Technische Universität München
Zentrum Mathematik

Arcs in Multibrot Sets, Locally Connected Julia Sets and Their Construction by Quasiconformal Surgery

Johannes Riedl

Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr.rer.nat.)

genehmigten Dissertation.

Vorsitzender: Univ.-Prof. Dr. H. Spohn

Prüfer der Dissertation: 1. Priv.-Doz. D. Schleicher, Ph.D.

2. Univ.-Prof. Dr. A. Leutbecher, em.

3. Univ.-Prof. Dr. J. Scheurle

Die Dissertation wurde am 14.09.2000 bei der Technischen Universität München

eingereicht und durch die Fakultät für Mathematik am 06.02.2001 angenommen.

Abstract

Let $\mathcal{M}_d$ denote the Multibrot set of degree $d \geq 2$. By quasiconformal surgery, we will construct homeomorphisms between certain subsets of the Multibrot sets. These homeomorphisms will be used to find paths in $\mathcal{M}_d$ from the origin to antenna tips of $\mathcal{M}_d$. For $d = 2$, we prove that every dyadic Misiurewicz point can be connected with the origin. This finishes a program started by B. Branner and A. Douady. Moreover, we transfer a result for real polynomials of G. Levin and S.v. Strien to complex polynomials and we prove that the Julia sets of the polynomials on the paths from above are locally connected.

Zusammenfassung

Mit $\mathcal{M}_d$ wird die Multibrotmenge vom Grad $d \geq 2$ bezeichnet. Wir konstruieren Homöomorphismen zwischen bestimmten Teilmengen von $\mathcal{M}_d$ mit Hilfe quasikonformer Chirurgie. Diese werden dazu verwendet, Wege in $\mathcal{M}_d$ vom Koordinatenursprung zu Antennenspitzen von $\mathcal{M}_d$ zu finden. Im Fall $d = 2$ wird gezeigt, daß so jeder dyadische Misiurewicz Punkt mit dem Koordinatenursprung verbunden werden kann. Dies vervollständigt ein von B. Branner und A. Douady begonnenes Programm. Zudem wird ein Ergebnis von G. Levin und S.v. Strien für reelle Polynome auf komplexe Polynome übertragen und der lokale Zusammenhang der Juliamenge der Polynome entlang dieser Wege gezeigt.

Contents

List of Figures

1 Introduction

Every holomorphic mapping $f : \mathbb{C} \to \mathbb{C}$ is either a polynomial or a transcendental function and defines via iteration a discrete complex dynamical system. Iterating f means to consider the n-fold composition of f by itself ($n \geq 0$):

$$f^n = \underbrace{f \circ \ldots \circ f}_{n}.$$

Other complex dynamical systems are given by holomorphic mappings $f : \mathbb{P} \to \mathbb{P}$; these are exactly the rational mappings, i.e. $f(z) = P(z)/Q(z)$ with complex polynomials P and $Q \not\equiv 0$. In the following we only discuss the dynamics of polynomials with one exception: later in the introduction we look at Newton's method and thus the dynamics of certain meromorphic functions to demonstrate the universality of the Multibrot sets.

A polynomial P of degree d is described by $d+1$ complex parameters $a_0, a_1, \ldots, a_d \in \mathbb{C}$ with $a_d \neq 0$:

$$P : \mathbb{C} \to \mathbb{C},\ z \mapsto \sum_{n=0}^{d} a_n z^n.$$

Conjugating f with a linear map $\varphi : \mathbb{C} \to \mathbb{C}$ means to look at the dynamics of f in a different coordinate system; thus a description of the dynamics of f gives a description of the dynamics of $\varphi^{-1} \circ f \circ \varphi$ and vice versa. It suffices to consider only normalized ($a_d = 1$) and centered ($a_{d-1} = 0$) polynomials, since every polynomial is conjugated to one of this type; the set of these polynomials can thus be described by $d-1$ complex parameters a_0, $a_1, \ldots, a_{d-2}$. The objects of interest in the dynamic plane of such polynomials are the filled-in Julia set K_f and the Julia set J_f, the boundary of K_f:

$$K_f := \{z \in \mathbb{C} \ :\ \text{the orbit } \left(f^n(z)\right)_{n\geq 0} \text{ is bounded }\} \quad \& \quad J_f := \partial K_f.$$

Another equivalent definition of the Julia set (which can be used for entire and rational mappings as well) is the following: a point $z \in \mathbb{C}$ belongs to the Julia set, iff for every neighborhood $U \subset \mathbb{C}$ of z, the family of restrictions $\{f^n|_U \ :\ n \in \mathbb{N}\}$ is not normal.

In this thesis we will discuss polynomials of degree $d \geq 2$ with only one critical point; these can be parameterized by one complex number $c \in \mathbb{C}$:

$$P_c : \mathbb{C} \to \mathbb{C},\ z \mapsto z^d + c;$$

Its Julia set is denoted by J_c (compare Figure 7). For a fixed degree $d \geq 2$, the parameter space of these polynomials is thus the complex plane, i.e. every parameter $c \in \mathbb{C}$ describes a polynomial P_c and therefore a discrete holomorphic dynamical system. One Question concerning this parameter space is for example: what is the locus of structural instability? This locus of structural instability is equal to the boundary of the Multibrot set $\mathcal{M}_d$ of degree d (Figure 6) which is defined by

$$\mathcal{M}_d := \{c \in \mathbb{C} \ :\ J_c \text{ is connected}\}.$$

This result is due to R. Mañé, P. Sad and D. Sullivan [MSS].

It turns out that the dynamic behavior of the critical points already describe the dynamics of the whole mapping: for example, the Julia set of a polynomial is connected if and only if all its critical points are contained in the filled-in Julia set. Thus a parameter $c \in \mathbb{C}$ belongs to $\mathcal{M}_d$ iff the unique critical point $z = 0$ of P_c is contained in K_c, i.e. iff it has bounded orbit.

The Multibrot sets have become projects of research for many people. The reason for this is that these sets do not only occur in the very special families of unicritical polynomials of degree d as introduced above but also in many other parameter spaces. This phenomenon is called the "universality" of the Multibrot sets (compare [McM2]). One immediate example for the appearance of small Multibrot sets (i.e. homeomorphic copies of the Multibrot sets) is $\mathcal{M}_d$ itself: in every neighborhood of any point in $\partial\mathcal{M}_d$ one finds infinitely many small Multibrot sets (therefore $\mathcal{M}_d$ is called "self-similar"). This can be explained by the theory of renormalization (Section 4.4) which is based on the theory of polynomial-like mappings by A. Douady and J.H. Hubbard [DH2].

Another example for the appearance of small Multibrot sets is Newton's method to find the zeros of a complex polynomial P. Here we examine the dynamics of the meromorphic Newton mapping

$$N_P(z) := z - \frac{P(z)}{P'(z)}.$$

To find a root of P we choose a point $z \in \mathbb{C}$ and look at the orbit $\left(N_P^n(z)\right)_n$ of z. If this sequence converges to a point $z_0 \in \mathbb{C}$, then z_0 is a root of P. Thus we are interested in the set of starting points, for which this sequence converges. That set is an open subset of $\mathbb{C}$ but it need not to be dense. For certain polynomials P, one finds copies of Julia sets J_c in the dynamic plane of N_P within the set of starting points not converging to a root of P. Example: Consider the family

$$p_\lambda \;:\; \mathbb{C} \to \mathbb{C},\; z \mapsto (z-1)\Big(z - (-1/2 + \lambda)\Big)\Big(z - (-1/2 - \lambda)\Big), \qquad \lambda \in \mathbb{C}$$

of polynomials of degree three and the corresponding family N_λ of Newton mappings. For the parameter $\lambda = -0.901... + 0.916...i$, we find subsets in the dynamics of N_λ (Figure 1) similar to the Julia sets shown in Figure 7. As we have mentioned above, the critical points of a holomorphic mapping determine the dynamic behavior. Since

$$N_P' = \frac{PP''}{(P')^2},$$

the critical points of N_λ are 1, $-1/2 - \lambda$, $-1/2 + \lambda$ and 0. The first three points are the roots of p_λ and thus fixed points of N_λ. Thus every point λ in the parameter space has one of the following properties:

- $N_\lambda^n(0)$ converges to 1,
- $N_\lambda^n(0)$ converges to $-1/2 - \lambda$,
- $N_\lambda^n(0)$ converges to $-1/2 + \lambda$ or
- $N_\lambda^n(0)$ does not converge to a root of p_λ.

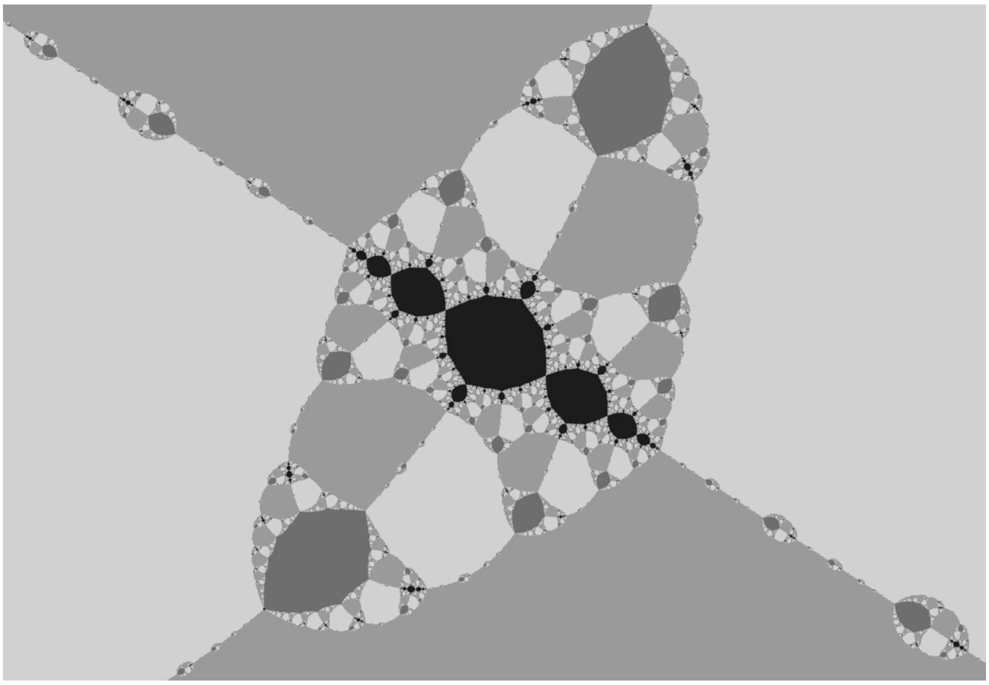

Figure 1: The dynamic plane of the Newton map N_λ for $\lambda = -0.901... + 0.916...i$: if the orbit of the point $z \in \mathbb{C}$ converges to 1 ($-1/2 + \lambda$, $-1/2 - \lambda$), then the corresponding point is marked grey (dark grey, light grey); otherwise, the point is marked black. The black set contains homeomorphic copies of a Julia set of a quadratic polynomial.

This is visualized in Figure 2. It turns out that the set of parameters λ, for which the orbit of the point 0 does not converge to a root of p_λ, contains infinitely many homeomorphic copies of the Mandelbrot set. This is again proven by renormalization theory. The correspondence of the Mandelbrot sets in the λ-space and the Julia sets in the dynamic spaces is as follows: every parameter λ in a just mentioned copy of the Mandelbrot set corresponds to a parameter $c(\lambda) \in \mathcal{M}_2$; in the dynamics of N_λ there occur homeomorphic copies of the filled-in Julia set of the polynomial $P_{c(\lambda)}$. Thus, thinking about the dynamics of Newton mappings often leads to the dynamics of the polynomials $z \mapsto z^d + c$ and their parameter spaces.

One of the main questions in holomorphic dynamics is whether the Multibrot sets are locally connected. The answer to this question is still open. If $\mathcal{M}_d$ was locally connected, we would have an abstract model for $\mathcal{M}_d$ homeomorphic to $\mathcal{M}_d$ itself. One knows that, if a compact connected set is locally connected, then it is path connected. Before trying to prove local connectivity of $\mathcal{M}_d$, one can ask which paths (or arcs = injective paths) in $\mathcal{M}_d$ can be constructed. Using a result of J.C. Yoccoz, D. Schleicher has proved that every point $c \in \mathcal{M}_2$ which is not infinitely renormalizable can be connected by a path to the center $c_0 := 0$ of the main cardioid of $\mathcal{M}_2$. Then B. Branner and A. Douady have constructed a path from c_0 to the dyadic Misiurewicz point c_1 in the 1/3-limb of $\mathcal{M}_2$ of lowest preperiod. They did so by constructing a homeomorphic embedding of the 1/2-Limb of $\mathcal{M}_2$ into the 1/3-Limb by quasiconformal surgery such that the parameter -2 is mapped to c_1. The 1/2-Limb contains a natural arc connecting the parameters 0 and -2.

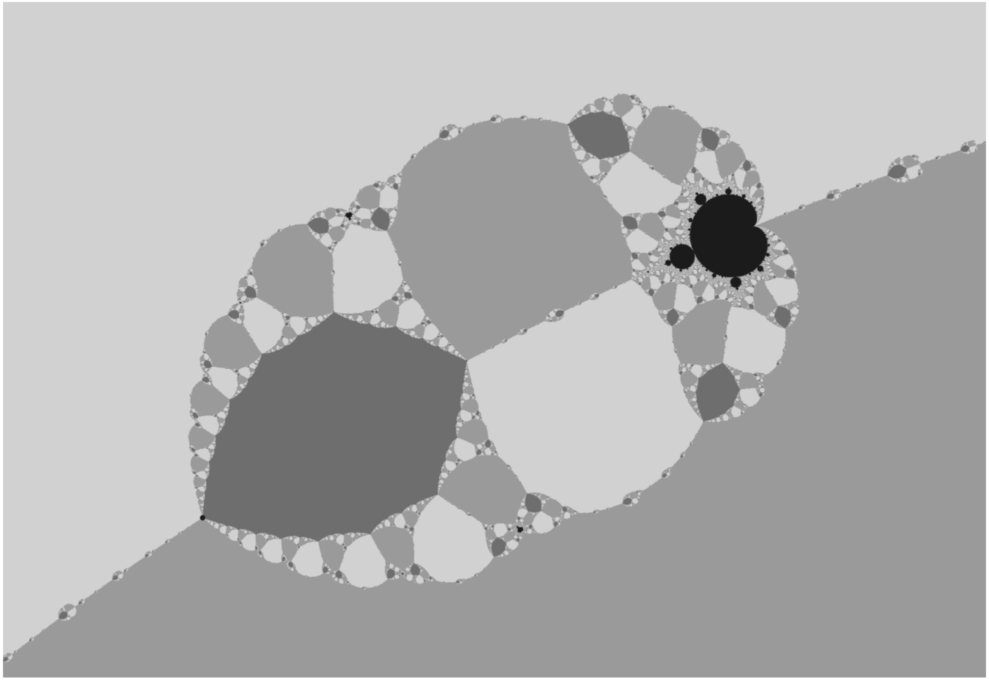

Figure 2: The parameter space of the Newton mappings N_λ for the polynomials p_λ: if the N_λ-orbit of the critical point 0 of N_λ converges to 1 ($-1/2+\lambda$, $-1/2-\lambda$) the corresponding parameter is marked grey (dark grey, light grey); if this orbit does not converge to one of the roots of p_λ, the parameter is marked black.

Then this homeomorphic embedding yields an arc from c_0 to c_1 which they call a **vein** of $\mathcal{M}_2$ (for the details, see [BD] or Section 5.3 of this paper). They asked for a method to construct veins from c_0 to any dyadic Misiurewicz point of $\mathcal{M}_2$. We will solve this problem by using a composition of three types of homeomorphisms:

Theorem 5.2 (Homeomorphic Trees at general Misiurewicz Points)
For $d = 2$, let $c_0 \in \mathcal{M}_2$ be a Misiurewicz point with preperiod k and period l such that $q \geq 3$ parameter rays land at c_0. Then for every integer $j \in \{1, \ldots, q-2\}$, there exists a homeomorphism

$$\chi : T_{c_0}^{(j)} \to T_{c_0}^{(j+1)}.$$

Theorem 5.17 (Homeomorphic Trees at Hyperbolic Components)
Let $H \subset \mathcal{M}_d$ be a hyperbolic component of period l and let p_1, p_2, q be positive integers with $p_1 \neq p_2$, $\gcd(p_1, q) = 1 = \gcd(p_2, q)$. Then for every $i \in \{1, \ldots, d-1\}$, there exists a mapping

$$\chi \ : \ T_H^{(p_1/q)}(i) \longrightarrow T_H^{(p_2/q)}(i)$$

which is continuous on the boundary and analytic on hyperbolic components.
In the case $d = 2$, the map χ is a homeomorphism.

Theorem 5.24 (Homeomorphic Embeddings of $1/q$-Sublimbs)
Let $H \subset \mathcal{M}_d$ be a hyperbolic component of period l and let $q \geq 2$. Then for every $i \in \{1, \ldots, d-1\}$, the $1/q$-sublimb in the i-th sector of H can be mapped into the $1/(q+1)$-sublimb of the i-th sector of H such that the dyadic Misiurewicz point(s) of lowest preperiod in the considered $1/q$-sublimb is (are) mapped to the dyadic Misiurewicz point(s) of lowest preperiod in the considered $1/(q+1)$-sublimb. This mapping is continuous on the boundary and analytic in the hyperbolic components.
In the case $d = 2$ this mapping is a homeomorphic embedding.

To prove the continuity of the mappings in these theorems we need to discuss Riemann mappings from the unit disc to Carathéodory discs (compare Definition 2.14) depending continuously on a parameter. The distance between two Carathéodory discs is measured by the Hausdorff distance of their boundaries. In Section 2.1 we introduce "uniform local connectivity" of a family of locally connected sets. This is the key to prove Theorem 2.17 stated below. Its proof is presented in Section 2.3, after discussing the dilatation of Riemann mappings and sequences of automorphisms of $\mathbb{D}$ (Section 2.2).

Theorem 2.17 (Continuous Dependence)
Let $\Lambda \subset \mathbb{C}$ be compact, $(U_\lambda)_{\lambda\in\Lambda}$ be a family of open simply connected domains and $f_\lambda : \mathbb{D} \to U_\lambda$ be biholomorphic mappings for $\lambda \in \Lambda$ with one of the following properties:

(1) the maps $\Lambda \to \mathbb{C},\ \lambda \mapsto f_\lambda(0)$ and $\Lambda \to S^1,\ \lambda \mapsto f'_\lambda(0)$ are continuous.

