h}$. We use the ordered basis

$$E_{-\beta_1},\dots,E_{-\beta_m},H_1,\dots,H_l,E_{\beta_1},\dots,E_{\beta_m}.$$

in the Poincaré-Birkhoff-Witt Theorem. The theorem says that

$$E_{-\beta_1}^{p_1}\cdots E_{-\beta_m}^{p_m}H_1^{k_1}\cdots H_l^{k_l}E_{\beta_1}^{q_1}\cdots E_{\beta_m}^{q_m}$$

is a basis of $U(\mathfrak{g})$. If we apply members of this basis to a nonzero highest weight vector v of V, we get control of a general member of $U(\mathfrak{g})v$. In fact, $E_{\beta_1}^{q_1}\cdots E_{\beta_m}^{q_m}$ will act as 0 if $q_1+\cdots+q_m>0$, and $H_1^{k_1}\cdots H_l^{k_l}$ will act as a scalar. Thus we have only to sort out the effect of $E_{-\beta_1}^{p_1}\cdots E_{-\beta_m}^{p_m}$, and most of the conclusions in the Theorem of the Highest Weight (Theorem 2.1) follow readily.

This completes the discussion of the universal enveloping algebra. The second tool used in the proofs of Theorems 2.1 and 2.2 is the Casimir element. For our complex semisimple Lie algebra $\mathfrak{g}$, the **Casimir element** Ω is the member

$$\Omega = \sum_{i,j} B(X_i,X_j)\tilde{X}_i\tilde{X}_j$$

of $U(\mathfrak{g})$, where $\{X_i\}$ is a basis of $\mathfrak{g}$ and $\{\tilde{X}_i\}$ is the dual basis relative to B. One shows that Ω is defined independently of the basis $\{X_i\}$ and is a member of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. (See [K3, Proposition 5.24].)

Theorem 2.6. *Let Ω be the Casimir element. Let $\{H_i\}_{i=1}^l$ be an orthonormal basis of $\mathfrak{h}_0$ relative to B, and choose root vectors E_α so that $B(E_\alpha, E_{-\alpha}) = 1$ for all roots α. Then*

(a) $\Omega = \sum_{i=1}^l H_i^2 + \sum_{\alpha\in\Delta} E_\alpha E_{-\alpha}$.
(b) *Ω operates by the scalar $|\lambda|^2 + 2\langle\lambda,\delta\rangle = |\lambda+\delta|^2 - |\delta|^2$ in an irreducible finite-dimensional representation of $\mathfrak{g}$ of highest weight λ, where δ is half the sum of the positive roots.*
(c) *the scalar by which Ω operates in an irreducible finite-dimensional representation of $\mathfrak{g}$ is nonzero if the representation is not trivial.*

Reference. [K3, Proposition 5.28].

The Casimir element is used in the proof of complete reducibility (Theorem 2.2). The key special case is that V has an irreducible invariant subspace of codimension 1 and dimension > 1. Then $\ker\Omega$ is the required invariant complement.

This completes the discussion of the Casimir element. The third tool used in the proofs of Theorems 2.1 and 2.2 is the theory of Verma modules. Fix a lexicographic ordering, and introduce $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$. For $\nu \in \mathfrak{h}^*$, make $\mathbb{C}$ into a 1-dimensional $U(\mathfrak{h})$ module $\mathbb{C}_\nu$ by defining an action of $\mathfrak{h}$ by $H(z) = \nu(H)z$ for $z \in \mathbb{C}$. Make $\mathbb{C}_\nu$ into a $U(\mathfrak{b})$ module by having $\bigoplus_{\alpha>0}\mathfrak{g}_\alpha$ act by 0. For $\mu \in \mathfrak{h}^*$, define the **Verma module** $V(\mu)$ by

$$V(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\mu-\delta},$$

where δ is half the sum of the positive roots. (The term "$-\delta$" in the definition is the usual convention and has the effect of simplifying calculations with the Weyl group.)

Verma modules have the following elementary properties:

(a) $V(\mu) \neq 0$,
(b) $V(\mu)$ is a universal highest weight module for highest weight modules of $U(\mathfrak{g})$ with highest weight $\mu - \delta$,
(c) each weight space of $V(\mu)$ is finite-dimensional,
(d) $V(\mu)$ has a unique irreducible quotient $L(\mu)$.

(See [K3, §V.3].)

The use of Verma modules allows one to prove the hard step of the Theorem of Highest Weight (Theorem 2.1), which is the existence of an irreducible finite-dimensional representation with given highest weight. In fact, if λ is dominant and algebraically integral, then $L(\lambda+\delta)$ is an irreducible representation with highest weight λ, and all that has to be proved is the finite-dimensionality.

The topic of the third main theorem on representation theory in this section is characters, which we treat for now as formal exponential sums. We continue with $\mathfrak{g}$ as a semisimple Lie algebra, $\mathfrak{h}$ as a Cartan subalgebra, Δ as the set of roots, and $W(\Delta)$ as the Weyl group. Introduce a lexicographic ordering, and let $\alpha_1, \dots, \alpha_l$ be the simple roots.

We regard the set $\mathbb{Z}^{\mathfrak{h}^*}$ of functions from $\mathfrak{h}^*$ to $\mathbb{Z}$ as an abelian group under pointwise addition. We write elements f of $\mathbb{Z}^{\mathfrak{h}^*}$ as $f = \sum_{\lambda\in\mathfrak{h}^*} f(\lambda)e^\lambda$. The **support** of such an f is defined to be the set of $\lambda \in \mathfrak{h}^*$ for which $f(\lambda) \neq 0$. Within $\mathbb{Z}^{\mathfrak{h}^*}$, let $\mathbb{Z}[\mathfrak{h}^*]$ be the subgroup of all f of finite support. The subgroup $\mathbb{Z}[\mathfrak{h}^*]$ has a natural commutative ring structure, which is determined by $e^\lambda e^\mu = e^{\lambda+\mu}$.

We introduce a larger ring, $\mathbb{Z}\langle\mathfrak{h}^*\rangle$. Let

$$Q^+ = \Big\{ \sum_{i=1}^{l} n_i\alpha_i \mid \text{ all } n_i \geq 0,\ n_i \in \mathbb{Z}\Big\}.$$

Then $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ consists of all $f \in \mathbb{Z}^{\mathfrak{h}^*}$ whose support is contained in the union of finitely many sets $\nu_i - Q^+$ with each $\nu_i \in \mathfrak{h}^*$. Then we have inclusions

$$\mathbb{Z}[\mathfrak{h}^*] \subseteq \mathbb{Z}\langle\mathfrak{h}^*\rangle \subseteq \mathbb{Z}^{\mathfrak{h}^*}.$$

Multiplication in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ is given by

$$\Big(\sum_{\lambda\in\mathfrak{h}^*} c_\lambda e^\lambda\Big)\Big(\sum_{\mu\in\mathfrak{h}^*} \tilde{c}_\mu e^\mu\Big) = \sum_{\nu\in\mathfrak{h}^*}\Big(\sum_{\lambda+\mu=\nu} c_\lambda \tilde{c}_\mu\Big)e^\nu.$$

If V is a representation of $\mathfrak{g}$ (not necessarily finite-dimensional), we say that V **has a character** (for present purposes) if V is the direct sum of its weight spaces under $\mathfrak{h}$, i.e., $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$, and if $\dim V_\mu < \infty$ for $\mu \in \mathfrak{h}^*$. In this case the **character** is

$$\mathrm{char}(V) = \sum_{\mu\in\mathfrak{h}^*} (\dim V_\mu)e^\mu$$

as a member of $\mathbb{Z}^{\mathfrak{h}^*}$. This definition is meaningful if V is finite-dimensional or if V is a Verma module.

The **Weyl denominator** is the member $d = e^\delta \prod_{\alpha\in\Delta^+}(1 - e^{-\alpha})$ of $\mathbb{Z}[\mathfrak{h}^*]$. In this expression, δ is again half the sum of the positive roots.

The **Kostant partition function** $\mathcal{P}$ is the function from Q^+ to the nonnegative integers that tells the number of ways, apart from order, that a member of Q^+ can be written as the sum of positive roots. By convention, $\mathcal{P}(0) = 1$. Define $K = \sum_{\gamma\in Q^+} \mathcal{P}(\gamma)e^{-\gamma} \in \mathbb{Z}\langle\mathfrak{h}^*\rangle$.

Lemma. *In the ring $\mathbb{Z}\langle\mathfrak{h}^*\rangle$, $Ke^{-\delta}d = 1$. Hence d^{-1} exists in $\mathbb{Z}\langle\mathfrak{h}^*\rangle$.*

Reference. [K3, Lemma 5.72].

Now we come to the third main theorem.

Theorem 2.7 (Weyl Character Formula)**.** *Let V be an irreducible finite-dimensional representation of the complex semisimple Lie algebra $\mathfrak{g}$ with highest weight λ. Then*

$$\mathrm{char}(V) = d^{-1} \sum_{w\in W(\Delta)} (\det w)e^{w(\lambda+\delta)}.$$

Reference. [K3, Theorem 5.75].

3. Compact Lie Groups and Real Forms of Complex Lie Algebras

This section deals with the structure theory of compact Lie groups and with the existence of compact real forms of complex semisimple Lie algebras. Some references for this material are [He], [K1], [K3], and [V].

Throughout this section, $\mathfrak{g}$ will denote a finite-dimensional complex Lie algebra, and $\mathfrak{g}_0$ will denote a finite-dimensional real Lie algebra. Let $Z_{\mathfrak{g}_0}$ be the center of $\mathfrak{g}_0$.

Let $\operatorname{Aut}\mathfrak{g}_0$ be the automorphism group of $\mathfrak{g}_0$ as a Lie algebra. This is a closed subgroup of $GL(\mathfrak{g}_0)$, hence a Lie subgroup. Its Lie algebra is $\operatorname{Der}\mathfrak{g}_0$. Let $\operatorname{Int}\mathfrak{g}_0$ be the analytic subgroup of $\operatorname{Aut}\mathfrak{g}_0$ with Lie algebra $\operatorname{ad}\mathfrak{g}_0$. If G is a connected Lie group with Lie algebra $\mathfrak{g}_0$, then $\operatorname{Ad}(G)$ is an analytic subgroup of $GL(\mathfrak{g}_0)$ with Lie algebra $\operatorname{ad}\mathfrak{g}_0$, hence equals $\operatorname{Int}\mathfrak{g}_0$. Thus $\operatorname{Int}\mathfrak{g}_0$ provides a way of forming $\operatorname{Ad}(G)$ without using a particular G. It is the group of inner automorphisms of G or $\mathfrak{g}_0$.

We begin with a discussion of real forms. If we regard $\mathfrak{g}$ as a real Lie algebra, then a real Lie subalgebra $\mathfrak{g}_0$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ as vector spaces is called a **real form** of $\mathfrak{g}$. To a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ is associated a **conjugation** of $\mathfrak{g}$, which is the $\mathbb{R}$ linear map that is 1 on $\mathfrak{g}_0$ and -1 on $i\mathfrak{g}_0$. This is an automorphism of $\mathfrak{g}$ as a real Lie algebra. If $\mathfrak{g}_0$ is given, then $\mathfrak{g}_0$ is a real form of its complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. If $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, then $\mathfrak{g}_0$ is semisimple if and only if $\mathfrak{g}$ is semisimple, as a consequence of Cartan's criterion for semisimplicity (Theorem 1.1).

Examples.

1) $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{su}(n)$, and $\mathfrak{su}(p,q)$ are real forms of $\mathfrak{sl}(n,\mathbb{C})$. Here $\mathfrak{su}(n)$ is the Lie algebra of n-by-n skew-Hermitian matrices of trace 0, and $\mathfrak{su}(p,q)$ consists of matrices $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ of trace 0 in which A and C are skew-Hermitian.

2) $\mathfrak{so}(n)$ is a real form of $\mathfrak{so}(n,\mathbb{C})$. Here $\mathfrak{so}(n)$ is the Lie algebra of n-by-n real skew-symmetric matrices.

3) $\mathfrak{so}(p,q)$ is isomorphic to a real form of $\mathfrak{so}(p+q,\mathbb{C})$ under conjugation by the block diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. Here $\mathfrak{so}(p,q)$ consists of real matrices $\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$ in which A and C are skew-symmetric. When we complexify and then conjugate by $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, we obtain $\mathfrak{so}(p+q,\mathbb{C})$.

4) $\mathfrak{sp}(n,\mathbb{R})$ and $\mathfrak{sp}(n,\mathbb{C}) \cap \mathfrak{u}(2n)$ are real forms of $\mathfrak{sp}(n,\mathbb{C})$.

The Lie algebra $\mathfrak{g}_0$ is said to be **reductive** if to each ideal $\mathfrak{a}_0$ in $\mathfrak{g}_0$ corresponds an ideal $\mathfrak{b}_0$ in $\mathfrak{g}_0$ with $\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{b}_0$.

Theorem 3.1. *The Lie algebra $\mathfrak{g}_0$ is reductive if and only if $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus Z_{\mathfrak{g}_0}$ with $[\mathfrak{g}_0, \mathfrak{g}_0]$ semisimple and $Z_{\mathfrak{g}_0}$ abelian.*

Reference. [K3, Corollary 1.53].

Now we consider the Lie algebra of a compact Lie group.

Theorem 3.2. *If G is a compact Lie group and $\mathfrak{g}_0$ is its Lie algebra, then*

(a) $\operatorname{Int}\mathfrak{g}_0$ *is compact.*
(b) $\mathfrak{g}_0$ *is reductive.*
(c) *the Killing form of $\mathfrak{g}_0$ is negative semidefinite.*

Furthermore let Z_G be the center of G, and let G_{ss} be the analytic subgroup of G with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. Then

(d) *G_{ss} has finite center.*
(e) *$(Z_G)_0$ and G_{ss} are closed subgroups.*
(f) *G is the commuting product $G = (Z_G)_0 G_{ss}$.*

Reference. [K3, §IV.4].

REMARKS. Conclusions (b) and (c) use the existence of a G invariant inner product on $\mathfrak{g}_0$, which is constructed using Haar measure on G. Conclusion (d) uses that G may be regarded as a Lie group of matrices; this fact is a consequence of the Peter-Weyl Theorem, which we do not review. See [K3, §IV.3].