(2) the maps $\Lambda \to \mathbb{C},\ \lambda \mapsto f_\lambda(0)$ and $\Lambda \to \mathbb{C},\ \lambda \mapsto f_\lambda(1)$ are continuous and $f_\lambda(1)$ is a simple boundary point of U_λ for all $\lambda \in \Lambda$.

(3) there are three points $\zeta_1,\ \zeta_2,\ \zeta_3 \in \partial\mathbb{D}$ such that the maps $\Lambda \to \mathbb{C},\ \lambda \mapsto f_\lambda(\zeta^j)$ are continuous $(1 \leq j \leq 3)$, have disjoint graphs and that the image points are simple boundary points of U_λ.

Then the mapping

$$\Lambda \times \overline{\mathbb{D}} \to \mathbb{C},\ (\lambda, z) \mapsto f_\lambda(z)$$

is continuous, if there exists a continuous motion $\Phi : \Lambda \times \partial U \to \mathbb{C}$ of a locally connected set ∂U such that $\partial U_\lambda = \Phi(\lambda, \partial U)$ for all $\lambda \in \Lambda$. In this case, the mappings

$$\Lambda \times \mathbb{D} \to \mathbb{C},\ (\lambda, z) \mapsto f_\lambda^{(k)}(z)$$

are continuous for all $k \geq 1$.

As an application we discuss the continuous dependence of moduli of quadrilaterals and annuli in Section 2.4; the statements there will be needed in the construction of the homeomorphisms in Section 5.

After giving two equivalent definitions for quasiconformal mappings in Section 3.1 we present some statements on quasiconformal homeomorphisms in Sections 3.2 and 3.3; they will be needed in the construction of the homeomorphisms in Section 5. In this context the measurable Riemann mapping theorem (Theorem 3.4) is the most important ingredient for quasiconformal surgery.

In Section 4, we give an introduction into the dynamics of complex polynomials and we make some further preparations for the construction of the homeomorphisms in the following section. In particular, we state well-known results about unicritical polynomials in the Sections 4.1, 4.3, 4.4 and 4.6. In Section 4.2, invariant sectors and their opening moduli are treated; this concept was introduced by B. Branner and A. Douady in [BD]. We introduce the concept of piecewise dynamic homeomorphisms in Section 4.5. Their existence turns out to determine the existence of the homeomorphisms around Misiurewicz points. These points are discussed in detail in Section 4.7: we distinguish between several types of Misiurewicz points and construct certain piecewise dynamic homeomorphisms. Looking at certain Misiurewicz points these piecewise dynamic homeomorphisms are not hard to find (compare Lemma 4.43), but the problem is, to show their existence for an arbitrary Misiurewicz point, the dynamics of which can be arbitrary complicated. For degree $d = 2$, we construct these piecewise dynamic homeomorphisms for every Misiurewicz point in $\mathcal{M}_2$. For higher degree $d > 2$, our construction still works for the α-type Misiurewicz points; there should be a similar construction for Misiurewicz points of period $l \geq 2$, but it is not clear yet how to generalize the construction in the proof of Lemma 4.49 to the case $d \geq 2$.

The first theorem in Section 5 describes certain homeomorphisms between parts of the branches behind a Misiurewicz point. This type of homeomorphisms is completely new and will be the main part of the thesis. The proof of Theorem 5.17 in Section 5.2 is based on an idea of D. Schleicher and will not be presented in detail. Theorem 5.24 in Section 5.3 is a slight generalization of a statement of B. Branner and A. Douady in [BD]. Therefore we will only sketch its proof.

Combining these three types of homeomorphisms we can connect the parameter 0 with any dyadic Misiurewicz point in $\mathcal{M}_2$ by an arc (Section 6.1). The proof of the existence of such paths does not use the result of Yoccoz that $\mathcal{M}_2$ is locally connected at all finitely renormalizable parameters $c \in \mathcal{M}_2$. Since the arguments of Yoccoz do not transfer to the case of higher degree $d > 2$, one could try to construct veins in $\mathcal{M}_d$ in the way presented here. This point is discussed in this section as well. Using the homeomorphisms from above we prove local connectivity for the Julia sets of many polynomials (Section 6.2); in particular, for degree $d > 2$ we prove local connectivity of Julia sets for which this was not known yet.

Acknowledgements. I want to thank Dierk Schleicher who has introduced me into the field of holomorphic dynamics. He made it possible for me to meet many interesting mathematicians at several conferences in this field. Most grateful I am to him for the numberless discussions on the topics of this thesis and for giving important comments to earlier versions.

Then I thank Prof. Leutbecher for his support during the last three years; in particular, I am very grateful for many discussions, comments and improvements on the second chapter about Riemann mappings in this thesis.

Moreover, I thank Wolf Jung for many discussions on the concept of quasiconformal surgery and for his useful comments on earlier versions of this thesis. He will finish his thesis "Homeomorphisms on Edges of the Mandelbrot set" [J] soon.

Finally, I thank Bodil Branner, Adrian Douady, Adam Epstein, John H. Hubbard, Mikhail Lyubich, Curtis T. McMullen, John Milnor and Mary Rees for many discussions on many topics of holomorphic dynamics.

2 Riemann Mappings

For every simply connected, open domain $U \subset \mathbb{C}$, $U \neq \mathbb{C}$ the Riemann mapping theorem guarantees the existence of a biholomorphic mapping

$$f : \mathbb{D} \to U$$

between the unit disc $\mathbb{D} := \{z \in \mathbb{C} \;:\; |z| < 1\}$ and U. Carathéodory has given a necessary and sufficient condition when this mapping can be extended continuously to the boundaries:

Theorem 2.1 (Carathéodory)
Let $U \subset \mathbb{C}$ be connected, simply connected, bounded and open. A biholomorphic mapping $f : \mathbb{D} \to U$ has a continuous extension to the closure $\overline{\mathbb{D}}$ if and only if the boundary ∂U of U is locally connected.

A proof of this theorem is given in [Car] or [Mi]. In [Mi, Lemma 17.13] one finds an equivalent description of local connectivity for compact sets (Lemma 2.5). Starting with this view of local connectivity we introduce the concept of uniform local connectivity (Definition 2.7). This will turn out to be the main ingredient in the discussion of convergence properties for sequences of Riemann mappings which have a continuous extension to the boundaries.

There have been made many investigations on the behavior of Riemann mappings if the domain varies analytically with a complex parameter λ (compare [Du], [PR], [Ro], [Schi]). We are interested in continuous variations of the domains and in the behavior on the boundaries of the extended Riemann mappings. Roughly spoken we prove in this section that the extended Riemann mapping varies continuously on $\overline{\mathbb{D}}$ if the corresponding image domain varies continuously; this is made precise in Theorem 2.17.

2.1 Local Connectivity

Definition 2.2 (Local Connectivity)
A topological space X is said to be **locally connected** *if for every point $x \in X$, every neighborhood of x contains a connected neighborhood of this point.*

Definition 2.3 (Carathéodory Discs)
A simply connected, bounded open domain $U \subset \mathbb{C}$ will be called a **Carathéodory Disc** *if the boundary ∂U of U is locally connected.*

Lemma 2.4 (Open Local Connectivity)
A compact topological space X is locally connected if and only if the components of open sets are always open.

Proof. Assume that for every open set its connected components are open. Choose a neighborhood N of a point $x \in X$. Then the component of $\text{int}(N)$ containing x is open and thus a connected neighborhood of x contained in N.

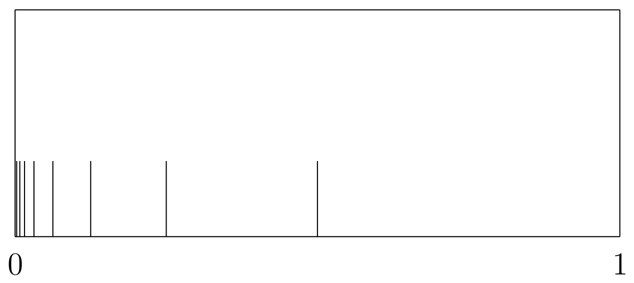

Figure 3: A not locally connected set in the complex plane bounding a simply connected domain

On the other hand, assume X to be locally connected. For every nonempty open set $V \subset X$ consider any connected component V' of V. By assumption we know that every point $x \in V'$ has a connected neighborhood $V'' \subset V$. Since V' is the largest connected subset of V which contains x, we have $V'' \subset V'$. In particular, for every point $x \in V'$, the component V' contains a connected neighborhood of x, i.e. the component V' is open. □

Lemma 2.5 (Local Connectivity and the Diameter of Connected Subsets)
Let (X, d) be a compact metric space. Then X is locally connected if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$, the open ball $B_\delta(x)$ is contained in the connected x-component of $B_\epsilon(x)$.

Proof. Let X be locally connected and $\epsilon > 0$. By Lemma 2.4, there exists for every $x \in X$ an open connected neighborhood $V_{\epsilon,x} \subset B_\epsilon(x)$ of x. Since X is compact, there exists a Lebesgue number $\delta > 0$ for the open covering

$$X = \bigcup_{x \in X} V_{\epsilon,x},$$

i.e. for every $x \in X$, there is a point $x' \in X$ such that $B_\delta(x) \subset V_{\epsilon,x'}$. Then δ has the property of the lemma.
On the other hand, if the ϵ-δ-condition of Lemma 2.5 holds, then clearly X is locally connected. □

The following statement can be found in [Mi, §17, Proof of Lemma 17.17]:

Lemma 2.6 (Locally Connected Sets are Path Connected)
Let (X, d) be a compact, connected and locally connected metric space, $\epsilon > 0$ and $\delta > 0$ be chosen as in Lemma 2.5. Then for all x_1, $x_2 \in X$ with distance less than δ, there exists a path of diameter less than 8ϵ connecting x_1 and x_2.

Definition 2.7 (Uniform Local Connectivity)
Let $(X_\lambda)_{\lambda \in \Lambda}$ be a family of compact, locally connected sets. If for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $\lambda \in \Lambda$ and any $x \in X_\lambda$ the ball $B_\delta(x)$ is contained in the x-component of $B_\epsilon(x)$ then the family $(X_\lambda)_{\lambda \in \Lambda}$ will be called **uniformly locally connected**.

Definition 2.8 (Continuous Motions)
Let Λ be a topological space and $X \subset \mathbb{C}$ compact. A continuous mapping

$$\Phi : \Lambda \times X \to \mathbb{C}$$

will be called a **continuous motion of** X*, if the map*

$$\Phi_\lambda : X \to \mathbb{C},\ z \mapsto \Phi(\lambda, z)$$

is injective for every $\lambda \in \Lambda$.

Remarks.
This definition of a continuous motion resembles that of a holomorphic motion (see for example [PR]).
All mappings Φ_λ are homeomorphisms to their images.

Lemma 2.9 (Continuity of the Inverse)
Let X, $\Lambda \subset \mathbb{C}$ be compact and $\Phi : \Lambda \times X \to \mathbb{C}$ a continuous motion of X. Then the mapping

$$\{(\lambda, z) \in \Lambda \times \mathbb{C} \ :\ z \in X_\lambda\} \to X,\ (\lambda, z) \mapsto (\Phi_\lambda)^{-1}(z)$$

is uniformly continuous.

Proof. Since Λ and X are compact, $\Lambda \times X$ is compact as well. Moreover

$$\Lambda \times X \to \{(\lambda, z) \in \Lambda \times \mathbb{C} \ :\ z \in X_\lambda\},\ (\lambda, z) \mapsto (\lambda, \Phi(\lambda, z))$$

is a continuous, bijective mapping on a compact set and thus a homeomorphism. The projection of its inverse to the second coordinate is continuous and thus uniformly continuous on the compact set $\{(\lambda, z) \in \Lambda \times \mathbb{C} \ :\ z \in X_\lambda\}$. □

Lemma 2.10 (Continuous Motion of Locally Connected Sets)
Let $\Lambda \subset \mathbb{C}$ be compact, $X \subset \mathbb{C}$ be compact and locally connected. Then for every continuous motion $\Phi : \Lambda \times X \to \mathbb{C}$ of X, the family of sets $X_\lambda = \Phi_\lambda(X)$, $\lambda \in \Lambda$, is uniformly locally connected.

Proof. Let $\epsilon > 0$. Since Φ is uniformly continuous on the compact set $\Lambda \times X$, there exists $\tilde{\epsilon} > 0$ such that

$$|\Phi(\lambda, x) - \Phi(\lambda, x')| < \epsilon$$

for all $\lambda \in \Lambda$ and x, $x' \in X$ with $|x - x'| < \tilde{\epsilon}$. Since X is locally connected, there exists a positive number $\tilde{\delta} > 0$ such that for every $x \in X$, $B_{\tilde{\delta}}(x)$ is contained in the x-component of $B_{\tilde{\epsilon}}(x)$ (Lemma 2.5). By Lemma 2.9, there exists $\delta > 0$ such that

$$|\Phi_\lambda^{-1}(y) - \Phi_\lambda^{-1}(y')| < \tilde{\delta}$$

for all $\lambda \in \Lambda$ and $y, y' \in \Phi_\lambda(X)$ with $|y - y'| < \delta$.

Then for every $\lambda \in \Lambda$, every δ-ball in X_λ is contained in the center component of the corresponding ϵ-ball: For every $\lambda \in \Lambda$ and $y \in X_\lambda$, we have

$$\Phi_\lambda^{-1}(B_\delta(y)) \subset B_{\tilde{\delta}}(\Phi_\lambda^{-1}(y))$$

and this $\tilde{\delta}$-ball is contained in the center component N of $B_{\tilde{\epsilon}}(\Phi_\lambda^{-1}(y))$. Therefore, $\Phi_\lambda(N)$ is connected, with

$$B_\delta(z) \subset \Phi_\lambda(N) \subset B_\epsilon(z).$$

□

2.2 Remarks on Riemann Mappings

Lemma 2.11 (Bounds on the Derivative of a Riemann Mapping)
Let $G \subset \mathbb{C}$ be a simply connected domain such that $B_r(0) \subset G \subset B_R(0)$ and let $\varphi : G \to \mathbb{D}$ denote the biholomorphic mapping normalized by $\varphi(0) = 0$ and $\varphi'(0) > 0$. Then

$$1/R \leq \varphi'(0) \leq 1/r.$$

PROOF. By the proof of the Riemann mapping theorem, we know that

$$\varphi'(0) = \sup\{\psi'(0)\}$$

where the supremum is taken over all holomorphic functions $\psi : G \to \mathbb{D}$ with $\psi(0) = 0$ and $\psi'(0) > 0$. One possible ψ is the restriction of $\mathbb{C} \to \mathbb{C}$, $z \mapsto z/R$:

$$\psi_0 : G \to \mathbb{D} \ , \ z \mapsto z/R$$

the derivative of which is equal to $1/R$ at the point $z = 0$. Thus we have found a lower bound for the derivative of φ at the origin. To find an upper bound for $\varphi'(0)$ we compute a lower bound for $(\varphi^{-1})'(0)$: We apply the same consideration to $G' := \varphi(B_r(0)) \subset \mathbb{D}$ and the mapping

$$1/r \cdot \varphi^{-1} : G' \to \mathbb{D}$$

instead of φ to get the upper bound:

$$1/r \cdot \frac{1}{\varphi'(0)} = (1/r \cdot \varphi^{-1})'(0) \geq 1.$$

□

Lemma 2.12 (Sequences of Automorphisms of the Unit Disc)
The limit function f of a pointwise convergent sequence of biholomorphic mappings $f_n : \mathbb{D} \to \mathbb{D}$ is an automorphism of $\mathbb{D}$ or a constant mapping $f \equiv e^{i\alpha} \in \partial\mathbb{D}$.

PROOF. Since the sequence f_n is pointwise convergent and bounded on $\mathbb{D}$, it converges indeed locally uniformly to a holomorphic function $f : \mathbb{D} \to \mathbb{D}$. By a theorem of Hurwitz, this mapping f is either biholomorphic or constant. If the limit function is constant, this constant is a boundary point of $\mathbb{D}$: assume that this is not the case. Then the sequence $(f_n(0))_n$ converges to a point $w \in \mathbb{D}$. For $n \in \mathbb{N}$ consider the following automorphism of $\mathbb{D}$:

$$M_n : \mathbb{D} \to \mathbb{D}, \ z \mapsto \frac{z - f_n(0)}{1 - z\overline{f_n(0)}}.$$

Then we have by the Schwarzian Lemma,

$$1 = |\frac{d}{dz}(M_n \circ f_n)(0)| = \frac{|f_n'(0)|}{1 - |f_n(0)|^2}$$

for all $n \in \mathbb{N}$ and thus $(f_n'(0))_n$ does not converge to 0 in contradiction to the fact that $\left(f_n|_{B_{1/2}(0)}\right)_n$ converges uniformly to a constant mapping. □

2.3 Continuous Dependence of Riemann Mappings

In this section we prove the main result about Riemann mappings (Theorem 2.17). The most important preparation is Theorem 2.15. There we discuss the behavior of sequences of Riemann mappings converging to an injective holomorphic mapping or to a constant mapping. In both cases we will determine the subset of $\overline{\mathbb{D}}$ on which this sequence converges locally uniformly.

Lemma 2.13 (Existence of Short Paths)
Let $g : \mathbb{D} \to B_R(0)$ be an injective holomorphic mapping. For every $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$, $\triangle\vartheta > 0$ and $\triangle r \in]0, 1/2[$, there exists $\varphi \in [\vartheta - \triangle\vartheta, \vartheta + \triangle\vartheta]$ such that the g-image of the straight line between $(1 - \triangle r)e^{i\varphi}$ and $e^{i\varphi}$ has length

$$l(\varphi) \leq R\sqrt{\frac{\pi}{\triangle\vartheta}} \cdot \sqrt{\triangle r}$$

In particular, the right hand side of this inequality tends to 0 for $\triangle r \to 0$.