Lemma. *If $\mathfrak{g}_0$ is semisimple, then* $\operatorname{Der}\mathfrak{g}_0 = \operatorname{ad}\mathfrak{g}_0$. *Hence* $\operatorname{Int}\mathfrak{g}_0 = (\operatorname{Aut}\mathfrak{g}_0)_0$, *and* $\operatorname{Int}\mathfrak{g}_0$ *is a closed subgroup of* $GL(\mathfrak{g}_0)$.

REFERENCE. [K3, Proposition 1.98].

REMARK. Since $\operatorname{Int}\mathfrak{g}_0$ is the group of inner automorphisms of $\mathfrak{g}_0$ and since $\operatorname{Int}\mathfrak{g}_0$ has Lie algebra $\operatorname{ad}\mathfrak{g}_0$, it is helpful to think of this lemma as saying that every derivation is inner.

Theorem 3.3. *If the Killing form of $\mathfrak{g}_0$ is negative definite, then* $\operatorname{Int}\mathfrak{g}_0$ *is compact.*

REFERENCE. [K3, Proposition 4.27].

Next we discuss compact real forms.

Theorem 3.4. *If $\mathfrak{g}_0$ is semisimple, then the following conditions are equivalent:*
(a) $\mathfrak{g}_0$ *is the Lie algebra of some compact Lie group.*
(b) $\operatorname{Int}\mathfrak{g}_0$ *is compact.*
(c) *the Killing form of $\mathfrak{g}_0$ is negative definite.*

PROOF. If G is compact connected with Lie algebra $\mathfrak{g}_0$, then $\operatorname{Ad}(G)$ is compact; hence (a) implies (b). Conversely if (b) holds, then $\operatorname{Int}\mathfrak{g}_0$ is a compact Lie group with Lie algebra $\operatorname{ad}\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is semisimple, $\operatorname{ad}\mathfrak{g}_0$ is isomorphic to $\mathfrak{g}_0$; thus (b) implies (a). If (b) holds, then the Killing form is negative semidefinite by Theorem 3.2, and it must be negative definite by Cartan's criterion for semisimplicity (Theorem 1.1). Thus (b) implies (c). Conversely (c) implies (b) by Theorem 3.3.

Let $\mathfrak{g}$ be semisimple. A real form $\mathfrak{g}_0$ of $\mathfrak{g}$ is said to be **compact** if the equivalent conditions of Theorem 3.4 hold. Here are some examples.

Examples. $\mathfrak{su}(n)$ is a compact real form of $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n)$ is a compact real form of $\mathfrak{so}(n,\mathbb{C})$, and $\mathfrak{sp}(n,\mathbb{C})\cap\mathfrak{u}(2n)$ is a compact real form of $\mathfrak{sp}(n,\mathbb{C})$.

Theorem 3.5. *Each complex semisimple Lie algebra has a compact real form.*

REFERENCE. [K3, Theorem 6.11].

This result is fundamental. The first step in the proof is to extend the vector space isomorphism $\varphi = -1$ of $\mathfrak{h}$ to an automorphism $\tilde\varphi$ of $\mathfrak{g}$, using the Isomorphism Theorem (Theorem 1.6). Then $\tilde\varphi$ is used to adjust the structural constants to produce a real form for which the Killing form is negative definite. Application of Theorem 3.4 completes the argument.

The next topic is maximal tori. The setting is that G is a compact connected Lie group, $\mathfrak{g}_0$ is its Lie algebra, $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$, and B is the negative of any $\operatorname{Ad}(G)$ invariant inner product on $\mathfrak{g}_0$. The **maximal tori** in G are defined to be the subgroups maximal with respect to the property of being compact connected abelian. The theorem below lists the first facts about maximal tori.

Theorem 3.6. *If G is a compact connected Lie group, then*

(a) *the maximal tori in G are exactly the analytic subgroups corresponding to the maximal abelian subalgebras of $\mathfrak{g}_0$.*

(b) *any two maximal abelian subalgebras of $\mathfrak{g}_0$ are conjugate via* $\mathrm{Ad}(G)$ *and hence any two maximal tori in G are conjugate via G.*

REFERENCE. [K3, Proposition 4.30 and Theorem 4.34].

Here are some standard examples of maximal tori.

Examples.

1) Let $G = SU(n)$, the special unitary group. The complexified Lie algebra is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. A maximal torus, its Lie algebra, and its complexified Lie algebra are

$$\begin{aligned} T &= \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \\ \mathfrak{t}_0 &= \mathrm{diag}(i\theta_1, \dots, i\theta_n) \\ \mathfrak{t} &= \text{standard Cartan subalgebra of } \mathfrak{sl}(n, \mathbb{C}). \end{aligned}$$

2) Let $G = SO(2n + 1)$, the rotation group. The complexified Lie algebra is $\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C})$. A maximal torus and its complexified Lie algebra are

$$\begin{aligned} &T \quad \text{from 2-by-2 blocks } \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix} \text{ and a} \\ &\quad\ \ \text{single 1-by-1 block } (1) \\ &\mathfrak{t} = \text{standard Cartan subalgebra of } \mathfrak{so}(2n + 1, \mathbb{C}). \end{aligned}$$

3) Let $G = Sp(n, \mathbb{C}) \cap U(2n)$. Here $Sp(n, \mathbb{C}) = \{x \in GL(2n, \mathbb{C}) \mid x^t J x = J\}$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ as earlier. The complexified Lie algebra of G is $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$. A maximal torus and its complexified Lie algebra are

$$\begin{aligned} T &= \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}, e^{-i\theta_1}, \dots, e^{-i\theta_n}) \\ \mathfrak{t} &= \text{standard Cartan subalgebra of } \mathfrak{sp}(n, \mathbb{C}). \end{aligned}$$

4) Let $G = SO(2n)$, the rotation group. The complexified Lie algebra is $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$.

$$\begin{aligned} &T \quad \text{from 2-by-2 blocks } \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix} \\ &\mathfrak{t} = \text{standard Cartan subalgebra of } \mathfrak{so}(2n, \mathbb{C}). \end{aligned}$$

The theory of Cartan subalgebras for the complex semisimple case extends to a complex reductive Lie algebras $\mathfrak{g}$ by just saying that the center of $\mathfrak{g}$ is to be adjoined to a Cartan subalgebra of the semisimple part of $\mathfrak{g}$.

Now let us extend the theory of Cartan subalgebras from the complex reductive case to the real reductive case. If $\mathfrak{g}_0$ is a real reductive Lie algebra, we call a Lie subalgebra of $\mathfrak{g}_0$ a **Cartan subalgebra** if its complexification is a Cartan subalgebra of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$. Using condition (c) in the definition of Cartan subalgebra for the complex semisimple Lie algebra, we readily see that if $\mathfrak{g}_0$ is the Lie algebra of a compact connected Lie group G and if $\mathfrak{t}_0$ is a maximal abelian subspace of $\mathfrak{g}_0$, then $\mathfrak{t}_0$ is a Cartan subalgebra. In this setting, we can form a root-space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Here $\mathfrak{g} = Z_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{t} = Z_{\mathfrak{g}} \oplus (\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}])$, and the root spaces $\mathfrak{g}_\alpha$ lie in $[\mathfrak{g}, \mathfrak{g}]$. Moreover, each root is the complexified differential of a multiplicative character ξ_α of the maximal torus T that corresponds to $\mathfrak{t}_0$, with

$$\operatorname{Ad}(t)X = \xi_\alpha(t)X \qquad \text{for } X \in \mathfrak{g}_\alpha.$$

The next results concern centralizers of tori. These results give the main control over connectedness of subgroups of semisimple and reductive groups.

Theorem 3.7. *If G is a compact connected Lie group and T is a maximal torus, then each element of G is conjugate to a member of T.*

Reference. [K3, Theorem 4.36].

This is a deep theorem. For $SU(n)$, it just amounts to the Spectral Theorem, but it becomes progressively more complicated for more complicated G. We list three immediate consequences.

Corollary.

(a) *Every element of a compact connected Lie group G lies in some maximal torus.*
(b) *The center Z_G of a compact connected Lie group lies in every maximal torus.*
(c) *For any compact connected Lie group G, the exponential map is onto G.*

With a supplementary argument and Theorem 3.7, we obtain

Theorem 3.8. *Let G be a compact connected Lie group, and let S be a torus of G. If g in G centralizes S, then there is a torus S' in G containing both S and g.*

Reference. [K3, Theorem 4.50].

This theorem is normally applied in either of the two forms in the following corollary.

Corollary.

(a) *In a compact connected Lie group, the centralizer of a torus is connected.*
(b) *A maximal torus in a compact connected Lie group is equal to its own centralizer.*

Let us introduce Weyl groups in this context. The notation is unchanged: G is compact connected, $\mathfrak{g}_0$ is the Lie algebra of G, $\mathfrak{g}$ is the complexification, T is a maximal torus, $\mathfrak{t}_0$ is the Lie algebra of T, $\mathfrak{t}$ is the complexification, $\Delta(\mathfrak{g}, \mathfrak{t})$ is the set of roots, and B is the negative of a G invariant inner product on $\mathfrak{g}_0$. Define $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}_0$. Roots are real on $\mathfrak{t}_{\mathbb{R}}$, hence are in $\mathfrak{t}_{\mathbb{R}}^*$. The form B, when extended to be complex bilinear, is positive definite on $\mathfrak{t}_{\mathbb{R}}$, yielding an inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{t}_{\mathbb{R}}^*$.

Let the root reflection s_α be defined on $\mathfrak{t}_{\mathbb{R}}^*$ by $s_\alpha(\lambda) = \lambda - \dfrac{2\langle \lambda, \alpha \rangle}{|\alpha|^2}\,\alpha$. The Weyl group $W(\Delta(\mathfrak{g}, \mathfrak{t}))$ is the group generated by all s_α for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$. This is a finite group.

We define $W(G, T)$ as the quotient of normalizer by centralizer

$$W(G, T) = N_G(T)/Z_G(T) = N_G(T)/T.$$

This also is a finite group. It follows from Theorems 3.7 and 3.6b that the conjugacy classes in G are parametrized by $T/W(G, T)$. (See [K3, Proposition 4.53].)

Theorem 3.9. *The group $W(G, T)$, when considered as acting on $\mathfrak{t}^*_{\mathbb{R}}$, coincides with $W(\Delta(\mathfrak{g}, \mathfrak{t}))$.*

Reference. [K3, Theorem 4.54].

Continuing with notation as above, we work with two notions of integrality. It is easy to see that the following two conditions on a member λ of $\mathfrak{t}^*$ are equivalent:

(1) Whenever $H \in \mathfrak{t}_0$ satisfies $\exp H = 1$, then $\lambda(H)$ is in $2\pi i\mathbb{Z}$.
(2) There is a multiplicative character ξ_λ of T with $\xi_\lambda(\exp H) = e^{\lambda(H)}$ for all $H \in \mathfrak{t}_0$.

When (1) and (2) hold, λ is said to be **analytically integral**. As before, we say that λ is **algebraically integral** if $\dfrac{2\langle \lambda, \alpha\rangle}{|\alpha|^2}$ is in $\mathbb{Z}$ for all $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$.

Theorem 3.10. *Analytic and algebraic integrality have the following eight properties:*

(a) *Weights of finite-dimensional representations of G are analytically integral. In particular, every root is analytically integral.*
(b) *Analytically integral implies algebraically integral.*
(c) *Fix a simple system of roots $\{\alpha_1, \dots, \alpha_l\}$. Then $\lambda \in \mathfrak{t}^*$ is algebraically integral if and only if $2\langle\lambda, \alpha_i\rangle/|\alpha_i|^2$ is in $\mathbb{Z}$ for each simple root α_i.*
(d) *If $\tilde{G}$ is a finite covering group of G, then the index of the group of analytically integral forms for G in the group of analytically integral forms for $\tilde{G}$ equals the order of the kernel of the covering homomorphism $\tilde{G} \to G$.*
(e) *The subgroup of $\mathbb{Z}$ combinations of roots in $\mathfrak{t}^*_{\mathbb{R}}$ is contained in the lattice of analytically integral forms, which in turn is contained in the subgroup of algebraically integral forms. If G is semisimple, all three subgroups are lattices.*
(f) *If G is semisimple, then the index of the lattice of $\mathbb{Z}$ combinations of roots in the lattice of algebraically integral forms is exactly the determinant of the Cartan matrix.*
(g) *If G is semisimple and Z_G is trivial, then every analytically integral form is a $\mathbb{Z}$ combination of roots.*
(h) *If G is simply connected and semisimple, then algebraically integral implies analytically integral.*

Reference. [K3, §§IV.7 and V.8].

Remarks. In the semisimple case, conclusion (e) identifies containments among three lattices in $\mathfrak{t}^*_{\mathbb{R}}$, and (f) says that the index of the smallest in the largest is the determinant of the Cartan matrix. Conclusions (g) and (h) give circumstances under which the middle lattice is equal to the smallest or the largest. The proof of (h) uses the existence result in the Theorem of the Highest Weight.

Theorem 3.11 (Weyl's Theorem)**.** *If G is a compact semisimple Lie group, then the fundamental group of G is finite. Consequently the universal covering group of G is compact.*

REFERENCE. [K3, Theorem 4.69].

Combining Weyl's Theorem with Theorem 3.10, we obtain the following consequence.

Corollary. *In a compact semisimple Lie group G,*

(a) *the order of the fundamental group of G equals the index of the group of analytically integral forms for G in the group of algebraically integral forms.*
(b) *if G is simply connected, then the order of the center Z_G of G equals the determinant of the Cartan matrix.*

Let us now rephrase the results about representations of complex semisimple Lie algebras as results about compact connected Lie groups. (See [K3, §V.8].)

Theorem 3.12 (Theorem of the Highest Weight)**.** *Let G be a compact connected Lie group with complexified Lie algebra $\mathfrak{g}$, let T be a maximal torus with complexified Lie algebra $\mathfrak{t}$, and let $\Delta^+(\mathfrak{g},\mathfrak{t})$ be a positive system for the roots. Apart from equivalence the irreducible finite-dimensional representations Φ of G stand in one-one correspondence with the dominant analytically integral linear functionals λ on $\mathfrak{t}$, the correspondence being that λ is the highest weight of Φ.*

In the context of representations of the compact connected group G, we can regard characters $\operatorname{char}(V) = \sum (\dim V_\lambda)e^\lambda$ as functions on $\mathfrak{t}_0$. The algebraic theory gives

$$d\operatorname{char}(V) = \sum_{w\in\Delta(\mathfrak{g},\mathfrak{t})} (\det w)e^{w(\lambda+\delta)}$$

in $\mathbb{Z}[\mathfrak{t}^*]$ for the semisimple case.