Proof. For Lebesgue almost all angles $\varphi \in [\vartheta - \triangle\vartheta, \vartheta + \triangle\vartheta]$, the length of the g-image of the straight line between 0 and $e^{i\varphi}$ is finite. For all these angles, the radial limit $\lim_{\rho\to 1} g(\rho e^{i\varphi})$ does exist and the length of the g-image of the straight line between $(1 - \triangle r)e^{i\varphi}$ and $e^{i\varphi}$ is denoted by

$$l(\triangle r, \varphi) := \int_{1-\triangle r}^{1} |g'(\rho e^{i\varphi})| d\rho.$$

Integrating over φ and using the Cauchy-Schwarz-inequality we get

$$\begin{aligned}
&\left(\int_{\vartheta-\triangle\vartheta}^{\vartheta+\triangle\vartheta} \int_{1-\triangle r}^{1} |g'(\rho e^{i\varphi})| d\rho d\varphi\right)^2 \leq \\
&\leq \int_{\vartheta-\triangle\vartheta}^{\vartheta+\triangle\vartheta} \int_{1-\triangle r}^{1} |g'(\rho e^{i\varphi})|^2 d\rho d\varphi \cdot \int_{\vartheta-\triangle\vartheta}^{\vartheta+\triangle\vartheta} \int_{1-\triangle r}^{1} d\rho d\varphi \\
&\leq \int_{\vartheta-\triangle\vartheta}^{\vartheta+\triangle\vartheta} \int_{1-\triangle r}^{1} |g'(\rho e^{i\varphi})|^2 \frac{\rho}{1/2} d\rho d\varphi \cdot \int_{\vartheta-\triangle\vartheta}^{\vartheta+\triangle\vartheta} \int_{1-\triangle r}^{1} d\rho d\varphi \\
&\leq 2 \cdot \mathrm{area}(B_R(0)) \cdot 2\triangle\vartheta \cdot \triangle r \\
&= 4R^2\pi\triangle\vartheta \cdot \triangle r.
\end{aligned}$$

Thus at least for one $\varphi \in [\vartheta - \triangle\vartheta, \vartheta + \triangle\vartheta]$ one has

$$l(\varphi) \leq R\sqrt{\frac{\pi}{\triangle\vartheta}}\sqrt{\triangle r}.$$

□

The distance between two nonempty compact sets V, $W \subset \mathbb{C}$ in the Hausdorff metric d_H is defined by

$$d_H(V, W) = \max\{\sup_{v\in V} \inf_{w\in W} |v - w| \ , \ \sup_{w\in W} \inf_{v\in V} |v - w|\}.$$

The distance between two open bounded simply connected subsets of $\mathbb{C}$ can be defined as the Hausdorff distance of their boundaries. This leads to the following definition:

Definition 2.14 (Sequences of simply connected Domains)
A sequence $(U_n)_n$ of simply connected, open and uniformly bounded domains $U_n \subset \mathbb{C}$ is called **convergent** *if there exists a simply connected (necessarily bounded) open domain U such that*

$$d_H(\partial U_n, \partial U) \longrightarrow 0, \quad n \to \infty.$$

Such a sequence will be called an **admissible sequence of Carathéodory-discs of size** *R, if in addition to that the family $(\partial U_n)_n$ is uniformly locally connected, $U_n \subset B_R(0)$ and ∂U is locally connected. The set $U =: \lim_{n\to\infty} U_n$ is called the limit of the sequence $(U_n)_n$.*

Remarks.

1. The condition that $d_H(\partial U_n, \partial U) \to 0$ for $n \to \infty$ is not equivalent to the condition that $d_H(U_n, U) \to 0$; this can be easily seen from

$$d_H\Big(\mathbb{D}\backslash]-1,0], \mathbb{D}\Big) = 0, \quad \text{but} \quad d_H\Big(\partial(\mathbb{D}\backslash]-1,0]), \partial\mathbb{D}\Big) = 1.$$

   This example shows that it is not enough to discuss the closure of the Carathéodory discs.

2. We will often consider Riemann mappings from the unit disc $\mathbb{D}$ to another simply connected, open domain $U \subset \mathbb{C}$. If ∂U is locally connected then by Theorem 2.1 this map can be extended continuously to $\overline{\mathbb{D}}$. In this case we will use the same notation for the Riemann mapping and its extension.

Theorem 2.15 (Sequences of Riemann Mappings)
Let $(U_n)_n$ be an admissible sequence of Carathéodory-discs of size R and limit U and let $(f_n\colon \mathbb{D} \to U_n)_n$ be a sequence of Riemann mappings which converges locally uniformly to a holomorphic mapping f.

1. *If f is not constant, then*
   - *(a) $U = f(\mathbb{D})$,*
   - *(b) $(f_n)_n$ is equicontinuous at every point $z_0 \in \partial\mathbb{D}$, i.e. for every $z_0 \in \partial\mathbb{D}$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that for all integers $n \geq 1$ and for all $z \in \overline{\mathbb{D}} \cap B_\delta(z_0)$*
   $$|f_n(z) - f_n(z_0)| < \epsilon,$$
   - *(c) the sequence $(f_n)_n$ converges uniformly to f on $\overline{\mathbb{D}}$.*

2. *If $f \equiv w$ is constant, then there exists $z_0 \in \partial\mathbb{D}$ and a subsequence $(f_{n_k})_k$, which converges locally uniformly to f on $\overline{\mathbb{D}} \setminus \{z_0\}$.*

Proof.

(1) In the case that f is not constant, we prove $U = f(\mathbb{D})$. For all n, let

$$\mu_n : U_n \to \mathbb{D}$$

be the biholomorphic mapping normalized by the condition

$$\mu_n(f_n(0)) = 0 \quad \text{and} \quad \mu_n'(f_n(0)) > 0.$$

Then there exists a subsequence of these mappings which converges locally uniformly to a mapping $\mu : U \to \mathbb{D}$ in the following sense: for any $z \in U$, there exists a radius $\epsilon > 0$ and a bound $N \in \mathbb{N}$ such that for all $n \geq N$, the ball $B := B_\epsilon(z)$ is contained in U_n and the sequence of restrictions $(\mu_n|_B)_{n\in\mathbb{N}}$ converges to $\mu|_B$ uniformly in the usual sense.

To prove this we use an exhaustion $(K_m)_m$ of U by compact subsets such that $K_j \subset \overset{\circ}{K}_{j+1}$ for all $j \geq 1$ and $\bigcup_{m=1}^{\infty} K_m = U$. There exists an increasing sequence $(n_m)_m$ such that

$$K_m \subset U_n$$

if $n \geq n_m$. Therefore we find for every $N \in \mathbb{N}$ a subsequence of $(\mu_n)_{n\geq n_1}$ denoted by $(\mu_{n(N,r)})_{r\geq 1}$ which converges uniformly on K_N such that $(\mu_{n(N+1,r)})_{r\geq 1}$ is a subsequence of it. Thus the sequence

$$\mu_{n(1,1)},\ \mu_{n(2,2)},\ \mu_{n(3,3)},\ \ldots$$

converges locally uniformly to a holomorphic mapping $\mu : U \to \mathbb{C}$ which is either injective or constant. If f was not constant, then μ cannot be constant:

$$z_\infty := \lim_{n\to\infty} f_n(0) = f(0) \in U$$

and therefore there exist positive bounds $s < S$ such that

$$B_s(f_n(0)) \subset U_n \subset B_S(f_n(0))$$

for large n. Lemma 2.11 yields

$$1/S \leq \mu_n'(f_n(0)) \leq 1/s$$

and thus

$$1/S \leq \mu'(z_\infty) \leq 1/s;$$

therefore μ cannot be constant. Let $k_j := n(j,j)$ for $j \in \mathbb{N}$. Then the sequence

$$\left(\mu_{k_j} \circ f_{k_j}\right)_j$$

of automorphisms of $\mathbb{D}$ (and thus of rotations about zero) converges to $\mu \circ f$ because for fixed $z \in \mathbb{D}$ we have

$$\begin{aligned} |(\mu_{k_j} \circ f_{k_j})(z) - (\mu \circ f)(z)| \quad &\leq \quad |(\mu_{k_j} \circ f_{k_j})(z) - (\mu \circ f_{k_j})(z)| \\ &\quad + |(\mu \circ f_{k_j})(z) - (\mu \circ f)(z)|. \end{aligned}$$

The continuity of μ and the convergence of $(f_{k_j}(z))_j$ makes the second term of the sum arbitrarily small if we choose j sufficiently large. The locally uniform convergence of $(\mu_{n_j})_j$ shows that the first term gets arbitrarily small if j is large enough.

Now we can finish the proof for $U = f(\mathbb{D})$. Obviously $f(\mathbb{D}) \subset U$. Suppose we have $f(\mathbb{D}) \neq U$, then the map $f : \mathbb{D} \to U$ is injective but not surjective and thus $\mu(U) \neq \mu \circ f(\mathbb{D}) = \mathbb{D}$. On the other hand, $(\mu_{k_j} \circ f_{k_j})_j$ is a sequence of automorphisms of the unit disc. By Lemma 2.12 and $(\mu_{k_j} \circ f_{k_j})(0) = 0$, every accumulation point of such a sequence is again an automorphism of $\mathbb{D}$, contradiction. This proves $U = f(\mathbb{D})$.

(2) Let $\epsilon > 0$ and $z_0 \in \partial\mathbb{D}$. Since the sequence $(\partial U_n)_n$ is uniformly locally connected, there exists by Lemma 2.6 a positive number $\rho \in]0, \epsilon/2[$ such that for all $n \geq 1$ any two points z_1, $z_2 \in \partial U_n$ with distance less than ρ can be joined by a path in ∂U_n of diameter less than $\epsilon/2$.

Since for $1 \leq n \leq \infty$ the sets ∂U_n are locally connected, the Riemann mappings $f_n : \mathbb{D} \to U_n$ can be extended continuously to the boundary $\partial\mathbb{D}$. But we do not know yet whether the sequence $f_n|_{\partial\mathbb{D}}$ converges to $f|_{\partial\mathbb{D}}$. By the continuity of f on $\overline{\mathbb{D}}$ we know however

$$f\left(B_{\tilde{\delta}}(z_0) \cap \overline{\mathbb{D}}\right) \subset B_{\rho/12}\left(f(z_0)\right)$$

for some $\tilde{\delta} > 0$. For suitably chosen $\triangle\varphi$, $\triangle\tilde{r} > 0$, the neighborhood $B_{\tilde{\delta}}(z_0) \cap \overline{\mathbb{D}}$ of z_0 in $\overline{\mathbb{D}}$ contains the circular rectangle

$$\{z \in \mathbb{C} : |\arg(z) - \arg(z_0)| < \triangle\varphi, 1 \geq |z| > 1 - \triangle\tilde{r}\}.$$

By Lemma 2.13, there exists $\triangle r \in]0, \triangle\tilde{r}[$ such that for every injective holomorphic mapping $g : \mathbb{D} \to B_R(0)$ and for every $\vartheta \in S^1$, there exists an injective path joining the arcs

$$\{z \in \mathbb{C} : |z| = 1, |\arg(z) - \vartheta| < \triangle\varphi/4\}$$

and

$$\{z \in \mathbb{C} : |z| = 1 - \triangle r, |\arg(z) - \vartheta| < \triangle\varphi/4\}$$

such that the g-image of that path has a length smaller than $\rho/4$. This holds in particular for

$$\vartheta \in \{\arg(z_0) \pm \frac{3}{4}\triangle\varphi\}.$$

Now we put

$$\delta := \min\{\triangle\varphi, \triangle r\}/2$$

and $A := \{z \in \mathbb{C} : |z| = 1 - \triangle r, |\arg(z) - \arg(z_0)| \leq \triangle\varphi\}$. Then A is compact and the sequence $(f_n)_n$ converges uniformly on A. Hence we can find an integer $\tilde{N}$ such that

$$|f_n(z) - f(z)| < \rho/6$$

for all $n \geq \tilde{N}$ and for all $z \in A$.

The choice of $\tilde{\delta}$ yields the following inequalities:

$$\mathrm{diam}(f_n(A)) \leq \mathrm{diam}(f(A)) + 2 \cdot \rho/6 < \rho/2.$$

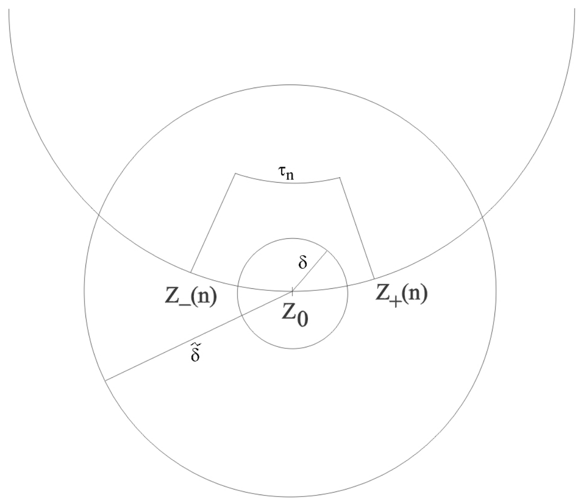

Figure 4: Construction of $z_-(n)$, $z_+(n)$ and τ_n.

Altogether for every $n \geq \tilde{N}$ there exist two points $z_-(n),\ z_+(n) \in S^1$ such that

$$\begin{aligned} \arg(z_-(n)) - \arg(z_0) &\in [-\triangle\varphi, -\triangle\varphi/2], \\ \arg(z_+(n)) - \arg(z_0) &\in [\triangle\varphi/2, \triangle\varphi] \end{aligned}$$

and there exists a Jordan curve

$$\tau_n \ : \ [0,1] \longrightarrow \overline{\mathbb{D}}$$

which connects the points $z_-(n)$ and $z_+(n)$ within $\mathbb{D} \setminus B_\delta(z_0)$ such that

$$\operatorname{diam}(f_n(\tau_n)) < \rho < \epsilon/2.$$

Let D_n^1, D_n^2 be the components of $\mathbb{D} \setminus \tau_n([0,1])$ such that the first one contains

$$\mathbb{D} \cap B_\delta(z_0).$$

Moreover let U_n^1, U_n^2 be the simply connected components of $U_n \setminus f_n\left(\tau_n([0,1])\right)$ such that

$$\operatorname{diam}(U_n^1) < \epsilon.$$

Then for every $n \geq \tilde{N}$,

$$\text{either} \quad f_n(D_n^1) = U_n^1 \quad \text{or} \quad f_n(D_n^1) = U_n^2.$$

In any case that $f_n(0) \in U_n^2$ for all but finitely many n an integer N can be chosen such that

$$f_n(D_n^1) = U_n^1 \tag{1}$$

for all $n \geq N$ and thus

$$\operatorname{diam}\left(f_n(B_\delta(z_0) \cap \overline{\mathbb{D}})\right) < 2 \cdot \epsilon/2 = \epsilon.$$

So far this construction makes even sense in the case that f is holomorphic but not necessarily biholomorphic. But if f is biholomorphic then Equation (1) is true for all n if ϵ at the beginning of part (2) of the proof has been chosen small enough. This proves part (1b).

(3) For all $z \in \mathbb{D}$ and $n \in \mathbb{N}$, the triangle inequality yields

$$|f_n(z_0) - f(z_0)| \leq |f_n(z_0) - f_n(z)| + |f_n(z) - f(z)| + |f(z) - f(z_0)|.$$

Since f is continuous, we get the third term arbitrarily small by choosing $|z - z_0|$ small enough. By the result of part (2) in this proof, the first term becomes arbitrarily small as well, independently of n. Since $(f_n)_n$ is locally uniformly convergent the second term becomes arbitrarily small, too, if n is sufficiently large. Thus the left hand side is arbitrarily small if n is large enough. This proves the pointwise convergence of $(f_n)_n$ against f on $\partial\mathbb{D}$ if $f_n(0) \in U_n^2$ for all but finitely many n. Together with part (2) it follows in this case that $(f_n)_n$ converges locally uniformly on $\overline{\mathbb{D}}$ and thus uniformly on $\overline{\mathbb{D}}$ to f. Therefore we are done if f was biholomorphic.

More precisely we have proved the following: if for every $z_0 \in \partial\mathbb{D}$, there exists $\epsilon > 0$ such that for all but finitely many n

$$f_n(0) \in U_n^2 = U_n^2(z_0, \epsilon),$$

then $(f_n)_n$ converges uniformly to f on $\overline{\mathbb{D}}$.

If f is constant no subsequence $(f_{n_k})_k$ converges uniformly on $\overline{\mathbb{D}}$ since

$$\operatorname{diam}(f_{n_k}(\partial\mathbb{D})) \to \operatorname{diam}(\partial U) \neq 0.$$

Therefore, by the above considerations, one can find a point $z_0 \in \partial\mathbb{D}$ such that for all $\epsilon > 0$, there exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ such that $f_{n_k}(0) \in U_{n_k}^1(\epsilon)$ for all k and thus

$$f_{n_k}(D_{n_k}^2) = U_{n_k}^1(\epsilon).$$

Since $\operatorname{diam}(U_{n_k}^1(\epsilon)) < \epsilon$ and $|w - f_{n_k}(0)| < \epsilon$ if k is large enough we know that

$$|w - f_{n_k}(z)| < 2\epsilon$$

whenever k is large enough and $z \in \overline{\mathbb{D}} \cap \overline{D_{n_k}^2(\epsilon)}$. By Cantor's argument on diagonal sequences one will find a subsequence of $(f_n)_n$, which converges locally uniformly to w on $\overline{\mathbb{D}} \setminus \{z_0\}$. □

Theorem 2.16 (Sequences of Normalized Riemann Mappings)
Let $(U_n)_n$ be an admissible sequence of Carathéodory-discs converging to U and $f_n : \mathbb{D} \to U_n$ a biholomorphic mapping for all $n \in \mathbb{N}$. Then the sequence of the continuous extensions of f_n to $\overline{\mathbb{D}}$ converges uniformly to the continuous extension of a biholomorphic mapping $f : \mathbb{D} \to U$ to $\overline{\mathbb{D}}$ if one of the following three conditions holds

(1) The sequence of the images $f_n(0)$, $n \in \mathbb{N}$ converges to an interior point $w_0 \in U$ and the sequence $(f_n'(0))_{n\geq 1}$ converges as well.

(2) The sequence of the images $f_n(0)$, $n \in \mathbb{N}$ converges to an interior point $w_0 \in U$ and the sequence of the images $(f_n(1))$ converges to a simple boundary point of U.

(3) For three pairwise distinct boundary points ζ_1, ζ_2, ζ_3 of $\mathbb{D}$, the sequences $(f_n(\zeta_k))_n$ converge to w_k $(k = 1, 2, 3)$ such that w_1, w_2, w_3 are pairwise distinct, simple boundary points of U.

Proof. By assumption, $(f_n)_n$ is a bounded and thus normal family of injective holomorphic mappings. In each case the limit function of any convergent subsequence cannot be constant in all three cases. Thus by Theorem 2.15 the limit function f of any convergent subsequence maps $\mathbb{D}$ biholomorphically to U such that

(1) $f(0) = w_0$ and $f'(0) = \lim_{n\to\infty} f_n'(0)$,

(2) $f(0) = w_0$ and $f(1)$ is a simple boundary point of U,

(3) $f(\zeta_k) = w_k$ $(1 \le k \le 3)$.

A biholomorphic mapping f between $\mathbb{D}$ and U having one of these properties is uniquely determined. Thus in each case all convergent subsequences of $(f_n)_n$ have the same limit function. Therefore $(f_n)_n$ converges to f and again by Theorem 2.15 $(f_n)_n$ converges uniformly to f on $\overline{\mathbb{D}}$ as claimed. □

Theorem 2.17 (Continuous Dependence)
Let $\Lambda \subset \mathbb{C}$ be compact, $(U_\lambda)_{\lambda\in\Lambda}$ be a family of open simply connected domains and $f_\lambda : \mathbb{D} \to U_\lambda$ be biholomorphic mappings for $\lambda \in \Lambda$ with one of the following properties:

(1) the maps $\Lambda \to \mathbb{C},\ \lambda \mapsto f_\lambda(0)$ and $\Lambda \to S^1,\ \lambda \mapsto f_\lambda'(0)$ are continuous.

(2) the maps $\Lambda \to \mathbb{C},\ \lambda \mapsto f_\lambda(0)$ and $\Lambda \to \mathbb{C},\ \lambda \mapsto f_\lambda(1)$ are continuous and $f_\lambda(1)$ is a simple boundary point of U_λ for all $\lambda \in \Lambda$.

(3) there are three points $\zeta_1,\ \zeta_2,\ \zeta_3 \in \partial\mathbb{D}$ such that the maps $\Lambda \to \mathbb{C},\ \lambda \mapsto f_\lambda(\zeta^j)$ are continuous $(1 \le j \le 3)$, have disjoint graphs and that the image points are simple boundary points of U_λ.

Then the mapping

$$\Lambda \times \overline{\mathbb{D}} \to \mathbb{C}, (\lambda, z) \mapsto f_\lambda(z)$$

is continuous if there exists a continuous motion $\Phi : \Lambda \times \partial U \to \mathbb{C}$ of a locally connected set ∂U such that $\partial U_\lambda = \Phi(\lambda, \partial U)$ for all $\lambda \in \Lambda$. In this case the mappings

$$\Lambda \times \mathbb{D} \to \mathbb{C},\ (\lambda, z) \mapsto f_\lambda^{(k)}(z)$$

are continuous for all $k \ge 1$.

Proof. Assume that there exists a continuous motion

$$\Phi : \Lambda \times \partial U \to \mathbb{C}$$

of a locally connected set ∂U such that $\partial U_\lambda = \Phi(\lambda, \partial U)$ for all $\lambda \in \Lambda$. Then by Lemma 2.10, the family $(\partial U_\lambda)_{\lambda\in\Lambda}$ is uniformly locally connected. Let $(\lambda_n, z_n) \to (\lambda, z)$. Then by the definition of a continuous motion the sets U_{λ_n} converge in the sense of Definition 2.14 to U_λ which is locally connected. By Theorem 2.16, the sequence $(f_{\lambda_n})_n$ converges uniformly to f_λ on $\overline{\mathbb{D}}$ and thus

$$f_{\lambda_n}(z_n) \to f_\lambda(z), \quad n \to \infty.$$

This proves the continuity of $\Lambda \times \overline{\mathbb{D}} \to \mathbb{C}, (\lambda, z) \mapsto f_\lambda(z)$. Moreover for every sequence $(f_n : \mathbb{D} \to \mathbb{C})_n$ of holomorphic functions which converges locally uniformly to $f : \mathbb{D} \to U$ all derivatives of f_n converge locally uniformly to the corresponding derivative of f by a theorem of Weierstraß. □

2.4 Moduli of Quadrilaterals and Annuli

Let $\gamma : S^1 \to \mathbb{C}$ be a closed Jordan curve and Q the bounded component of $\mathbb{C} \setminus \gamma(S^1)$. Every choice of pairwise distinct points $z_1,\ z_2,\ z_3,\ z_4 \in \gamma(S^1)$ defines a quadrilateral $Q(z_1, z_2, z_3, z_4)$ with the vertices $z_1, \ldots, z_4$; in the following we only consider quadrilaterals such that the sequence of vertices follows the positive orientation of the boundary curve $\gamma(S^1)$. There exists a unique biholomorphic mapping φ from Q to the upper half plane, which can be extended continuously to the boundaries, such that

$$\begin{aligned} \varphi(z_1) &= 0, \\ \varphi(z_2) &= \infty, \\ \varphi(z_3) &= -1. \end{aligned}$$

Defining

$$\begin{aligned} k &:= \varphi(z_4) \in]-1, 0[, \\ a &:= \int_0^\infty \frac{1}{\sqrt{w(w+1)(w-k)}}\, dw, \\ b &:= \int_0^k \frac{1}{\sqrt{w(w+1)(w-k)}}\, dw \end{aligned}$$

we have the biholomorphic mapping

$$q : \mathbb{H} := \{z \in \mathbb{C} \ :\ \mathrm{Im}(z) > 0\} \longrightarrow \{z \in \mathbb{C} \ :\ 0 < \mathrm{Re}(z) < a, 0 < \mathrm{Im}(z) < b\},$$

$$z \mapsto \int_0^z \frac{1}{\sqrt{w(w+1)(w-k)}}\, dw.$$

Definition 2.18 (Moduli of Quadrilaterals)
The quotient b/a is called the **modulus of the quadrilateral** $Q(z_1, z_2, z_3, z_4)$ *and it is denoted by* $\mathrm{mod}\,(Q(z_1, z_2, z_3, z_4))$.

Let $A \subset \mathbb{C}$ be an annulus, i.e. an open connected subset of $\mathbb{C}$ such that the fundamental group of A is isomorphic to $\mathbb{Z}$. If both connected components of $\mathbb{C} \setminus A$ consist of a single point, then A can be mapped biholomorphically to the annulus $\mathbb{C}\setminus\{0\}$; if exactly one of the components of $\mathbb{C}\setminus A$ consists of a single point, then A can be mapped biholomorphically to the annulus $\mathbb{C}\setminus\overline{\mathbb{D}}$. In both cases the modulus $\mathrm{mod}(A)$ of A is defined to be ∞. Otherwise, A can be mapped biholomorphically to a standard annulus

$$A_R := \{z \in \mathbb{C} \ :\ |z| \in]1, R[\ \}$$

with a uniquely determined radius $R \in]1, \infty[$ and the modulus of A is defined by

$$\mathrm{mod}(A) := \log(R)/2\pi.$$

Lemma 2.19 (Moduli are Biholomorphically Invariant)

(1) Let $Q(z_1, z_2, z_3, z_4)$ *and* $Q'(z_1', z_2', z_3', z_4')$ *be quadrilaterals and*

$$f : Q(z_1, z_2, z_3, z_4) \to Q'(z_1', z_2', z_3', z_4')$$

biholomorphic such that $f(z_j) = z_j'$ $(1 \le j \le 4)$*. Then*

$$\operatorname{mod}\left(Q(z_1, z_2, z_3, z_4)\right) = \operatorname{mod}\left(Q'(z_1', z_2', z_3', z_4')\right).$$

(2) Let A *and* $A' \subset \mathbb{C}$ *be annuli and* $f : A \to A'$ *biholomorphic. Then*

$$\operatorname{mod}(A) = \operatorname{mod}(A').$$

Proof. The first part follows from the fact that there is no automorphism of the upper half plane different from the identity which has the three fixed points ∞, 0 and -1. The second part follows from the uniqueness of R in the definition of $\operatorname{mod}(A)$. □

Remark. The reason why we discuss annuli together with quadrilaterals is the following: consider an annulus A which is bounded by two simple closed curves γ_1 and γ_2. Then we can connect every point in γ_1 with any point in γ_2 by a simple curve γ and A can be considered as the quadrilateral $A \setminus \gamma$ with two of its sides identified. Thus a fundamental domain in the universal covering space of A is mapped biholomorphically to the interior of a quadrilateral Q describing A. By the definitions above, we have $\operatorname{mod}(Q) = \operatorname{mod}(A)$.

Theorem 2.20 (Continuous Dependence of Moduli)
Let $\Lambda \subset \mathbb{C}$ *be compact,* $Q(z_1, z_2, z_3, z_4)$ *be a quadrilateral and*

$$\Phi : \Lambda \times \partial Q \to \mathbb{C}$$

a continuous motion of ∂Q*. Then the mapping*

$$\Lambda \to \mathbb{R}^+, \lambda \mapsto \operatorname{mod}\Big(Q_\lambda\Big(\Phi(\lambda, z_1), \Phi(\lambda, z_2), \Phi(\lambda, z_3), \Phi(\lambda, z_4)\Big)\Big)$$

is continuous. The analogous statement holds for the modulus of annuli which depend continuously on a parameter.

Proof. For every $\lambda \in \Lambda$, consider the biholomorphic mapping

$$f_\lambda : \overline{\mathbb{D}} \to \overline{Q_\lambda}$$

such that $f_\lambda(-i) = \Phi(\lambda, z_1)$, $f_\lambda(1) = \Phi(\lambda, z_2)$ and $f_\lambda(i) = \Phi(\lambda, z_3)$. Then by Theorem 2.17, the mapping

$$\Lambda \times \overline{\mathbb{D}} \to \mathbb{C}, (\lambda, z) \mapsto f_\lambda(z)$$

is continuous.

Using the biholomorphic mapping

$$g : \mathbb{D} \to \{z : \mathrm{Im}(z) > 0\},\ g(z) := -\frac{1+i}{2} \cdot \frac{z+i}{z-1},$$

the map

$$k : \Lambda \to]-1, 0[: \lambda \mapsto g\Big((f_\lambda)^{-1}\Big(\Phi(\lambda, z_4)\Big)\Big)$$

is continuous and the claim follows from

$$\mathrm{mod}\Big(Q_\lambda\Big(\Phi(\lambda, z_1), \Phi(\lambda, z_2), \Phi(\lambda, z_3), \Phi(\lambda, z_4)\Big)\Big) =$$

$$\left(\int_0^\infty \frac{1}{\sqrt{z(z+1)(z-k(\lambda))}}\, dz\right)\left(\int_{k(\lambda)}^0 \frac{1}{\sqrt{(-z)(z+1)(z-k(\lambda))}}\, dz\right)^{-1}. \qquad \square$$

Lemma 2.21 (Small Moduli)
For every $L > 0$, there exists a function

$$M_L\ :]0, \infty[\to \mathbb{R}_0^+$$

with the following properties:

(1) For $z_1,\ z_2,\ z \in \mathbb{C}$ with $|z_1 - z_2| = L$, $|z_1 - z| = \epsilon$ and for every annulus $A_\epsilon \subset \mathbb{C}$ disconnecting $\{z_1, z_2\}$ from $\{z, \infty\}$ the modulus of A_ϵ is smaller than $M_L(\epsilon)$ and

(2) $\lim_{\epsilon \to 0} M_L(\epsilon) = 0$.

Proof. Since the modulus of an annulus A and the modulus of $\psi(A)$ are equal for every biholomorphic mapping $\psi : A \to A'$, we can assume that $z_1 = 0$, $z_2 = -1$ and $|z_1 - z| = \epsilon/L$. By the considerations in [Ahl, page 35 ff] the modulus of an annulus A, such that $\{0, -1\}$ is contained in the bounded component and z in the unbounded component of $\mathbb{C} \setminus A$, can be estimated by

$$\mathrm{mod}(A) \leq \frac{1}{2\pi} \log(|\alpha|) =: M_L(\epsilon)$$

where $|\alpha|$ is chosen such that

$$1 = \frac{4|\alpha|\epsilon/L}{(1 - |\alpha|)^2},$$

i.e.

$$|\alpha| = -2\frac{\epsilon}{L} + 1 + \sqrt{\left(2\frac{\epsilon}{L} - 1\right)^2 - 1}.$$

In the limit $\epsilon \to 0$ the equality above shows that $|\alpha| \to 1$ and therefore

$$\lim_{\epsilon \to 0} M_L(\epsilon) = 0. \qquad \square$$

3 Quasiconformal Mappings

In this section we give a short introduction into the theory of quasiconformal mappings and we state some theorems which will be needed in later chapters. A detailed discussion of quasiconformal mappings can be found for example in [Ahl] or [LV].

3.1 Definitions of Quasiconformal Mappings

Definition 3.1 (Geometric Definition)
*Let $G \subset \mathbb{C}$ be a domain in the complex plane, ψ an orientation preserving homeomorphism from G to its image and $K \in [1, \infty[$. Then ψ is called K-**quasiconformal** if*

$$\frac{\operatorname{mod}\Big(\psi(Q)\Big(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4)\Big)\Big)}{\operatorname{mod}(Q(z_1, z_2, z_3, z_4))} \leq K$$

for all quadrilaterals $Q(z_1, z_2, z_3, z_4)$ the closure of which is contained in G.

Definition 3.2 (Analytic Definition)
*Let $G \subset \mathbb{C}$ be a domain in the complex plane, ψ an orientation preserving homeomorphism form G to its image and $K \in [1, \infty[$. Then ψ is called K-**quasiconformal** if*

(1) ψ is absolutely continuous on lines in G and

(2) for almost all $z \in G$, we have

$$\max_\alpha |\partial_\alpha \psi(z)| \leq K \min_\alpha |\partial_\alpha \psi(z)|,$$

where $\partial_\alpha \psi$ denotes the dilatation of ψ in direction $e^{i\alpha}$.

Remark. Both definitions are equivalent, i.e. if a map is K-quasiconformal in the sense of Definition 3.1 then it is K-quasiconformal in the sense of Definition 3.2 and vice versa. A proof can be found in [Ahl, Chapter II].

As indicated in the second definition, a K-quasiconformal homeomorphism $\psi : G \to \psi(G)$ is partial differentiable almost everywhere and thus it is differentiable almost everywhere (compare [Ahl]). Thus the complex dilatation

$$\mu_\psi = \frac{\psi_{\overline{z}}}{\psi_z} = \frac{1/2(\psi_x + i\psi_y)}{1/2(\psi_x - i\psi_y)}$$

of ψ is defined almost everywhere and its absolut value is bounded by

$$|\mu_\psi| \leq \frac{K-1}{K+1} < 1.$$

Every K-quasiconformal homeomorphism defines a field of ellipses and thus an almost complex structure on G: for almost every $x \in G$ the map ψ induces an $\mathbb{R}$-linear map $T_x\psi$ from the tangent space of $G \subset \mathbb{C}$ at x to the tangent space of $\psi(G) \subset \mathbb{C}$ at $\psi(x)$ such that the image of a circle is an ellipse; this ellipse is uniquely determined up to a positive real factor.

The ratio of the major and the minor axis is given by

$$D_\psi = \frac{|\psi_z| + |\psi_{\overline{z}}|}{|\psi_z| - |\psi_{\overline{z}}|} = \frac{1+|\mu_\psi|}{1-|\mu_\psi|} \in \ [1, \infty[$$

and the direction of the major axis (respectively minor axis) is given by $1/2 \cdot \arg(\mu_\psi)$ (respectively $1/2 \cdot \arg(\mu_\psi) + \pi/2$).

This defines an ellipse field on G and thus an almost complex structure σ; the standard complex structure σ_0 is given by a field of circles.

On the other hand every measurable ellipse field on G defines an almost complex structure σ there and in this context it is important to know when there exists a quasiconformal homeomorphism $\psi : G \to \psi(G)$ which maps σ to σ_0, i.e. when there exists a quasiconformal homeomorphism integrating the almost complex structure σ. This answers Theorem 3.4 and Theorem 3.5 below.

3.2 Construction of Certain Quasiconformal Mappings

Lemma 3.3 (Quasiconformal Deformation of Triangles)
Let $\alpha \in \,]0, \pi[\,$, w, $z \in \mathbb{C} \setminus \{0\}$ and r, $s \in \mathbb{R}^+$. Then the mapping φ between the triangles $\triangle(0, w, re^{i\alpha}w)$ and $\triangle(0, z, se^{i\alpha}z)$ given by stretching along radial lines is a quasiconformal homeomorphism.

Proof. There exists $\epsilon > 0$ and a differentiable function

$$k : [0, \alpha] \to [\epsilon, 1/\epsilon]$$

such that φ can be written in the form

$$\varphi : \triangle(0, w, re^{i\alpha}w) \longrightarrow \triangle(0, z, se^{i\alpha}z)\ ,\ z \mapsto k(\arg(z))z.$$

Thus φ is differentiable as well and using the identity $\arg(z) = \arctan(y/x)$ we find

$$D\varphi(x,y) = \begin{pmatrix} k'(\arg(z))\frac{-yx}{x^2+y^2} + k(\arg(z)) & k'(\arg(z))\frac{x^2}{x^2+y^2} \\ k'(\arg(z))\frac{-y^2}{x^2+y^2} & k'(\arg(z))\frac{yx}{x^2+y^2} + k(\arg(z)) \end{pmatrix}$$

and

$$\det\Big((D\varphi)^t(D\varphi)(z)\Big) = k(\arg(z))^4, \quad \mathrm{Tr}\Big((D\varphi)^t(D\varphi)(z)\Big) = 2k(\arg(z))^2.$$

Therefore the eigenvalues of $(D\varphi)^t(D\varphi)$ are bounded from above and from below independent on z. This proves the claim. □

For proofs of the following two theorems see [Ahl, Chapter V.B and V.C]:

Theorem 3.4 (Measurable Riemann Mapping Theorem)
Let $u : \mathbb{P} \to \mathbb{D}$ be a measurable function such that $||u||_\infty < 1$, i.e. there exists $k \in [0, 1[$ such that $|u(z)| \le k$ for Lebesgue almost all $z \in \mathbb{P}$. Then there exists a unique quasiconformal mapping φ fixing the points 0, 1 and ∞ such that

$$\mu_\varphi := \frac{(\partial\varphi/\partial\overline{z})d\overline{z}}{(\partial\varphi/\partial z)dz} = u\frac{d\overline{z}}{dz}.$$ □

Theorem 3.5 (Dependence on a Parameter)
Let $\Lambda \subset \mathbb{C}$ be open and $(u_\lambda : \mathbb{C} \to \mathbb{C})_{\lambda\in\Lambda}$ be a family of measurable functions such that for every $z \in \mathbb{C}$ the mapping $\Lambda \to \mathbb{C}$, $\lambda \mapsto u_\lambda(z)$ is continuous (holomorphic). Assume moreover that there exists $k \in [0, 1[$ such that $||u_\lambda||_\infty \leq k$. Then there exists a family of $K = \frac{1+k}{1-k}$-quasiconformal mappings $(\varphi_\lambda)_{\lambda\in\Lambda}$ such that for every $z \in \mathbb{C}$, the map $\lambda \mapsto \varphi_\lambda(z)$ is continuous (holomorphic) and $\mu_{\varphi_\lambda} = u_\lambda \frac{d\bar{z}}{dz}$. □

3.3 Near Translations

In this section we introduce near translations. These describe in some situations the boundary behavior of biholomorphic mappings between strips; this concept was investigated by B. Bielefeld in [Bi] and by B. Branner and A. Douady in the errata of [BD].