We can pass from the algebraic result in $\mathbb{Z}[\mathfrak{t}^*]$ to the group case for G semisimple by using the evaluation homormorphism at each point of $\mathfrak{t}_0$ and addressing analytic integrality. Then we can extend the group result to general compact connected G. One shows that the element $\delta \in \mathfrak{t}^*$ (half the sum of the positive roots) has $2\langle\delta,\alpha_i\rangle/|\alpha_i|^2 = 1$ for simple α_i, hence is algebraically integral. Nevertheless, δ is not always analytically integral; it is not analytically integral in $SO(3)$, for example. A sufficient compensation for this failure is that $\delta - w\delta$ is always analytically integral for all w. Consequently we are able to obtain the following group version of the Weyl Character Formula.

Theorem 3.13 (Weyl Character Formula)**.** *Let G be a compact connected Lie group, let T be a maximal torus, let $\Delta^+ = \Delta^+(\mathfrak{g},\mathfrak{t})$ be a positive system for the roots, and let $\lambda \in \mathfrak{t}^*$ be analytically integral and dominant. Then the character χ_λ of the irreducible finite-dimensional representation of G with highest weight λ is given by*

$$\chi_\lambda = \frac{\sum_{w\in W}(\det w)\xi_{w(\lambda+\delta)-\delta}(t)}{\prod_{\alpha\in\Delta^+}(1-\xi_{-\alpha}(t))}$$

at every $t \in T$ where no ξ_α takes the value 1 on t. If G is simply connected, then this formula can be rewritten as

$$\chi_\lambda = \frac{\sum_{w\in W}(\det w)\xi_{w(\lambda+\delta)}(t)}{\xi_\delta(t)\prod_{\alpha\in\Delta^+}(1-\xi_{-\alpha}(t))}.$$

Before concluding the treatment of compact groups, let us mention that much of the theory for compact connected Lie groups can be obtained directly, without first addressing complex semisimple Lie algebras. Weyl carried out such a program, using integration as the tool. Here is the formula that Weyl used.

Theorem 3.14 (Weyl Integration Formula)**.** *Let T be a maximal torus of the compact connected Lie group G, and let invariant measures on G, T, and G/T be normalized so that*

$$\int_G f(x)\,dx = \int_{G/T}\Big[\int_T f(xt)\,dt\Big]\,d(xT)$$

for all continuous f on G. Then every Borel function $F \geq 0$ on G has

$$\int_G F(x)\,dx = \frac{1}{|W(G,T)|}\int_T\Big[\int_{G/T} F(gtg^{-1})\,d(gT)\Big]\,|D(t)|^2\,dt,$$

where

$$|D(t)|^2 = \prod_{\alpha\in\Delta^+} |1-\xi_\alpha(t^{-1})|^2.$$

Reference. [K3, Theorem 8.60].

4. Structure Theory of Noncompact Semisimple Groups

This section deals with the structure theory of noncompact semisimple Lie groups and with the definition and first properties of reductive Lie groups. Some references for this material are [He], [K1], [K3], and [W].

The theory begins with the development of Cartan involutions. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, and let B be the Killing form. (Later we shall allow other forms in place of the Killing form.) A source of many examples of real semisimple Lie algebras is as follows.

Theorem 4.1. *If $\mathfrak{g}_0$ is a real Lie algebra of real or complex or quaternion matrices closed under conjugate transpose, then $\mathfrak{g}_0$ is reductive. If also $Z_{\mathfrak{g}_0} = 0$, then $\mathfrak{g}_0$ is semisimple.*

Reference. [K3, Proposition 1.56].

Examples. The following examples are classical Lie algebras that satisfy the hypotheses of Theorem 4.1 for all n, p, and q. For appropriate values of n, p, and q, these examples are semisimple.

1) Compact Lie algebras: $\mathfrak{su}(n)$, $\mathfrak{so}(n)$, and $\mathfrak{sp}(n,\mathbb{C})\cap\mathfrak{u}(2n)\cong\mathfrak{sp}(n)$.

2) Complex Lie algebras: $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, and $\mathfrak{sp}(n,\mathbb{C})$.

3) Other Lie algebras: $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{H})$, $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{so}(p,q)$, $\mathfrak{su}(p,q)$, $\mathfrak{sp}(p,q)$, and $\mathfrak{so}^*(2n)$. Here $\mathfrak{sl}(n,\mathbb{H})$ refers to quaternion matrices for which the real part of the trace is 0, and $\mathfrak{sp}(p,q)$ refers to quaternion matrices preserving a Hermitian form of signature (p,q).

An involution θ of $\mathfrak{g}_0$ (understood to respect brackets) such that the symmetric bilinear form

$$B_\theta(X,Y) = -B(X,\theta Y)$$

is positive definite is called a **Cartan involution** of $\mathfrak{g}_0$. Correspondingly there is a **Cartan decomposition** of $\mathfrak{g}_0$ given by

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

The subspaces $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are understood to be the $+1$ and -1 eigenspaces of θ; they satisfy the bracket relations

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0.$$

Moreover, B is negative on $\mathfrak{k}_0$, B is positive on $\mathfrak{p}_0$, and $B(\mathfrak{k}_0, \mathfrak{p}_0) = 0$.

Examples.

1) If $\mathfrak{g}_0$ is as in the list of examples above, then θ can be taken to be negative conjugate transpose.

2) Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{u}_0$ be a compact real form of $\mathfrak{g}$, and let τ be the corresponding conjugation of $\mathfrak{g}$. If $\mathfrak{g}$ is regarded as a real Lie algebra, then τ is a Cartan involution of $\mathfrak{g}$.

The main tool for handling Cartan involutions is Theorem 4.2 below. This is a result of Berger that improves on the original result of Cartan.

Theorem 4.2. *Let θ be a Cartan involution of $\mathfrak{g}_0$, and let σ be any involution. Then there exists φ in $\operatorname{Int}\mathfrak{g}_0$ such that $\varphi\theta\varphi^{-1}$ commutes with σ.*

REFERENCE. [K3, Theorem 6.16].

Corollary.

(a) *$\mathfrak{g}_0$ has a Cartan involution.*
(b) *Any two Cartan involutions of $\mathfrak{g}_0$ are conjugate via $\operatorname{Int}\mathfrak{g}_0$.*
(c) *If $\mathfrak{g}$ is a complex semisimple Lie algebra, then any two compact real forms of $\mathfrak{g}$ are conjugate via $\operatorname{Int}\mathfrak{g}$.*
(d) *If $\mathfrak{g}$ is a complex semisimple Lie algebra and is considered as a real Lie algebra, then the only Cartan involutions of $\mathfrak{g}$ are the conjugations with respect to the compact real forms of $\mathfrak{g}$.*

REFERENCE. [K3, §VI.2].

SKETCH OF PROOF. For (a), Theorem 4.2 is applied to $\mathfrak{g}$ made real, using θ from a compact real form and σ from conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Conclusion (b) is immediate, and (c) is a special case of (b). Conclusion (d) follows from (b) and the fact that such a conjugation exists (Theorem 3.5).

If $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$, then $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$. Conversely Theorem 3.3 shows that if $\mathfrak{h}_0$ and $\mathfrak{q}_0$ are the $+1$ and -1 eigenspaces of an involution σ, then σ is a Cartan involution if the real form $\mathfrak{h}_0 \oplus i\mathfrak{q}_0$ of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ is compact.

These considerations allow B to be generalized a little. Fix an involution θ of $\mathfrak{g}_0$, and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the eigenspace decomposition relative to θ. We suppose that B is any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}_0$ with $B(\theta X, \theta Y) = B(X, Y)$ such that $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite. Then B is negative definite on $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$, and it follows that $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is compact. Consequently θ is a Cartan involution. In this setting we allow B to be used in place of the Killing form.

Notice in this case that B is negative definite on a maximal abelian subspace of $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$, hence positive definite on the real subspace of a Cartan subalgebra of $(\mathfrak{g}_0)^{\mathbb{C}}$ where roots are real-valued. Therefore B has the correct "sign" on $(\mathfrak{g}_0)^{\mathbb{C}}$ for the theory of complex semisimple Lie algebras to be applicable.

By a **semisimple Lie group**, we mean a connected Lie group whose Lie algebra is semisimple. The next theorem gives the **global Cartan decomposition** of a semisimple Lie group.

Theorem 4.3. *Let G be a semisimple Lie group, let θ be a Cartan involution of its Lie algebra $\mathfrak{g}_0$, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}_0$. Then*

(a) *there exists a Lie group automorphism Θ of G with differential θ, and Θ has $\Theta^2 = 1$.*
(b) *the subgroup of G fixed by Θ is K.*
(c) *the mapping $K \times \mathfrak{p}_0 \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism onto.*
(d) *K is closed.*
(e) *K contains the center Z of G.*
(f) *K is compact if and only if Z is finite.*
(g) *when Z is finite, K is a maximal compact subgroup of G.*

REFERENCE. [K3, Theorem 6.31].

Example. When G is an analytic group of matrices and θ is negative conjugate transpose, Θ is conjugate transpose inverse. The content of (c) is that G is stable under the polar decomposition of matrices. Thus (c) of the theorem may be regarded as a generalization of the polar decomposition to all semisimple Lie groups.

This completes the discussion of Cartan involutions. For most of the remainder of this section, we shall use the following notation. Let G be a semisimple Lie group, let $\mathfrak{g}_0$ be its Lie algebra, let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$, let θ be a Cartan involution of $\mathfrak{g}_0$, and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Let B as above be a θ invariant nondegenerate symmetric bilinear form on $\mathfrak{g}_0$ such that B_θ is positive definite.

The next topic will be restricted roots and the Iwasawa decomposition. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$. **Restricted roots** are the nonzero $\lambda \in \mathfrak{a}_0^*$ such that the space $(\mathfrak{g}_0)_\lambda$ defined as

$$\{X \in \mathfrak{g}_0 \mid (\operatorname{ad} H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}_0\}$$

is nonzero. Let Σ be the set of restricted roots. Define $\mathfrak{m}_0 = Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$. Restricted roots and the corresponding restricted-root spaces have the following elementary properties:

(a) $\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus \bigoplus_{\lambda \in \Sigma} (\mathfrak{g}_0)_\lambda$,
(b) $[(\mathfrak{g}_0)_\lambda, (\mathfrak{g}_0)_\mu] \subseteq (\mathfrak{g}_0)_{\lambda+\mu}$,
(c) $\theta(\mathfrak{g}_0)_\lambda = (\mathfrak{g}_0)_{-\lambda}$,
(d) Σ is a root system in $\mathfrak{a}_0^*$.

Introduce a lexicographic ordering in $\mathfrak{a}_0^*$, and define

$$\Sigma^+ = \{\text{positive restricted roots}\}$$

$$\mathfrak{n}_0 = \bigoplus_{\lambda\in\Sigma^+} (\mathfrak{g}_0)_\lambda.$$

The subspace $\mathfrak{n}_0$ of $\mathfrak{g}_0$ is a nilpotent Lie subalgebra.

Theorem 4.4 (Iwasawa decomposition of Lie algebra). *The semisimple Lie algebra $\mathfrak{g}_0$ is a vector-space direct sum $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$. Here $\mathfrak{a}_0$ is abelian, $\mathfrak{n}_0$ is nilpotent, $\mathfrak{a}_0\oplus\mathfrak{n}_0$ is a solvable Lie subalgebra of $\mathfrak{g}_0$, and $\mathfrak{a}_0\oplus\mathfrak{n}_0$ has $[\mathfrak{a}_0\oplus\mathfrak{n}_0, \mathfrak{a}_0\oplus\mathfrak{n}_0] = \mathfrak{n}_0$.*

Reference. [K3, Proposition 6.43].

Theorem 4.5 (Iwasawa decomposition of Lie group). *Let G be a semisimple group, let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ be an Iwasawa decomposition of the Lie algebra $\mathfrak{g}_0$ of G, and let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$. Then the multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism onto. The groups A and N are simply connected.*

Reference. [K3, Theorem 6.46].

Roots and restricted roots are related to each other. If $\mathfrak{t}_0$ is a maximal abelian subspace of $\mathfrak{g}_0$, then $\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ (see [K3, Proposition 6.47]). Roots are real-valued on $\mathfrak{a}_0$ and imaginary-valued on $\mathfrak{t}_0$. The nonzero restrictions to $\mathfrak{a}_0$ of the roots turn out to be the restricted roots (see [K3, §VI.4]). Roots and restricted roots can be ordered compatibly by taking $\mathfrak{a}_0$ before $i\mathfrak{t}_0$.

The next theorem describes the effect of altering the choices that have been made in obtaining the Iwasawa decomposition.

Theorem 4.6.

(a) *If $\mathfrak{a}_0$ and $\mathfrak{a}_0'$ are two maximal abelian subspaces of $\mathfrak{p}_0$, then there is a member k of K with $\operatorname{Ad}(k)\mathfrak{a}_0' = \mathfrak{a}_0$. Consequently the space $\mathfrak{p}_0$ satisfies $\mathfrak{p}_0 = \bigcup_{k\in K} \operatorname{Ad}(k)\mathfrak{a}_0$.*

(b) *Any two choices of $\mathfrak{n}_0$ are conjugate by Ad of a member of $N_K(\mathfrak{a}_0)$.*

(c) *Define $W(G, A) = N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0)$. The Lie algebra of the normalizer $N_K(\mathfrak{a}_0)$ is $\mathfrak{m}_0$, and therefore $W(G, A)$ is a finite group.*

(d) *$W(G, A)$ coincides with $W(\Sigma)$.*

Reference. [K3, §VI.5].

Remarks. Already we know from the Corollary to Theorem 4.2 that any two Cartan decompositions of $\mathfrak{g}_0$ are conjugate via $\operatorname{Int}\mathfrak{g}_0$. Therefore any two choices of K are conjugate in G. Conclusion (a) of the theorem says that with K fixed, any two choices of $\mathfrak{a}_0$ are conjugate, and conclusion (b) says that with K and $\mathfrak{a}_0$ fixed, any two choices of $\mathfrak{n}_0$ are conjugate. Therefore any two Iwasawa decompositions are conjugate.