Definition 3.6 (Near Translations)
Let r_0, $r_1 \in \mathbb{R}$. A diffeomorphism $\beta :]-\infty, r_0] \to]-\infty, r_1]$ is said to be **near translation** *if there are positive constants L, δ such that*

(a) $|\beta(x) - x| \leq L$ *and*

(b) $\delta < \beta'(x) < 1/\delta$

for all $x \in]-\infty, r_0[$.

Lemma 3.7 (About Near Translations)

(1) Let $\beta :]-\infty, r] \to]-\infty, s]$ and $\tilde\beta :]-\infty, s] \to]-\infty, t]$ be near translations. Then $\tilde\beta \circ \beta$ and β^{-1} are near translations.

(2) Let $\beta :]-\infty, r] \to]-\infty, s]$ be a diffeomorphism such that for some constant $\sigma > 0$, one has $\beta(x) - \sigma = \beta(x - \sigma)$ for all $x \in]-\infty, r]$. Then β is near translation.

Proof.

(1) Let L, $\tilde L$, δ, $\tilde\delta > 0$ such that for all suitable x

$$|\beta(x) - x| \leq L \qquad |\tilde\beta(x) - x| \leq \tilde L$$
$$\delta < \beta'(x) < 1/\delta \qquad \tilde\delta < \tilde\beta'(x) < 1/\tilde\delta.$$

Then
$$|\tilde\beta \circ \beta(x) - x| \leq |\tilde\beta(\beta(x)) - \beta(x)| + |\beta(x) - x| \leq \tilde L + L \quad \text{and}$$

$$\tilde\delta\delta \leq \tilde\beta'(\beta(x))\beta'(x) = (\tilde\beta \circ \beta)'(x) = \tilde\beta'(\beta(x))\beta'(x) \leq 1/(\tilde\delta\delta)$$

for all $x \in]-\infty, r]$; thus $\tilde\beta \circ \beta$ is also near translation. Similarly one proves that $\beta^{-1} :]-\infty, s] \to]-\infty, r]$ is near translation as well.

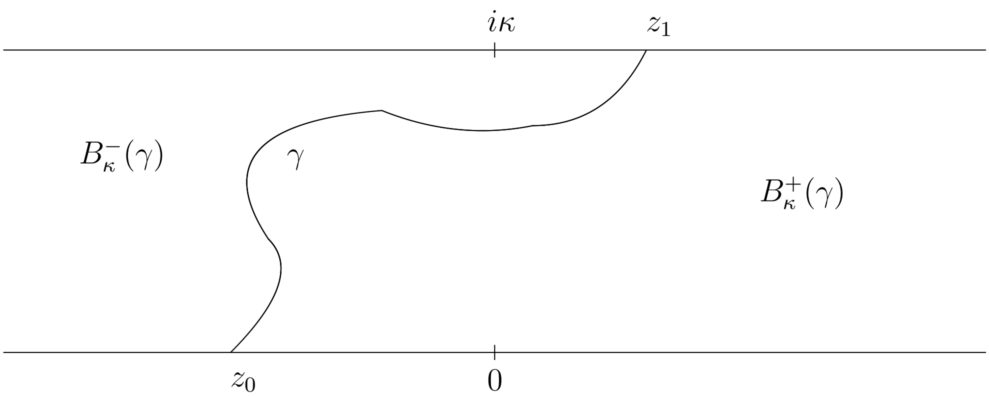

Figure 5: Straight Strips and Cutting Lines

(2) If the functional equation $\beta(x) - \sigma = \beta(x - \sigma)$ holds for all $x \in]-\infty, r]$, then

$$L := \max_{x \in [r-\sigma, r]} |\beta(x) - x| \quad \in \quad]0, \infty[\quad \text{and}$$

$$\delta := \max_{x \in [r-\sigma, r]} \{\beta'(x), 1/\beta'(x)\} \quad \in \quad [1, \infty[.$$

This proves the claim. □

Remark.
By the second part of this lemma, every translation

$$\beta :]-\infty, r[\to]-\infty, r+\sigma[, \; x \mapsto x + \sigma$$

is near translation.

Definition 3.8 (Strips)
Let $\kappa > 0$. The set

$$B_\kappa := \{z \in \mathbb{C} \; : \; 0 < \operatorname{Im}(z) < \kappa\}$$

is called **straight strip of width** κ. *Every simple curve $\gamma : [0,1] \to \overline{B_\kappa}$ with $\gamma(]0,1[) \subset B_\kappa$ and*

$$z_0 := \lim_{t \to 0} \gamma(t) \in \mathbb{R} \qquad \text{and} \qquad z_1 := \lim_{t \to 1} \gamma(t) \in i\kappa + \mathbb{R}$$

cuts B_κ into two connected components; γ is called a **cutting line of B_κ connecting z_0 and z_1**. *The connected component of $B_\kappa \setminus \gamma(]0,1[)$ containing $]-\infty, z_0[$ on its boundary is denoted by $B_\kappa^-(\gamma)$ and the other component is denoted by $B_\kappa^+(\gamma)$.*

Lemma 3.9 (Quasiconformal Continuation of Near Translations)
Let $\tilde\kappa$, $\kappa > 0$, $g :]-\infty, \tilde r] \to]-\infty, r]$ and $h :]-\infty, \tilde s] \to]-\infty, s]$ be near translations, $\tilde\gamma$ a cutting line of $B_{\tilde\kappa}$ connecting $\tilde r$ and $\tilde s + i\tilde\kappa$ and let γ be a cutting line of B_κ connecting r and $s + i\kappa$. Then there exists a quasiconformal homeomorphism

$$G : B_{\tilde\kappa}^-(\tilde\gamma) \to B_\kappa^-(\gamma)$$

such that $G|_{]-\infty, \tilde r]} = g$ and $G|_{i\tilde\kappa +]-\infty, \tilde s]} = i\kappa + h$.
If κ, $\tilde\kappa$, g, g', h and h' depend continuously on a parameter, then G depends also continuously on the parameter. The complex dilatation of G can be estimated independently on the parameter on every compact subset of the parameter space.

PROOF. Since g and h are near translations, there exist δ, $L > 0$ such that

$$|g(x) - x| \le L\ ,\ \ |h(x) - x| \le L$$

$$\delta < g'(x) < 1/\delta\ , \quad \delta < h'(x) < 1/\delta.$$

We choose $\rho \in\ ]-\infty, \min\{\tilde{r}, r\}[$ such that with

$$\begin{array}{lll} \tilde{\gamma}_\rho :]0,1[\to B_{\tilde{\kappa}} & , & t \mapsto \rho + i\tilde{\kappa}t, \\ \gamma_\rho :]0,1[\to B_\kappa & , & t \mapsto (1-t)g(\rho) + th(\rho) + i\kappa t, \end{array}$$

we have $B^-_{\tilde{\kappa}}(\tilde{\gamma}_\rho) \subset B^-_{\tilde{\kappa}}(\tilde{\gamma})$ and $B^-_\kappa(\gamma_\rho) \subset B^-_\kappa(\gamma)$. Let

$$G : B^-_{\tilde{\kappa}}(\tilde{\gamma}_\rho) \to B^-_\kappa(\gamma_\rho),\ (x,y) \mapsto \left((1-\frac{y}{\tilde{\kappa}})g(x) + \frac{y}{\tilde{\kappa}}h(x), y\frac{\kappa}{\tilde{\kappa}}\right).$$

Then $G|_{]-\infty,\rho]} = g|_{]-\infty,\rho]}$ and $G|_{i\tilde{\kappa}+]-\infty,\rho]} = i\kappa + h|_{]-\infty,\rho]}$; by construction, G is a homeomorphism. Now we prove that G is quasiconformal:

$$\begin{aligned} G_{\overline{z}}(x,y) &= \frac{1}{2}(G_x + iG_y) \\ &= \frac{1}{2}\left[\left((1-\frac{y}{\tilde{\kappa}})g'(x) + \frac{y}{\tilde{\kappa}}h'(x),\ 0\right) + \left((-\frac{\kappa}{\tilde{\kappa}},\ -\frac{1}{\tilde{\kappa}}g(x) + \frac{1}{\tilde{\kappa}}h(x))\right)\right] \\ &= \frac{1}{2}\left((1-\frac{y}{\tilde{\kappa}})g'(x) + \frac{y}{\tilde{\kappa}}h'(x) - \frac{\kappa}{\tilde{\kappa}},\ -\frac{1}{\tilde{\kappa}}g(x) + \frac{1}{\tilde{\kappa}}h(x))\right) \end{aligned}$$

$$\begin{aligned} G_z(x,y) &= \frac{1}{2}(G_x - iG_y) \\ &= \frac{1}{2}\left[\left((1-\frac{y}{\tilde{\kappa}})g'(x) + \frac{y}{\tilde{\kappa}}h'(x),\ 0\right) - \left(-\frac{\kappa}{\tilde{\kappa}},\ -\frac{1}{\tilde{\kappa}}g(x) + \frac{1}{\tilde{\kappa}}h(x))\right)\right] \\ &= \frac{1}{2}\left((1-\frac{y}{\tilde{\kappa}})g'(x) + \frac{y}{\tilde{\kappa}}h'(x) + \frac{\kappa}{\tilde{\kappa}},\ \frac{1}{\tilde{\kappa}}g(x) - \frac{1}{\tilde{\kappa}}h(x))\right) \end{aligned}$$

$$\begin{aligned} \frac{|G_{\overline{z}}|^2}{|G_z|^2}(x,y) &= \frac{\left((1-\frac{y}{\tilde{\kappa}})g'(x) + \frac{y}{\tilde{\kappa}}h'(x) - \frac{\kappa}{\tilde{\kappa}}\right)^2 + \left(\frac{1}{\tilde{\kappa}}g(x) - \frac{1}{\tilde{\kappa}}h(x))\right)^2}{\left((1-\frac{y}{\tilde{\kappa}})g'(x) + \frac{y}{\tilde{\kappa}}h'(x) + \frac{\kappa}{\tilde{\kappa}}\right)^2 + \left(\frac{1}{\tilde{\kappa}}g(x) - \frac{1}{\tilde{\kappa}}h(x))\right)^2} \\ &= 1 - \frac{4\frac{\kappa}{\tilde{\kappa}}\left((1-\frac{y}{\tilde{\kappa}})g'(x) + \frac{y}{\tilde{\kappa}}h'(x)\right)}{\left((1-\frac{y}{\tilde{\kappa}})g'(x) + \frac{y}{\tilde{\kappa}}h'(x) + \frac{\kappa}{\tilde{\kappa}}\right)^2 + \left(\frac{1}{\tilde{\kappa}}g(x) - \frac{1}{\tilde{\kappa}}h(x))\right)^2} \\ &\le 1 - \frac{4\frac{\kappa}{\tilde{\kappa}}\delta}{(\frac{1}{\delta} + \frac{\kappa}{\tilde{\kappa}})^2 + 4L^2\frac{1}{\tilde{\kappa}^2}} < 1 \end{aligned}$$

It remains to prove that G can be continued quasiconformally to $B^-_{\tilde{\kappa}}(\tilde{\gamma})$ such that

$$G(z) = \begin{cases} g(z) & : \ z \in [\rho, \tilde{r}] \\ h(z) & : \ z \in [\rho, \tilde{s}] \\ \gamma(\tilde{\gamma}^{-1}(z)) & : \ z \in \tilde{\gamma}([0,1]) \end{cases}$$

This is proven in [Ahl, Chapter IV]. The continuity statement follows immediately from the explicit construction above and from the explicit construction of quasiconformal homeomorphisms with a certain boundary behavior again in [Ahl, Chapter IV]. □

Corollary 3.10 (Riemann Mappings between Strips)
Let $\kappa > 0$ and $\tilde{\gamma}$ (respectively γ) be a cutting line of B_κ connecting $\tilde{z}_0$ and $\tilde{z}_1$ (respectively z_0 and z_1). Moreover, consider the unique homeomorphism

$$\Psi : \overline{B_\kappa^-(\tilde{\gamma})} \to \overline{B_\kappa^-(\gamma)}$$

with

$$\Psi(-\infty) = -\infty, \quad \Psi(\tilde{z}_0) = z_0, \quad \Psi(\tilde{z}_1) = z_1$$

such that its restriction to $B_\kappa^-(\tilde{\gamma})$ yields a biholomorphic mapping to $B_\kappa^-(\gamma)$. Then there exists $r > 0$ such that Ψ is of the form

$$\Psi(z) = z + \frac{\kappa}{\pi}\log(r) + O\left(\exp\left(z\frac{\pi}{\kappa}\right)\right) \qquad (z \to \infty).$$

If κ, $\tilde{z}_0$, z_0, $\tilde{z}_1$, z_1, $\tilde{\gamma}$ and γ depend continuously on a parameter then r and Ψ depend continuously on this parameter as well.

Proof. The mapping

$$e : B_\kappa \to \mathbb{H} := \{z \in \mathbb{C} \,:\, \mathrm{Im}(z) > 0\},\; z \mapsto \exp\left(z\frac{\pi}{\kappa}\right)$$

is biholomorphic. For $\tilde{U} := e\left(B_\kappa^-(\tilde{\gamma})\right)$ and $\tilde{V} := e\left(B_\kappa^-(\gamma)\right)$, the biholomorphic mapping

$$\widehat{\Psi} := e \circ \Psi \circ e^{-1} \;:\; \tilde{U} \longrightarrow \tilde{V}$$

can be extended continuously to the boundaries. Then

$$\widehat{\Psi}(0) = 0 \quad \text{and} \quad \widehat{\Psi}\left(\mathbb{R} \cap \partial\tilde{U}\right) = \mathbb{R} \cap \partial\tilde{V}.$$

By the Schwarzian reflection principle, this yields a biholomorphic mapping $\widehat{\Psi}$ between the Jordan domains

$$U := \tilde{U} \cup \;]e(\tilde{z}_1), e(\tilde{z}_0)[\; \cup \{z \in \mathbb{C} \,:\, \overline{z} \in \tilde{U}\}$$

and

$$V := \tilde{V} \cup \;]e(z_1), e(z_0)[\; \cup \{z \in \mathbb{C} \,:\, \overline{z} \in \tilde{V}\}.$$

These sets are bounded by the curves

$$[0,2] \to \mathbb{C}\;,\; t \mapsto \begin{cases} e(\tilde{\gamma}(t)) & : \; 0 \le t \le 1 \\ \overline{e(\tilde{\gamma}(2-t))} & : \; 1 < t \le 2 \end{cases}$$

and

$$[0,2] \to \mathbb{C}\;,\; t \mapsto \begin{cases} e(\gamma(t)) & : \; 0 \le t \le 1 \\ \overline{e(\gamma(2-t))} & : \; 1 < t \le 2 \end{cases}.$$

Therefore there exists $r > 0$ such that

$$\widehat{\Psi}(z) = rz + O(z^2) \qquad (z \to 0).$$

Transferring this formula back to B_κ yields:

$$\begin{aligned} \Psi(z) &= e^{-1}\left((re(z) + O(e(z)^2)\right) \\ &= z + \frac{\kappa}{\pi} log(r) + O\left(\frac{1}{r}\exp\left(z\frac{\pi}{\kappa}\right)\right). \end{aligned}$$

The continuity statement follows immediately from Theorem 2.17. □

Remark. By this Corollary

(1) $\lim_{|z|\to\infty}(\Psi(z) - z)$ exists,

(2) $\lim_{|z|\to\infty} \Psi'(z) = 1$

and thus the mappings

$$\begin{aligned}]-\infty, \tilde{z}_0[\to \mathbb{R} \quad &, \quad t \mapsto \Psi(t) \\ ]-\infty, \tilde{z}_1 - i\kappa[\to \mathbb{R} \quad &, \quad t \mapsto \Psi(t + i\kappa) - i\kappa \end{aligned}$$

are near translation.

4 Julia Sets and Multibrot Sets

In this section we first give a short introduction to the dynamics of unicritical complex polynomials and their parameter spaces. Then we introduce the concept of piecewise dynamic homeomorphisms and in this context we prove some statements which turn out to be important in Section 5

4.1 Introduction

For an integer $d \geq 2$, a unicritical polynomial $P(z)$ of degree d is a complex polynomial of degree d which has exactly one critical point. By conjugation with an affine map, this polynomial can be assumed to have the form $P_c : z \mapsto z^d + c$; the parameter c is uniquely defined up to multiplication with a $(d-1)$-th root of unity. In the following let $\zeta := e^{2\pi i/d}$ and $\tilde{\zeta} := e^{2\pi i/(d-1)}$.

Lemma 4.1 (Unicritical Polynomials)
Every unicritical polynomial P of degree $d \geq 2$ is affinely conjugate to a polynomial P_c; the parameter c is uniquely determined up to multiplication by a $(d-1)$-th root of unity.

Proof. The existence of such an affine conjugation is easy. To prove uniqueness we argue as follows: let c, $c' \in \mathbb{C}$ be such that P_c is affinely conjugate to $P_{c'}$. Then there are $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$ such that

$$\begin{aligned} z^d + c &= a\left(\left(\frac{1}{a}z - \frac{b}{a}\right)^d + c'\right) + b \\ &= a^{1-d}(z-b)^d + ac' + b. \end{aligned}$$

Therefore $a^{d-1} = 1$, $b = 0$ and $ac' = c$. □

Definition 4.2 (Julia Sets and Multibrot Sets)
The **filled-in Julia set** K_P *of a polynomial P is defined as the set of points with bounded orbit, i.e.*

$$K_P := \left\{ z \in \mathbb{C} \;:\; \left(P^n(z) = P \circ \ldots \circ P(z)\right)_{n \geq 1} \text{ is bounded} \right\}.$$

Its boundary is called the **Julia set** J_P *of P. For every parameter $c \in \mathbb{C}$ and a fixed degree $d \geq 2$, let*

$$K_c := K_{P_c} \quad \text{and} \quad J_c := J_{P_c}.$$

The **Multibrot set** $\mathcal{M}_d$ *is defined by*

$$\mathcal{M}_d := \{ c \in \mathbb{C} \;:\; J_c \text{ is connected} \}.$$

In the case $d = 2$ this set is also called the **Mandelbrot set**.