Now let us study Cartan subalgebras and subgroups. We know that $\mathfrak{g}_0$ always has a Cartan subalgebra. Namely if $\mathfrak{t}_0$ is any maximal abelian subspace of $\mathfrak{m}_0$, then $\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. However, Cartan subalgebras are not necessarily unique up to conjugacy, as the following example shows.

Example. The Lie algebra $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$ has two Cartan subalgebras nonconjugate via $\operatorname{Int}\mathfrak{g}_0$, namely all $\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ and all $\begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}$. Every Cartan subalgebra of $\mathfrak{g}_0$ is conjugate via $\operatorname{Int}\mathfrak{g}_0$ to one of these.

In a complex Lie algebra $\mathfrak{g}$, any two Cartan subalgebras are conjugate via $\operatorname{Int}\mathfrak{g}$. Therefore, despite the nonconjugacy, any two Cartan subalgebras of $\mathfrak{g}_0$ have the same dimension. This dimension is called the **rank** of $\mathfrak{g}_0$.

Let us mention some properties of Cartan subalgebras of $\mathfrak{g}_0$ (see [K3, §VI.6]). Any Cartan subalgebra is conjugate via $\operatorname{Int}\mathfrak{g}_0$ to a θ stable Cartan subalgebra.

If $\mathfrak{h}_0$ is a θ stable Cartan subalgebra, we can decompose $\mathfrak{h}_0$ according to $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ as $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ with $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$. It is appropriate to think of $\mathfrak{t}_0$ as the compact part of $\mathfrak{h}_0$ and $\mathfrak{a}_0$ as the noncompact part. Define $\mathfrak{h}_0$ to be **maximally compact** if its compact part has maximal dimension among all θ stable Cartan subalgebras, or to be **maximally noncompact** if its noncompact part has maximal dimension. The Cartan subalgebra $\mathfrak{h}_0$ constructed after the Iwasawa decomposition is maximally noncompact. If $\mathfrak{t}_0$ is a maximal abelian subspace of $\mathfrak{k}_0$, then $\mathfrak{h}_0 = Z_{\mathfrak{g}_0}(\mathfrak{t}_0)$ is maximally compact.

Among θ stable Cartan subalgebras $\mathfrak{h}_0$ of $\mathfrak{g}_0$, the maximally noncompact ones are all conjugate via K, and the maximally compact ones are all conjugate via K. Hence the constructions in the previous paragraph yield all maximally compact and maximally noncompact θ stable Cartan subalgebras.

Up to conjugacy by $\operatorname{Int}\mathfrak{g}_0$, there are only finitely many Cartan subalgebras of $\mathfrak{g}_0$. In fact, any θ stable Cartan subalgebra, up to conjugacy, can be transformed into any other θ stable Cartan subalgebra by a sequence of **Cayley transforms**, which change a Cartan subalgebra of $\mathfrak{g}_0$ only within a subalgebra $\mathfrak{sl}(2,\mathbb{R})$. Within the $\mathfrak{sl}(2,\mathbb{R})$, the change is essentially the change between the two types in the example above. The relevant $\mathfrak{sl}(2,\mathbb{R})$'s for the Cayley transforms are the ones corresponding to particular kinds of roots.

By definition a **Cartan subgroup** of G is the centralizer in G of a Cartan subalgebra of $\mathfrak{g}_0$. In order to analyze noncompact semisimple groups, one wants an analog of the result Theorem 3.7 in the compact case that every element is conjugate to a member of a maximal torus.

For this purpose we introduce the regular elements of G. Let l be the common dimension of all Cartan subalgebras of $\mathfrak{g}_0$, and write

$$\det((\lambda+1)1_n - \operatorname{Ad}(x)) = \lambda^n + \sum_{j=0}^{n-1} D_j(x)\lambda^j.$$

We call $x \in G$ **regular** if $D_l(x) \neq 0$. Let G' be the set of all regular elements in G.

Theorem 4.7. *Let $(\mathfrak{h}_1)_0, \dots, (\mathfrak{h}_r)_0$ be a maximal set of nonconjugate θ stable Cartan subalgebras of $\mathfrak{g}_0$, and let $H_1, \dots, H_r$ be the corresponding Cartan subgroups of G. Then*

(a) $G' \subseteq \bigcup_{i=1}^r \bigcup_{x\in G} xH_ix^{-1}$.
(b) *each member of G' lies in just one Cartan subgroup of G.*

Reference. [K3, Theorem 7.108].

REMARKS. By the theorem the regular elements are conjugate to members of Cartan subgroups. This fact turns out to be good enough to give an analog of the Weyl Integration Formula for noncompact semisimple groups. We omit the details.

This completes our discussion of Cartan subalgebras and Cartan subgroups. We turn now to the topic of parabolic subalgebras and parabolic subgroups. The notation remains unchanged.

First we introduce two subgroups M and N^-. The group N^- is often called $\overline{N}$ in the literature. The subgroup M of G is defined by $M = Z_K(\mathfrak{a}_0)$. Its Lie algebra is $\mathfrak{m}_0 = Z_{\mathfrak{k}_0}(\alpha_0)$, and M normalizes each restricted-root space $(\mathfrak{g}_0)_\lambda$.

It follows from the Iwasawa decomposition (Theorem 4.5) that MAN is a closed subgroup of G. It and its conjugates in G are called **minimal parabolic** subgroups. Its Lie algebra is $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$, a **minimal parabolic** subalgebra of $\mathfrak{g}_0$.

Let $\mathfrak{n}_0^- = \bigoplus_{\lambda\in\Sigma^+}(\mathfrak{g}_0)_{-\lambda} = \theta\mathfrak{n}_0$, and let $N^- = \Theta N$ be the corresponding analytic subgroup of G. Here is a handy integral formula used in analysis on G; for $g = SL(2,\mathbb{R})$, it amounts to an arctangent substitution for passing from the circle to the line.

Theorem 4.8. *Write elements of $G = KAN$ as $g = \kappa e^{H(g)}n$. Let 2ρ be the sum of the members of Σ^+ with multiplicities counted. Then there exists a normalization of Haar measures such that*

$$\int_K f(k)\,dk = \int_{N^-} f(\kappa(\bar{n}))e^{-2\rho H(\bar{n})}\,d\bar{n}$$

for all continuous f on K that are right invariant under M.

REFERENCE. [K3, Proposition 8.46].

The next theorem gives the double-coset decomposition of G relative to the subgroup MAN.

Theorem 4.9 (Bruhat decomposition)**.** *Let $\{\tilde{w}\}$ be a set of representatives in K for the members w of $W(G,A)$, and let $[\tilde{w}]$ be the image of $\tilde{w}$ in $W(G,A)$. Then*

$$G = \bigcup_{[\tilde{w}]\in W(G,A)} MAN\tilde{w}MAN$$

disjointly.

REFERENCE. [K3, Theorem 7.40].

The existence half of the following decomposition is an immediate consequence of the global Cartan decomposition (Theorem 4.3) and the conjugacy of the various choices for $\mathfrak{a}_0$ (Theorem 4.6).