A proof of the following statement for degree $d = 2$ can be found in [CG] or [McM1]; a proof for the general case is given in [E]:

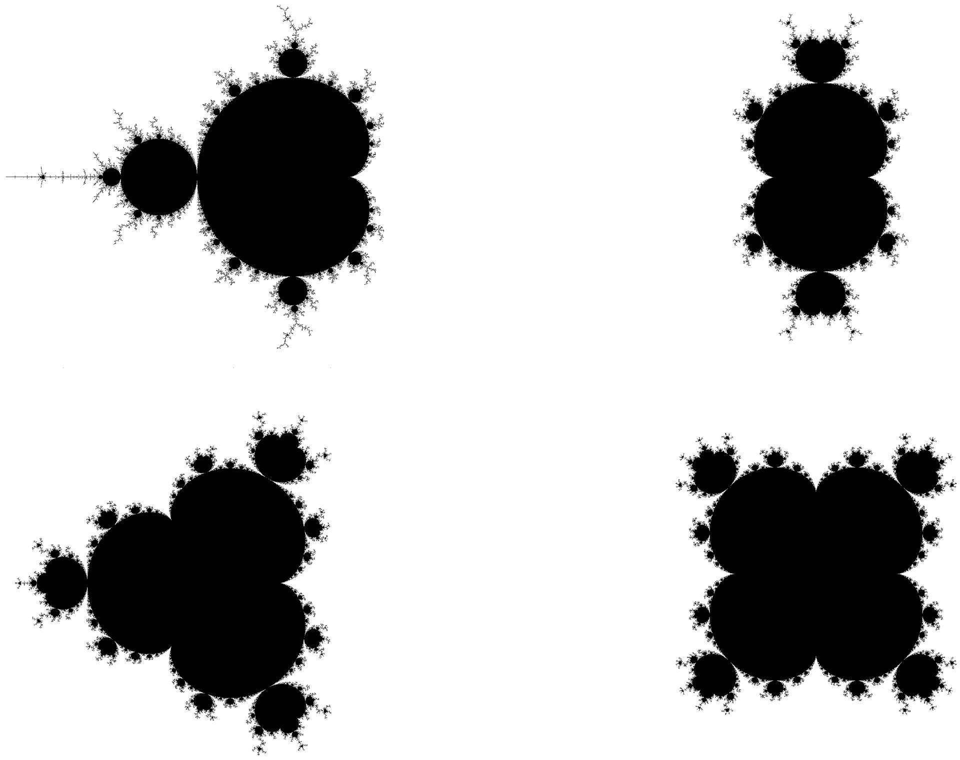

Figure 6: The Multibrot sets $\mathcal{M}_d$ of degree $d = 2$ (upper left), $d = 3$ (upper right), $d = 4$ (lower left) and $d = 5$ (lower right).

Lemma 4.3 (Properties of $\mathcal{M}_d$)
The Multibrot set $\mathcal{M}_d$ is nonempty, compact, full and connected. □

Definition 4.4 (Period and Preperiod)
For $c \in \mathcal{M}_d$, let z be a preperiodic point of P_c. Then the smallest non negative integers k and l with

$$P_c^k(z) = P_c^{k+l}(z)$$

are called the **preperiod** *and* **period** *of z. In the case $k = 0$, the point z is said to be* **attractive** *(***repelling***) if*

$$\left|\frac{\partial}{\partial z} P_c^l(z)\right| < 1 \quad \left(\left|\frac{\partial}{\partial z} P_c^l(z)\right| > 1\right)$$

Remark. It may be unusual to define the period of strictly preperiodic points. But this makes the statements in the later sections much easier to talk about.

If the polynomial P_c has an attractive periodic point of period l, then the parameter c is contained in the interior of $\mathcal{M}_d$; the corresponding connected component H of $\mathrm{int}(\mathcal{M}_d)$ is called a **hyperbolic component of period** l. On the other hand, for every parameter $c \in H$, the polynomial P_c has an attractive periodic orbit $\{z_c, P_c(z_c), \ldots, P_c^{l-1}(z_c)\}$. Its multiplier defines a mapping

$$\mu_H \;:\; H \to \mathbb{D}\,,\; c \mapsto \frac{\partial}{\partial z} P_c^l(z_c).$$

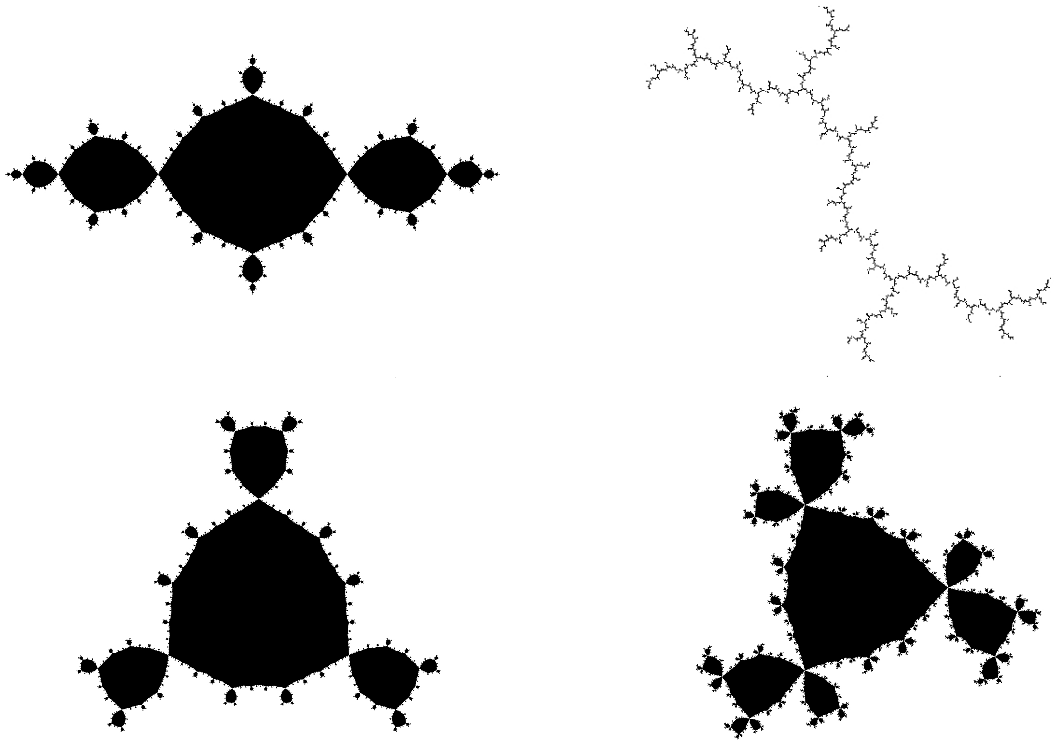

Figure 7: Filled-in Julia sets of the polynomials $z \mapsto z^2 - 1$ (upper left), $z \mapsto z^2 + i$ (upper right), $z \mapsto z^3 + i$ (lower left) and $z \mapsto z^3 - 0.5575\ldots + 0.5403\ldots i$ (lower right).

Theorem 4.5 (The Multiplier Map)
The mapping $\mu_H : H \to \mathbb{D}$ is a $(d-1)$-fold covering map with one branch point. It extends continuously to a neighborhood of $\overline{H}$. □

The unique branch point of μ_H is called the **center** c_0 of H; this is the only parameter $c \in H$ such that the critical point of P_c is periodic. It is conjectured that every component of $\mathrm{int}(\mathcal{M}_d)$ is hyperbolic. This would follow from

Conjecture 4.6 (MLC)
The Multibrot set $\mathcal{M}_d$ is locally connected.

For every parameter $c \in \mathcal{M}_d$, the **Böttcher mapping**

$$\Phi_c : \mathbb{P} \setminus K_c \to \mathbb{P} \setminus \overline{\mathbb{D}}$$

is the unique biholomorphic mapping $\mathbb{P} \setminus K_c \to \mathbb{P} \setminus \overline{\mathbb{D}}$ conjugating P_c to

$$\mathbb{P} \setminus \overline{\mathbb{D}} \to \mathbb{P} \setminus \overline{\mathbb{D}},\ z \mapsto z^d$$

such that $\lim_{z\to\infty} \Phi_c(z)/z = 1$. The Green's function $G_c : \mathbb{P} \to \mathbb{R}_0^+$ is defined by

$$G_c(z) := \begin{cases} \log|\Phi_c(z)| & : \quad z \in \mathbb{P} \setminus K_c \\ 0 & : \quad z \in K_c \end{cases} .$$

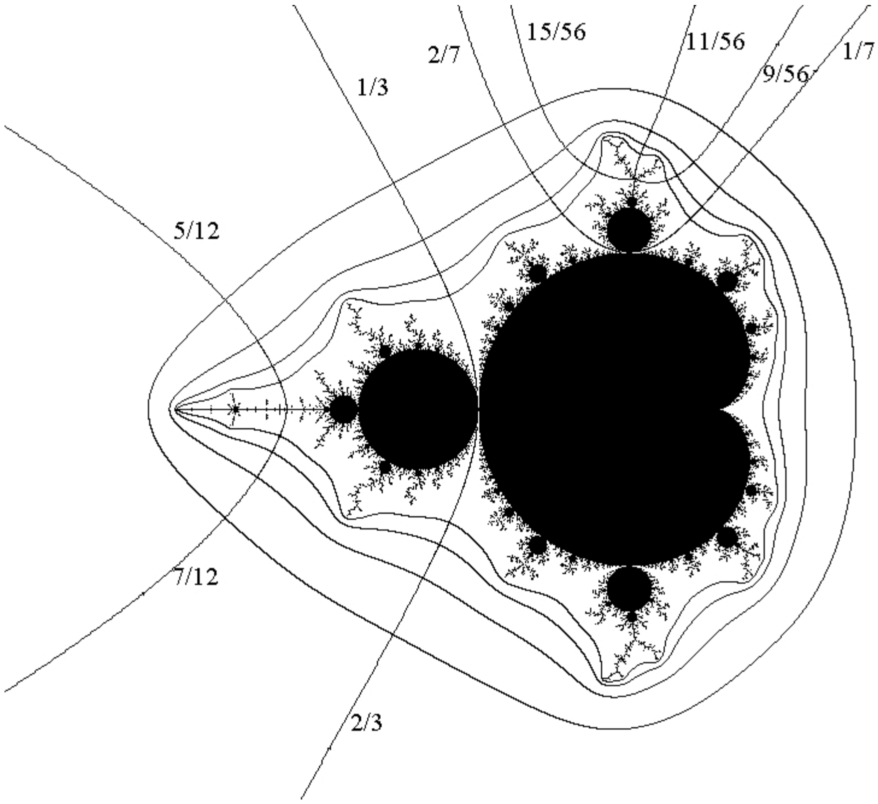

Figure 8: Parameter rays and equipotentials for the Mandelbrot set.

For $\eta > 0$, let

$$X_c^\eta := \{z \in \mathbb{C} \ : \ G_c(z) < \eta\}.$$

The polar coordinates in the complement of the closed unit disc are transferred by Φ_c^{-1} to the complement of K_c; we thus get dynamically defined coordinates $\mathbb{P} \setminus K_c$: the dynamic ray with angle $\vartheta \in \mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ is given by

$$\mathrm{R}_\vartheta := \{\Phi_c^{-1}(re^{2\pi i\vartheta}) \ : \ 1 < r < \infty\}$$

and the equipotential with potential $\rho > 0$ is given by

$$\mathrm{E}_\rho := \{\Phi_c^{-1}(e^\rho e^{2\pi i\vartheta}) \ : \ \vartheta \in \mathbb{S}^1\} = G_c^{-1}(\rho).$$

From the conjugation property of Φ_c we conclude that

$$P_c(\mathrm{R}_\vartheta) = \mathrm{R}_{d\vartheta} \quad \text{and} \quad P_c(\mathrm{E}_\rho) = \mathrm{E}_{d\rho}.$$

Since every rational angle in $\mathbb{S}^1$ is periodic or preperiodic under the map

$$\mathbb{S}^1 \to \mathbb{S}^1, \ \vartheta \mapsto d\vartheta,$$

every dynamic ray with rational angle is periodic or preperiodic in the dynamics of P_c. Often it is important to know whether or not a dynamic ray lands, i.e. whether the limit

$$\lim_{r \to 1} \Phi_c^{-1}(re^{2\pi i\vartheta})$$

exists or not.

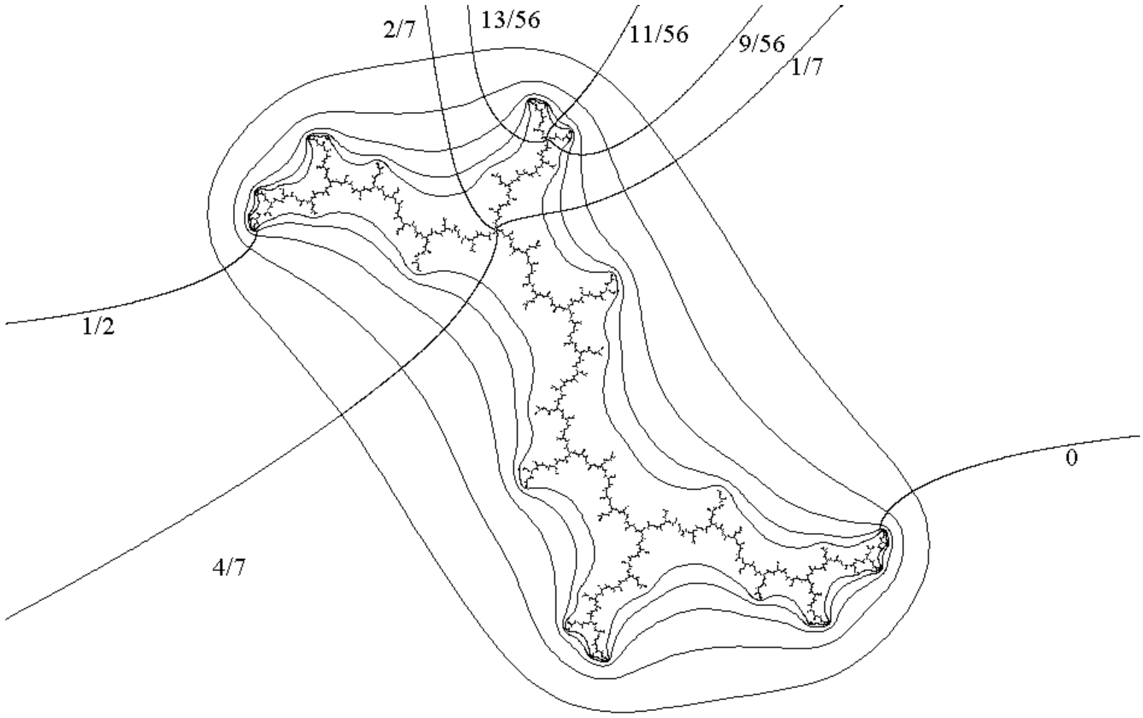

Figure 9: Dynamic rays and equipotentials for the Julia set of a quadratic polynomial.

If the filled-in Julia set of P_c is locally connected, then the theorem of Carathéodory (Theorem 2.1) tells us that the Böttcher map has a continuous extension to $\partial K_c = J_c$ and thus all dynamic rays land. If K_c is not locally connected not all the dynamic rays must land; but the rays with rational angles land in any case (Lemma 4.7). Of course, two different dynamic rays do not intersect but they can land at the same point. Later we will often look for **pinching points**, i.e. landing points of at least two dynamic rays. For every point $z \in J_c$, we denote by

$$\Theta(z) \quad \text{and} \quad \text{rays}(z)$$

the angles of the dynamic rays landing at z and the union of all dynamic rays landing at z together with z itself. It is important to know which dynamic rays land and which points are the landing points of dynamic rays (see for example [Mi, §18]):

Lemma 4.7 (Landing Properties of Dynamic Rays)
Let $c \in \mathcal{M}_d$. Then

(1) every periodic (preperiodic) dynamic ray lands at a periodic (preperiodic) point in J_c;

(2) every periodic (preperiodic) point in J_c is the landing point of finitely many dynamic rays. □

Remark. For a repelling periodic point z of period l disconnecting K_c we denote its periodic orbit by $z_0, \ldots, z_{l-1}$ such that

(1) $P_c(z_j) = \begin{cases} z_{j+1} & : \ j \in \{0, \ldots, l-1\} \\ z_0 & : \ j = l \end{cases}$;

(2) z_1 is the **characteristical point**, i.e. the connected component of $K_c \setminus \{z_1\}$ containing the critical value contains none of the points $z_0, z_2, \ldots, z_{l-1}$. The point z_0 is called the **pre-characteristical point** of this orbit.

The dynamic rays landing at z_1 divide the complex plane into several connected components; the one containing the critical value is called the **characteristic sector at** z_1.

Even for $c \in \mathbb{C} \setminus \mathcal{M}_d$ an injective holomorphic mapping Φ_c exists in a simply connected neighborhood U_∞ of infinity, which conjugates P_c to $z \mapsto z^d$ such that $\lim_{z\to\infty} \Phi_c(z)/z = 1$. Using the conjugation property

$$\Phi_c(P_c(z)) = \Phi_c(z)^d,$$

we can extend the map Φ_c inductively to $P_c^{-m}(U_\infty)$ as long as $0 \notin P_c^{-m}(U_\infty)$. In particular, it turns out that $\Phi_c(c)$ is defined. This yields a mapping

$$\Phi : \mathbb{P} \setminus \mathcal{M}_d \to \mathbb{P} \setminus \overline{\mathbb{D}}$$

which indeed is biholomorphic (for a proof see for example [DH1], [CG]). Considering Φ instead of Φ_c one can define parameter rays and equipotentials in the complement of the Multibrot set analogously as in the dynamic plane. Some landing properties of the parameter rays can be found in Lemma 4.12.

Now we give a short description of the structure of the Multibrot sets using parameter rays at rational angles. The details in the case $d = 2$ can be found in [Schl4] and the general case $d \geq 2$ is treated in [E]. Consider a hyperbolic component H of period l and a parameter $c \in \partial H$ such that $\mu_H(c) = 1$. If c is the landing point of two parameter rays, then it is called **essential root of** H; if c is the landing point of exactly one parameter ray, then it is called a **non-essential root of** H. The following lemma is proved in [E]:

Lemma 4.8 (Roots of Hyperbolic Components)
Every hyperbolic component of period $l > 1$ has exactly one essential root and $d - 2$ non-essential roots. □

For every hyperbolic component H, the parameter rays landing at all its roots together with the $d-1$ paths $\mu_H^{-1}([0,1])$ from the center to the roots of H divide the parameter plane into d connected components. One of them contains the parameter 0; the intersections of $\mathcal{M}_d$ with one of the $d-1$ other components are called the **sectors of** H. Thus every hyperbolic component defines $d-1$ sectors. The restriction of the multiplier map μ to the intersection of H with any sector defines natural coordinates there; these are called the **internal coordinates** of the intersection of H with the considered sector of H.

The points $c \in \partial H$ with multiplier

$$\mu_H(c) = e^{2\pi i p/q}, \quad p/q \in \mathbb{Q}/\mathbb{Z}$$

are called **parabolic parameters**. Every parabolic parameter c with $p/q \neq 0$ is the essential root of a hyperbolic component $\widetilde{H}$ of a period $m \in l\mathbb{N}$. Then c is the landing point of exactly two periodic parameter rays. These rays together with their landing point c split the parameter plane into two connected components. The one not containing the parameter 0 is called a p/q-**wake** of H. The intersection of a p/q-wake with $\mathcal{M}_d$ is said to be a p/q-**sublimb** of H; every sector of H contains exactly one p/q-wake and thus one p/q-sublimb.

Definition 4.9 (Misiurewicz Points)
A parameter $c \in \mathcal{M}_d$ is called a **Misiurewicz point** *if the critical value $z = c$ of P_c is strictly preperiodic. In this case the preperiod and period of c are defined to be the preperiod and period of the critical value $c = P_c(0)$ of P_c. A Misiurewicz point is called a* **dyadic Misiurewicz point** *if it is the landing point of a parameter ray with angle $\frac{a}{(d-1)d^n} \in \mathbb{S}^1$ (a, $n \in \mathbb{N}$).*

Every Misiurewicz point c is contained in the boundary of $\mathcal{M}_d$ and the Misiurewicz points are dense in $\partial\mathcal{M}_d$. Then $\mathcal{M}_d \setminus \{c\}$ may or may not be connected (it is iff c sits on an antenna tip of $\mathcal{M}_d$). See Section 4.7 for a more detailed discussion of Misiurewicz points.

The landing properties of the rational parameter rays can be described as follows (for details and the proof we refer to [E]):

Theorem 4.10 (Structure Theorem for Multibrot Sets)

(1) Every periodic parameter ray lands at a parabolic parameter of $\mathcal{M}_d$.

(2) Every parabolic parameter is the landing point of exactly one or exactly two periodic parameter rays.

(3) Every strictly preperiodic parameter ray lands at a Misiurewicz point of $\mathcal{M}_d$.

(4) Every Misiurewicz point is the landing point of at least one but finitely many preperiodic parameter rays. □

Later we need to know for which parameters certain rational dynamic rays land together. The proofs of the following two statements can be found in [Schl4]:

Lemma 4.11 (Dynamic Rays with a Common Endpoint)
Let $c_0 \in \mathcal{M}_d$ and ϑ_1, $\vartheta_2 \in \mathbb{Q}/\mathbb{Z}$ such that in the dynamic plane of P_{c_0} the dynamic rays with angles ϑ_1 and ϑ_2 land together at a repelling periodic point z. Assume that the orbit of z does not contain the critical point. Then the considered dynamic rays land together for all parameters c in a small neighborhood of c_0 and the landing point $z = z(c)$ depends continuously on the parameter. □

Lemma 4.12 (Dynamic Rays and Parameter Rays)

(1) Let $c \in \mathcal{M}_d$ be the landing point of two periodic parameter rays with angles ϑ_1 and ϑ_2 of period l. Then the parameters in the wake bounded by the parameter rays with angles ϑ_1 and ϑ_2 are exactly the ones for which the dynamic rays with these angles land together.

(2) For $c \in \mathcal{M}_d$, consider a point $z \in K_c$ of period l such that $q \geq 2$ dynamic rays land at z. Let ϑ_1 and ϑ_2 denote the angles of the two dynamic rays bounding the characteristic sector at the characteristic point of the orbit $\{z, P_c(z), \ldots, P_c^{l-1}(z)\}$. Then the parameter rays with angles ϑ_1 and ϑ_2 land at a common parameter c_0, which separates the parameter c from 0.

(3) Let $c \in \mathcal{M}_d$ be the landing point of $q \geq 2$ strictly preperiodic parameter rays with angles $\vartheta_1, \ldots, \vartheta_q$. Then the set of parameters for which the dynamic rays with angles $\vartheta_1, \ldots, \vartheta_q$ land at a common point is bounded by two parameter rays with angles in $\{d^i\vartheta_j \ : \ i \geq 1, 1 \leq j \leq q\}$. The Misiurewicz point c is contained in the interior of this wake. □

For every parameter $c \in \mathcal{M}_d$, the dynamic rays with angles

$$\frac{0}{d-1}, \ \frac{1}{d-1}, \ \ldots, \ \frac{d-2}{d-1}$$

land at distinct fixed points of P_c denoted by $\beta_0, \ldots, \beta_{d-2}$ (compare [GM]; if $d = 2$, then we simply omit the index and write β for the corresponding fixed point). Since the fixed points of the polynomial P_c are the solutions of the equation

$$z^d - z + c = 0,$$

we have found $d - 1$ of the d fixed points and there is exactly one fixed point remaining: it is called the α-fixed point of P_c. Assuming that c is not contained in the main cardioid of $\mathcal{M}_d$, this fixed point is the landing point of at least two dynamic rays.

A combinatorial description of the Multibrot sets is given by the internal addresses which have been investigated by D. Schleicher (compare [LS]). The internal address of a parameter $c \in \mathcal{M}_d$ is an increasing sequence of integers which starts with entry $n^{(1)} = 1$ and may be finite or infinite. This sequence is constructed as follows: consider all pairs of parameter rays which separate the parameter 0 from the parameter c. Then there is one ray pair of minimal period $n^{(2)}$ with this property; let $c^{(2)}$ be their common landing point. Then there exists a pair of parameter rays with minimal period $n^{(3)}$ such that $c^{(2)}$ is separated from c by this ray pair; denote their common landing point by $c^{(3)}$ .... If c is contained in the closure of a hyperbolic component of $\mathcal{M}_d$ then this procedure ends after finitely many steps. Altogether the **internal address** of c is

$$n^{(1)} \to n^{(2)} \to n^{(3)} \to \ldots.$$

The angled internal address of a parameter c is constructed as follows: for every $j \geq 2$, the parameter $c^{(j)}$ is the essential root of a hyperbolic component $H^{(j)}$. Let

$$s_j \in \{1, \ldots, d-1\} \quad \text{and} \quad p_j/q_j \in \mathbb{Q}/\mathbb{Z}$$

such that c is contained in the p_j/q_j-sublimb of the s_j-th sector of $H^{(j)}$. Then the **angled internal address** of c is defined by the sequence

$$n^{(1)}_{p_1/q_1}(s_1) \to n^{(2)}_{p_2/q_2}(s_2) \to n^{(3)}_{p_3/q_3}(s_3) \to \ldots .$$

Definition 4.13 (Stars and Regions)
For $c \in \mathcal{M}_d$, let z, z_1, …, $z_n \in J_c$ be landing points of at least two dynamic rays each and $w \in K_c$.

(a) We say that z **separates** *z_1* **from** *z_2 if these points are contained in two different connected components of $J_c \setminus \{z\}$.*

(b) Moreover,

$$]z_1, z_2[\ := \ \{z' \in J_c \ ; \ z' \text{ separates } z_1 \text{ from } z_2\};$$

The sets $[z_1, z_2[$, $]z_1, z_2]$, $[z_1, z_2]$ are defined analogously.

(c) A point $z \in K_c$ is called a **pinching point** *if $K_c \setminus \{z\}$ is not connected.*

*(d) The connected component of $\mathbb{C} \setminus \big(\text{rays}(z) \cup \ldots \cup \text{rays}(\zeta^{d-1} z)\big)$ containing the critical point is called the z-**star**; we denote it by $\text{star}(z)$.*

(e) The **region before** *z is the connected component of $\mathbb{C} \setminus \text{rays}(z)$ containing the critical point; it is denoted by $\mathcal{R}_0(z)$.*

(f) The set $\mathbb{C} \setminus \overline{\mathcal{R}_0(z)}$ is called the **region behind** *z; it is denoted by $\mathcal{R}_b(z)$.*

(g) For $w \in \mathbb{C} \setminus \text{rays}(z)$, the connected component of $\mathbb{C} \setminus \text{rays}(z)$ containing w is called the **region behind** *z* **pointing to** *w. It is denoted by $\mathcal{R}_w(z)$.*

(h) The connected component of

$$\mathbb{C} \setminus \Big(\text{rays}(z_1) \cup \ldots \cup \text{rays}(z_n)\Big)$$

which contains all points z_1, …, z_n on the boundary is called the **region between the points** *z_1, …, z_n. We denote this set by $\mathcal{R}(z_1, \ldots, z_n)$.*

Remark. Later we will consider mappings between two regions $\mathcal{R}(z_1, z_2)$ and $\mathcal{R}(\tilde{z}_1, \tilde{z}_2)$; these mappings will map these regions to each other such that z_1 is mapped to $\tilde{z}_1$ and z_2 is mapped to $\tilde{z}_2$.

Definition 4.14 (Branches)

(a) Let $c \in \mathcal{M}_d$ and $z \in K_c \setminus \{0\}$ be the landing point of at least two dynamic rays. Then the connected components of $K_c \setminus \{z\}$ not containing the critical point are called the **branches of K_c behind** *z. The connected component containing the critical point is called the* **branch of K_c before** *z.*

(b) For every parameter $c \in \mathcal{M}_d$, which is the landing point of at least two parameter rays the connected components of $\mathcal{M}_d \setminus \{c\}$ not containing the parameter 0 are called the **branches of $\mathcal{M}_d$ behind** *c. The component containing the parameter 0 is called the* **branch of $\mathcal{M}_d$ before** *c.*

4.2 Sectors and Their Opening Moduli

In this section we introduce invariant sectors and their opening moduli as a slight generalization of the sectors around the α-fixed point for quadratic polynomials in [BD]. In the examples below we define some types of sectors. Some of them contain parts of the filled-in Julia set and these are more difficult to handle than the sectors which are completely contained in $\mathbb{C} \setminus K_c$.

For $c \in \mathcal{M}_d$, consider a preperiodic point $z_0 \in K_c$ of preperiod k and period l which does not contain the critical point in its orbit such that $P_c^k(z_0)$ is repelling. Then for every $n \geq 1$, the iterate P_c^{nl} maps a small neighborhood of $P_c^k(z_0)$ biholomorphically to a small neighborhood of $P_c^k(z_0)$. In other words, the n-th iterate of

$$Q := P_c^{-k} \circ P_c^l \circ P_c^k$$

maps a neighborhood U_0 of z_0 biholomorphically to a neighborhood $V_0 \supset U_0$ of z_0 if we choose the branch of P_c^{-k} correctly. We assume that ∂U_0 and ∂V_0 are Jordan curves and that they do not intersect (for example by using discs in linearizing coordinates). Obviously,

$$x \sim y \quad :\Leftrightarrow \quad \exists k \in \mathbb{Z} \quad : \quad x = Q^k(y)$$

defines an equivalence relation on V_0. The quotient $V_0/_\sim$ is an annulus and we consider certain parts of this annulus now:

Definition 4.15 (Opening Modulus)
Let w_l, $w_r \in \partial V_0$ be two distinct points and

$$\gamma_l,\ \gamma_r\ :\ [0,1] \to \overline{V_0}$$

be two simple curves such that

$$\gamma_l(0) = \gamma_r(0) = z_0, \quad \gamma_l(1) = w_l, \quad \gamma_r(1) = w_r$$

and

$$\gamma_l(]0,1[),\ \gamma_r(]0,1[)\ \subset\ V_0$$

which intersect only in z. Then these two curves disconnect V_0 into two **sectors at** *z_0. Let S be the one to the left of γ_r. The curves γ_l and γ_r will be called the* **left** *and* **right boundary curves** *and the points z_0, w_l and w_r will be called the* **vertices of** *S. The sector S is called a* **Q^n-invariant sector** *if*

$$Q^{-n}(S) = S \cap Q^{-n}(V_0).$$

In this case $S/_\sim$ is a (Q^n-invariant) annulus. If n has been chosen minimal, its modulus is called the **opening modulus** *of S and is denoted by*

$$\operatorname{mod}(S).$$

Remark. This modulus is independent on the choice of U_0. Later we will often talk about Riemann mappings between sectors S and S' at the same point z_0. Then we always have in mind the unique Riemann mapping which maps the vertices z_0, w_l, w_r of S to the vertices z_0, w_l', w_r' of S':

$$z_0 \mapsto z_0, \quad w_l \mapsto w_l', \quad w_r \mapsto w_r'.$$

An easy consequence of Theorem 2.20 is the following:

Lemma 4.16 (Continuous Dependence of Opening Moduli)
Let $\Lambda \subset \mathbb{C}$ and for every $c \in \Lambda$, let $z_c \in K_c$ be a point of preperiod k and period l depending continuously on the parameter c such that $P_c^k(z_c)$ is repelling. Suppose that $(S_c)_{c\in\Lambda}$ is a family of $Q_c^n := (P_c^{-k} \circ P_c^{nl} \circ P_c^k)$-invariant sectors at z_c for an integer $n \geq 1$ such that $(\partial S_c)_{c\in\Lambda}$ is a continuous motion of a locally connected set ∂S. Then $\operatorname{mod}(S_c)$ depends continuously on the parameter c. □

Examples:
For $c \in \mathcal{M}_d$, let z_0 be a point of preperiod k and period l such that $P_c^k(z_0)$ is repelling and $0 \notin \{z_0,\, P_c(z_0),\, \ldots\}$.

(1) Let $\vartheta \in \mathbb{S}^1$ be the angle of a dynamic ray landing at z_0. Then for every width $w > 0$ and for every potential $\eta > 0$ with $w\eta < \pi$, let

$$S_w^\eta(\vartheta) := \left\{\Phi_c^{-1}\left(\exp(z)\right) \in \mathbb{C} \;:\; \Big|\mathrm{Im}(z) - 2\pi\vartheta\Big| < w\Big|\mathrm{Re}(z)\Big|,\; 0 < \mathrm{Re}(z) < \eta\right\}.$$

There exists $n \geq 1$ such that the dynamic ray with angle ϑ is preperiodic with preperiod k and period nl. The sector $S_w^\eta(\vartheta)$ is Q^n-invariant: this sector is one of the two connected components of

$$X_c^\eta \Big\backslash \Big(\begin{array}{l} \{\Phi_c^{-1}\left(\exp(x + i(2\pi\vartheta - wx))\right) \;:\; 0 < x < \eta\} \;\cup \\ \{\Phi_c^{-1}\left(\exp(x + i(2\pi\vartheta + wx))\right) \;:\; 0 < x < \eta\} \end{array} \Big).$$

Since the angle ϑ has period nl under the mapping $\mathbb{S}^1 \to \mathbb{S}^1,\; t \mapsto dt$, we conclude from $\Phi_c \circ P_c \circ \Phi_c^{-1}(z) = z^d$:

$$\begin{aligned} &Q^n\left(\left\{\Phi_c^{-1}\left(\exp(x + i(2\pi\vartheta \pm wx))\right) \;:\; 0 < x < \eta/d^{nl}\right\}\right) \\ &= \left\{\Phi_c^{-1}\left(\exp(x + i(2\pi\vartheta - wx))\right) \;:\; 0 < x < \eta\right\}. \end{aligned}$$

This proves that the sector $S_w^\eta(\vartheta)$ is Q^n-invariant.

For a small perturbation of the parameter c the dynamic ray with angle ϑ still lands at a point $z(c)$ of preperiod k and period l (compare Lemma 4.11) such that $z = z(c)$ depends continuously on the parameter. If $w = w(c)$ and $\eta = \eta(c)$ also depend continuously on c, then

$$\operatorname{mod}\left(S_{w(c)}^{\eta(c)}(\vartheta)\right)$$

depends continuously on c (Lemma 4.16). Since the opening modulus of such a sector does not depend on the choice of η, the continuity condition on η is no longer necessary to assure that the opening modulus depends continuously on the parameter. Thus we simply omit the upper index η and for every $w > 0$, we can choose for example the potential $\pi/2w$ to define the opening modulus of $S_w(\vartheta)$.

(2) If $q \geq 2$ dynamic rays with angles $\vartheta_1 < \ldots < \vartheta_q$ land at z, then the connected components of $\mathbb{C} \setminus \mathrm{rays}(z)$ will be denoted by

$$\Sigma(\vartheta_1, \vartheta_2), \quad \ldots \quad \Sigma(\vartheta_{q-1}, \vartheta_q), \quad \Sigma(\vartheta_q, \vartheta_1).$$

There are a smallest integer $n \geq 1$ and neighborhoods U, $V \subset \mathbb{C}$ of z such that for every $j \in \{1, \ldots, q\}$,

$$Q^n \ : \ U \cap \Sigma(\vartheta_j, \vartheta_{j+1}) \to V \cap \Sigma(\vartheta_j, \vartheta_{j+1})$$

is bijective; here $\vartheta_{q+1} := \vartheta_1$. Thus every sector $\Sigma(\vartheta_j, \vartheta_{j+1})$ is Q^n-invariant and has a well defined opening modulus which does not depend on j by Lemma 2.19.

By the same arguments as in Example (1), the opening modulus of the sector $\Sigma(\vartheta_j, \vartheta_{j+1})$ depends continuously on the parameter c.

(3) If $q \geq 2$ dynamic rays with angles $\vartheta_1 < \ldots < \vartheta_q$ land at $z(c)$, then

$$X_c^\eta \Big\backslash \left(\bigcup_{j=1}^{q} \overline{S_w^\eta(\vartheta_j)} \right)$$

has exactly q connected components. The component between the sectors $S_w^\eta(\vartheta_j)$ and $S_w^\eta(\vartheta_{j+1})$ will be denoted by

$$\Sigma_w^\eta(\vartheta_j, \vartheta_{j+1}).$$

There are a smallest integer $n \geq 1$ and neighborhoods U, $V \subset \mathbb{C}$ of z such that for every $j \in \{1, \ldots, q\}$,

$$Q^n \ : \ U \cap \Sigma_w^\eta(\vartheta_j, \vartheta_{j+1}) \to V \cap \Sigma_w^\eta(\vartheta_j, \vartheta_{j+1}))$$

is bijective. Thus every sector $\Sigma_w^\eta(\vartheta_j, \vartheta_{j+1})$ has a well defined opening modulus which does not depend on j by Lemma 2.19.

By the same arguments as in Example (1), the opening modulus of the sector $\Sigma_{w(c)}^{\eta(c)}(\vartheta_j, \vartheta_{j+1})$ depends continuously on the parameter c if $w = w(c)$ depends continuously on c; again the opening modulus of these sectors does not depend on the equipotential boundary and thus we can omit index η when we are interested only in the opening modulus of these sectors.