Theorem 4.10 (KAK decomposition)**.** *Every element in G has a decomposition as k_1ak_2 with $k_1, k_2 \in K$ and $a \in A$. In this decomposition, a is uniquely determined up to conjugation by a member of $W(G,A)$. If a is fixed as* $\exp H$ *with $H \in \mathfrak{a}_0$ and if $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$, then k_1 is unique up to right multiplication by a member of M.*

Before considering general parabolic subalgebras and subgroups, we mention special features of the "complex case." Suppose that the real semisimple Lie algbra

Lie algebra $\mathfrak{g}_0$ is actually complex, i.e., that there exists a linear map $J : \mathfrak{g}_0 \to \mathfrak{g}_0$ such that $J[X,Y] = [JX,Y] = [X,JY]$ and $J^2 = -1$. The corresponding group G then has an invariant complex structure and is called a **complex semisimple group**. Any choice of $\mathfrak{k}_0$ is a compact real form of $\mathfrak{g}_0$, and $\mathfrak{p}_0 = J\mathfrak{k}_0$. The Lie algebra $\mathfrak{m}_0$ is $J\mathfrak{a}_0$, and $\mathfrak{a}_0 \oplus J\mathfrak{a}_0$ is a complex Cartan subalgebra of the complex Lie algebra $\mathfrak{g}_0$. Each restricted root space has real dimension 2 and is a root space for $\mathfrak{a}_0 \oplus J\mathfrak{a}_0$. The group M is connected, all Cartan subalgebras are complex and are conjugate, and all Cartan subgroups are connected.

Returning to an arbitrary real semisimple Lie algebra $\mathfrak{g}_0$, let us now give the definitions of general parabolic subalgebras and subgroups. A **Borel subalgebra** of our complex semisimple Lie algebra $\mathfrak{g}$ is defined to be a subalgebra of the form $\mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$, where $\mathfrak{h}$ is a Cartan subalgebra and Δ^+ is a positive system of roots. A **parabolic subalgebra** of $\mathfrak{g}$ is a subalgebra containing a Borel subalgebra.

Theorem 4.11. *The parabolic subalgebras containing a given Borel subalgebra may be parametrized as follows. Let* Π *be the set of simple roots defining the set* Δ^+ *of positive roots that determine the Borel subalgebra. If* Π' *is any subset of* Π*, then there is a parabolic subalgebra corresponding to* Π'*, namely*

$$\begin{aligned}\mathfrak{p}_{\Pi'} &= \Big(\mathfrak{h} \oplus \bigoplus_{\alpha\in\operatorname{span}(\Pi')} \mathfrak{g}_\alpha\Big) \oplus \Big(\bigoplus_{\substack{\text{other}\\ \alpha\in\Delta^+}} \mathfrak{g}_\alpha\Big)\\ &= \textbf{Levi subalgebra} \;\oplus \textbf{nilpotent radical}\;.\end{aligned}$$

All parabolic subalgebras containing the given Borel subalgebra are of this form.

Reference. [K3, Proposition 5.90].

Now let us consider $\mathfrak{g}_0$. Suppose above that $\mathfrak{h} = (\mathfrak{h}_0)^{\mathbb{C}}$ with $\mathfrak{h}_0$ constructed from the Iwasawa decomposition and with Δ^+ consistent with Σ^+. Then one can show that the parabolic subalgebras of $\mathfrak{g}$ that are complexifications are the complexifications of all subalgebras of $\mathfrak{g}_0$ containing a minimal parabolic $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$.

We can parametrize these by subsets of simple restricted roots as follows. The formulas look similar to those in Theorem 4.11. Let Φ be a subset of simple restricted roots. Define

$$\begin{aligned}(\mathfrak{q}_\Phi)_0 &= \Big(\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \bigoplus_{\lambda\in\operatorname{span}(\Phi)} (\mathfrak{g}_0)_\lambda\Big) \oplus \Big(\bigoplus_{\substack{\text{other}\\ \lambda\in\Sigma^+}} (\mathfrak{g}_0)_\lambda\Big)\\ &= ((\mathfrak{m}_\Phi)_0 \oplus (\mathfrak{a}_\Phi)_0) \oplus (\mathfrak{n}_\Phi)_0,\end{aligned}$$

where $(\mathfrak{a}_\Phi)_0 = \bigcap_{\lambda\in\Phi} \ker\lambda$ and $(\mathfrak{m}_\Phi)_0$ is the orthocomplement of $(\mathfrak{a}_\Phi)_0$ in $(\mathfrak{m}_\Phi)_0 \oplus (\mathfrak{a}_\Phi)_0$. See [K3, §VII.7]. The decomposition $(\mathfrak{q}_\Phi)_0 = ((\mathfrak{m}_\Phi)_0 \oplus (\mathfrak{a}_\Phi)_0) \oplus (\mathfrak{n}_\Phi)_0$ is called the **Langlands decomposition** of $(\mathfrak{q}_\Phi)_0$.

The corresponding parabolic subgroup is the normalizer $Q_\Phi = N_G((\mathfrak{q}_\Phi)_0)$. This is a closed subgroup of G, being a normalizer. It has a **Langlands decomposition** $Q_\Phi = M_\Phi A_\Phi N_\Phi$, with the factors defined as follows: $(M_\Phi)_0$, A_Φ, N_Φ are to be connected, and $M_\Phi = M(M_\Phi)_0$. See [K3, §VII.7].

Finally we mention reductive Lie groups. Any representation theory done for the semisimple group G needs to be done also for all M_Φ, but M_Φ is not necessarily connected and $(M_\Phi)_0$ is not necessarily semisimple. One wants a class of groups containing interesting semisimple groups and closed under passage to the M_Φ's. Such groups are usually called **reductive Lie groups**.

There are various definitions, depending on the author. Here is the definition of G in the **Harish-Chandra class**:

(a) $\mathfrak{g}_0$ is reductive,
(b) G has finitely many components,
(c) the analytic subgroup of G corresponding to $[\mathfrak{g}_0, \mathfrak{g}_0]$ has finite center, and
(d) the action of every $\mathrm{Ad}(g)$ on $(\mathfrak{g}_0)^{\mathbb{C}}$ is in $\mathrm{Int}\,\mathfrak{g}$.

These groups have a number of important properties that we state in a qualitative form. First, $\mathfrak{g}_0$ has a Cartan involution θ. Second, G has a corresponding global Cartan decomposition. Third, the centralizer in G of any abelian θ stable subalgebra of $\mathfrak{g}_0$ is again in the class. Fourth, M meets every component of G. Fifth, the basic decompositions extend from the semisimple finite-center case to the reductive case. See [K3, §VII.2].

References

[He] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[Hu] J. E. Humphreys, *Introduction to Lie Algebras and Repressentation Theory*, Springer-Verlag, New York, 1972.

[J] N. Jacobson, *Lie Algebras*, Interscience Publishers, New York, 1962; second edition, Dover Publications, New York, 1979.

[K1] A. W. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton University Press, Princeton, N.J., 1986.

[K2] A. W. Knapp, *Lie Groups, Lie Algebras, and Cohomology*, Princeton University Press, Princeton, N.J., 1988.

[K3] A. W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, 1996.

[V] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Prentice-Hall, Englewood Cliffs, N.J., 1974; second edition, Springer-Verlag, New York, 1984.

[W] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups I*, Springer-Verlag, New York, 1972.

Department of Mathematics, State University of New York, Stony Brook, New York 11794, U.S.A.

E-mail address: aknapp@ccmail.sunysb.edu