Lemma 4.17 (Maps between invariant Sectors)
Let S and S' be two Q^n-invariant sectors around a preperiodic point z_0. Then there exists a quasiconformal map $\varphi : S \to S'$ such that

$$\varphi^{-1} \circ Q^n \circ \varphi(z) = Q^n(z) \quad \textit{for all} \quad z \in Q^{-n}(S) \cap S.$$

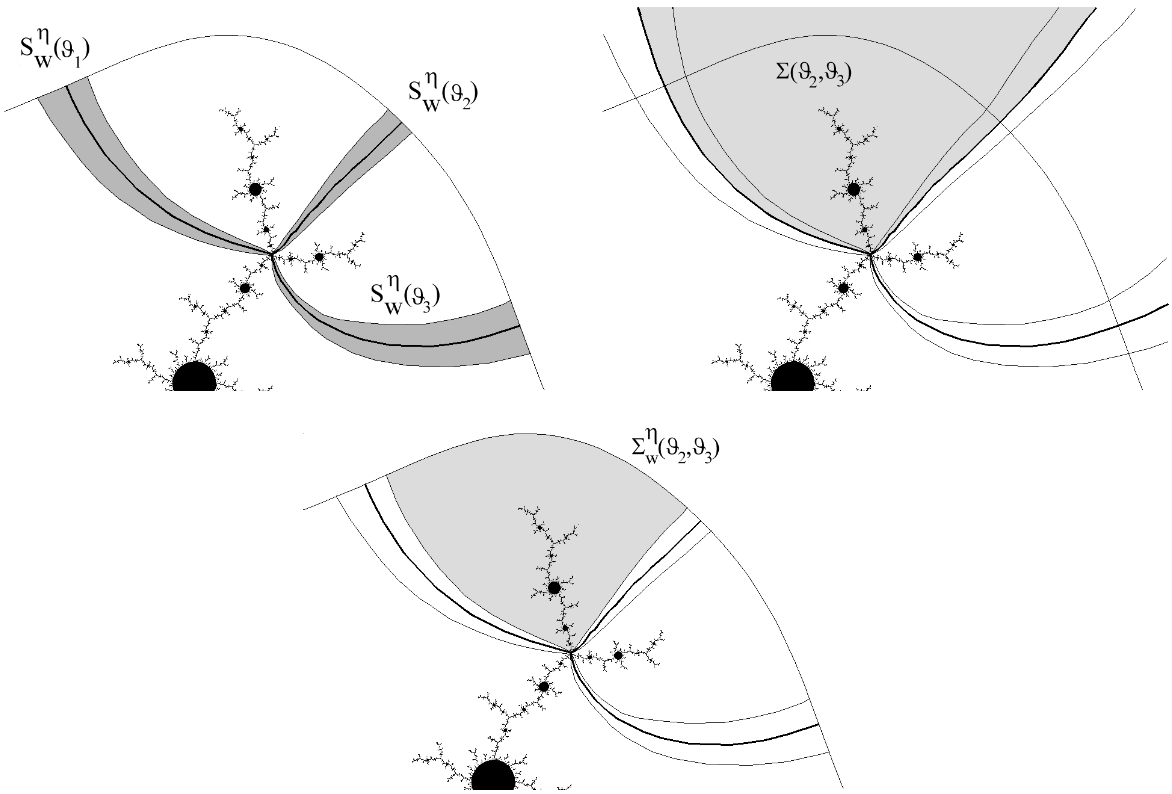

Figure 10: There are shown the three types of P_c^3-invariant sectors around a point of preperiod 3 and period 1 considered in the examples. The sectors are marked in grey.

Proof. Choose a quasiconformal homeomorphism between the following quadrilaterals

$$\varphi : \overline{S \setminus Q^{-n}(S)} \longrightarrow \overline{S' \setminus Q^{-n}(S')}$$

such that

$$\varphi^{-1} \circ Q^n \circ \varphi(z) = Q^n(z)$$

for all $z \in Q^{-n}(\partial S) \cap S$. Then we use the repulsive dynamics of Q^n at z_0 to define φ on the whole sector S: for every point $z \in S$, there exists a unique integer $m \geq 0$ such that $Q^{mn}(z) \in S \setminus Q^{-n}(S)$; let

$$\varphi(z) := Q^{-mn}\Big(\varphi(Q^{mn}(z))\Big).$$

Then

$$\varphi : S \to S'$$

defines a homeomorphism with the above conjugation property. Since we started with a quasiconformal mapping and since the complex dilatation of φ on S is bounded by the complex dilatation of φ on $Q^{mn}(z) \in S \setminus Q^{-n}(S)$, the map $\varphi : S \to S'$ is a quasiconformal homeomorphism. □

Lemma 4.18 (Limits of Opening Moduli)
Let $c \in \mathcal{M}_d$ and $z_0 \in K_c$ be a point of preperiod k and period l such that $q \geq 2$ dynamic rays with angles $\vartheta_1 < \ldots < \vartheta_q$ land at z_0. Then for every $j \in \{1, \ldots, q\}$, the function

$$]0, \infty[\to]0, \infty[\,, \; w \mapsto \mathrm{mod}(S_w(\vartheta_j))$$

is increasing, the function

$$]0, \infty[\to]0, \infty[\,, \; w \mapsto \mathrm{mod}(\Sigma_w(\vartheta_j, \vartheta_{j+1}))$$

is decreasing and

$$\lim_{w \to 0} \mathrm{mod}(S_w(\vartheta_j)) = 0.$$

If there exists a sequence of pinching points in $]z_0, 0[$ converging to z_0, then

$$\lim_{w \to \infty} \mathrm{mod}(\Sigma_w(\vartheta_j, \vartheta_{j+1})) = 0.$$

Proof. Since

$$S_w^\eta(\vartheta_j) \subset S_{w'}^\eta(\vartheta_j)$$

and

$$\Sigma_w^\eta(\vartheta_j, \vartheta_{j+1}) \supset \Sigma_{w'}^\eta(\vartheta_j, \vartheta_{j+1}),$$

for $w < w'$ and $\eta < w'/\pi$, the function $w \mapsto \mathrm{mod}(S_w(\vartheta_j))$ is increasing and the function $w \mapsto \mathrm{mod}(\Sigma_w(\vartheta_j, \vartheta_{j+1}))$ is decreasing. To prove $\lim\limits_{w \to 0} \mathrm{mod}(S_w(\vartheta_j)) = 0$ we proceed as follows: by

$$\varphi \; : \; z \mapsto \log(\Phi_c(z) e^{-2\pi i \vartheta}),$$

the sector $S_w^\eta(\vartheta)$ is mapped to the straight sector $S^\eta := \{x + iy \; : \; 0 < x < \eta, \; |y| < wx\}$ with vertices 0, $\eta + iw\eta$ and $\eta - iw\eta$. Denote by m the period of the rays with angles $d^k\vartheta_1, \ldots, d^k\vartheta_q$ (if $q \geq 3$, then $m = ql$; in the case $q = 2$ we have $m \in \{l, 2l\}$). Moreover,

$$S^{\eta/d^m} \to S^\eta, \; z \mapsto \varphi \circ Q^n \circ \varphi^{-1}(z) = d^m z.$$

By Lemma 2.19, the opening modulus of $S_w(\vartheta)$ is equal to the opening modulus of S^η considering the dynamics $z \mapsto d^m z$ in S^η. Consider the quadrilateral

$$\mathcal{Q} := \{z \in S^\eta \; : \; d^{-m}\eta < |z| < \eta\}$$

with vertices at $d^{-m}\eta \cdot \exp(i \arctan(w))$, $\eta \cdot \exp(i \arctan(w))$, $\eta \cdot \exp(-i \arctan(w))$ and $d^{-m}\eta \cdot \exp(-i \arctan(w))$. Then

$$\mathrm{mod}(S_w^\eta(\vartheta)) = \mathrm{mod}(\mathcal{Q}) = \frac{2 \arctan(w)}{m \cdot \log(d)} \to 0 \quad \text{for } w \to 0;$$

the second equality follows from the fact that Q is mapped by

$$\mathbb{C} \setminus \mathbb{R}_0^- \to \mathbb{C} \; : \; z \mapsto \log(z)$$

to a rectangle with edges of length $2 \arctan(w)$ and $m \cdot \log(d)$.

It is a little bit more difficult to argue that $\lim_{w\to\infty} \Sigma_w(\vartheta_j, \vartheta_{j+1}) = 0$, since there is no dynamically defined mapping between a fundamental domain of $\Sigma_w(\vartheta_j, \vartheta_{j+1})/\sim$ and a rectangle; thus we cannot simply calculate the opening modulus of the considered sector as before. But by increasing the width w, the sector $\Sigma_w(\vartheta_j, \vartheta_{j+1})$ gets thinner, at least at all positions where pinching points occur. This can be made precise as follows: let z be a pinching point of $\Sigma_w(\vartheta_j, \vartheta_{j+1}) \cap K_c$ near z_0. Then the distance of the points of potential w on the dynamic rays landing at z converges to zero for $w \to \infty$. Thus the euclidean distance of the boundary components of the annulus $A_w := \Sigma_w(\vartheta_j, \vartheta_{j+1})/_\sim$ tends to zero for $w \to \infty$. Since the multiplier of Q^n at z_0 is strictly bigger than one, the bounded component of $\mathbb{C} \setminus A_w$ has an euclidean diameter bounded away from zero independent on w. By Lemma 2.21, we have

$$\lim_{w\to\infty} \operatorname{mod}(A_w) = 0. \qquad \square$$

Lemma 4.19 (Quasiconformal Continuations)
Let $c \in \mathcal{M}_d$, $z_0 \in K_c$, $n \geq 1$ and Q_c be as before and assume that $P_c^k(z_0)$ is repelling. Moreover, let S, $\tilde{S}$, $S_0 \subset S$ and $\tilde{S}_0 \subset \tilde{S}$ be Q_c^n-invariant sectors around z_0 such that

(1) the z_0-component of $\mathbb{P} \setminus (S \cup \tilde{S})$ contains ∞,

(2) $S \setminus \overline{S_0}$ is the disjoint union of two Q_c^n-invariant sectors S_r and S_l,

(3) $\tilde{S} \setminus \overline{\tilde{S}_0}$ is the disjoint union of two Q_c^n-invariant sectors $\tilde{S}_r$ and $\tilde{S}_l$,

(4) $\operatorname{mod}(S_0) = \operatorname{mod}(\tilde{S}_0)$,

(5) the right and left boundary curves of the sectors S, $\tilde{S}$, S_0 and $\tilde{S}_0$ are analytic curves.

Denote by γ_r (respectively $\tilde{\gamma}_r$) the right boundary curve of S (respectively $\tilde{S}$) and by γ_l (respectively $\tilde{\gamma}_l$) the left boundary curve of S (respectively $\tilde{S}$). Consider a mapping

$$g : \gamma_l \cup S_0 \cup \gamma_r \to \tilde{\gamma}_l \cup \tilde{S}_0 \cup \tilde{\gamma}_r, \qquad \text{with } g(z_0) = z_0$$

which is defined by

(1) the Riemann mapping $S_0 \to \tilde{S}_0$ sending the vertices of S_0 to the vertices of $\tilde{S}_0$ and

(2) a composition of the mappings P_c, certain branches of P_c^{-1} and $z \mapsto \zeta z$ on both curves γ_r and γ_l.

Then there exists a homeomorphic extension

$$g : \overline{S} \to \overline{\tilde{S}}$$

such that its restriction $g : S \to \tilde{S}$ is a quasiconformal homeomorphism.
If the boundaries of the sectors above depend continuously on the parameter c (in the sense of Definition 2.8), then this extension g depends continuously on c as well and the maximal complex dilatation $\overline{\partial}g/\partial g$ of g is locally uniformly bounded away from 1.

PROOF. Let $\mu \neq 0$ be the multiplier of Q_c^n at z_0. There exist a neighborhood U of z_0 and a biholomorphic mapping (the linearizing map of Q_c at z_0)

$$\lambda : U \to \mathbb{D}$$

such that

$$\lambda \circ Q_c^n \circ \lambda^{-1} \; : \; B_{1/\mu}(0) \to \mathbb{D}, \; y \mapsto \mu y.$$

We can assume that S and $\tilde{S}$ are contained in U (otherwise we pull S and $\tilde{S}$ back by the dynamics of Q_c^n to get smaller Q_c^n-invariant sectors $Q_c^{-mn}(S)$, $Q_c^{-mn}(\tilde{S}) \subset U$ and continue with these smaller sectors instead of S and $\tilde{S}$). Thus the images

$$\lambda(S) \quad \text{and} \quad \lambda(\tilde{S})$$

are sectors in $\mathbb{D}$ which are invariant under the map

$$\mathbb{D} \to \mathbb{D}, \; y \mapsto y/\mu.$$

Since the 0-component of $\mathbb{P}\setminus(\lambda(S)\cup\lambda(\tilde{S}))$ contains ∞ by the first assumption, there exists a holomorphic branch of the logarithm in a connected subset of $\mathbb{C}$ containing $\lambda(S)\cup\lambda(\tilde{S})$. It maps $\lambda(S)$ and $\lambda(\tilde{S})$ to strips $\mathcal{B}$ and $\tilde{\mathcal{B}}$ which are both invariant under $y \mapsto y - \log(\mu)$:

$$\begin{aligned} \mathcal{B} &= \underbrace{\overline{\log(\lambda(S_l))}}_{=:\mathcal{B}_l} \cup \underbrace{\overline{\log(\lambda(S_0))}}_{=:\mathcal{B}_0} \cup \underbrace{\overline{\log(\lambda(S_r))}}_{=:\mathcal{B}_r} \\ \tilde{\mathcal{B}} &= \underbrace{\overline{\log(\lambda(\tilde{S}_l))}}_{=:\tilde{\mathcal{B}}_l} \cup \underbrace{\overline{\log(\lambda(\tilde{S}_0))}}_{=:\tilde{\mathcal{B}}_0} \cup \underbrace{\overline{\log(\lambda(\tilde{S}_r))}}_{=:\tilde{\mathcal{B}}_r} \end{aligned}$$

Conjugating g with $\log \circ \lambda$ yields a mapping

$$\widehat{g} : \log(\lambda(\gamma_l)) \cup \mathcal{B}_0 \cup \log(\lambda(\gamma_r)) \longrightarrow \log(\lambda(\tilde{\gamma}_l)) \cup \tilde{\mathcal{B}}_0 \cup \log(\lambda(\tilde{\gamma}_r)).$$

Since g is conjugated to itself by an iterate of Q_c^n on both curves γ_l and γ_r, there are

$$\sigma_l, \; \sigma_r \in \log(\mu)\mathbb{Z} \setminus \{0\}$$

with

$$\begin{aligned} \widehat{g}(y - \sigma_r) = \widehat{g}(y) - \sigma_r \qquad &\text{for all } \; y \in \log(\lambda(\gamma_r)) \quad \text{and} \\ \widehat{g}(y - \sigma_l) = \widehat{g}(y) - \sigma_l \qquad &\text{for all } \; y \in \log(\lambda(\gamma_l)). \end{aligned}$$

We have to find a homeomorphic extension

$$\widehat{g} : \mathcal{B} \to \tilde{\mathcal{B}}$$

such that its restriction $\widehat{g} : \operatorname{int}(\mathcal{B}) \to \operatorname{int}(\tilde{\mathcal{B}})$ is a quasiconformal homeomorphism. We will construct such an extension

$$\mathcal{B}_r \to \tilde{\mathcal{B}}_r;$$

the extension $\mathcal{B}_l \to \tilde{\mathcal{B}}_l$ can be constructed analogously. The map $z \mapsto z + \log(\mu)$ is an automorphism of both strips

$$\begin{aligned} \mathcal{B}_{r,\infty} &:= \{y \in \mathbb{C} \; : \; \text{there exists } n \in \mathbb{N} \text{ such that } y - n\log(\mu) \in \mathcal{B}_r\} \supset \mathcal{B}_r \quad \text{and} \\ \tilde{\mathcal{B}}_{r,\infty} &:= \{y \in \mathbb{C} \; : \; \text{there exists } n \in \mathbb{N} \text{ such that } y - n\log(\mu) \in \tilde{\mathcal{B}}_r\} \supset \tilde{\mathcal{B}}_r. \end{aligned}$$

Therefore there are $\tilde{\kappa}$, $\kappa > 0$ and biholomorphic mappings

$$\begin{aligned}\psi &: \mathcal{B}_{r,\infty} \to B_\kappa \\ \tilde{\psi} &: \tilde{\mathcal{B}}_{r,\infty} \to B_{\tilde{\kappa}}\end{aligned}$$

such that

$$\psi(\psi^{-1}(y) - \sigma_r) = y - 1 \quad \text{and} \quad \tilde{\psi}(\tilde{\psi}^{-1}(y) - \sigma_r)) = y - 1$$

where B_κ and $B_{\tilde{\kappa}}$ are the straight strips of width κ and $\tilde{\kappa}$ from Definition 3.8. Since the boundaries of the considered regions are locally connected, $\tilde{\psi}$ and ψ can be extended continuously to the closure of the strips and the equations above also hold if y is contained in one of the boundary curves. It remains to prove the existence of a quasiconformal homeomorphism

$$h : \psi(\mathcal{B}_r) \to \tilde{\psi}(\tilde{\mathcal{B}}_r)$$

such that

$$h(y) = \tilde{\psi} \circ \widehat{g} \circ \psi^{-1}(y)$$

for all $y \in \psi(\partial\mathcal{B}_r)$ with $\operatorname{Im}(y) \in \{0, \kappa\}$. There exist $a, b \in \mathbb{R}$ such that

$$\begin{aligned}\mathbb{R} \cap \psi(\partial\mathcal{B}_r) &= \,]-\infty, a] \quad \text{and} \\ (\mathbb{R} + i\kappa) \cap \psi(\partial\mathcal{B}_r) &= \,]-\infty, b] + i\kappa.\end{aligned}$$

By Lemma 3.9, it remains to prove that

$$h|_{]-\infty,a]} \quad \text{and} \quad h|_{]-\infty,b]+i\kappa} \circ (\mathrm{id}_{]-\infty,b]} + i\kappa) - i\tilde{\kappa}$$

are near translations. This can be seen as follows: for all $y \in \,]-\infty, a]$,

$$\begin{aligned}h(y-1) &= \tilde{\psi} \circ \widehat{g} \circ \psi^{-1}(y-1) \\ &= \tilde{\psi} \circ \widehat{g}(\psi^{-1}(y) - \sigma_r) \\ &= \tilde{\psi}(\widehat{g}(\psi^{-1}(y)) - \sigma_r) \\ &= \tilde{\psi} \circ \widehat{g} \circ \psi^{-1}(y) - 1 \\ &= h(y) - 1.\end{aligned}$$

Thus by Lemma 3.7, the map $h|_{]-\infty,a]}$ is near translation. To check that the second map is near translation as well, let

$$\begin{aligned}\mathcal{B}_{0,\infty} &:= \{y \in \mathbb{C} : \text{there exists } n \in \mathbb{N} \text{ such that } y - n\log(\mu) \in \mathcal{B}_0\} \supset \mathcal{B}_0 \quad \text{and} \\ \tilde{\mathcal{B}}_{0,\infty} &:= \{y \in \mathbb{C} : \text{there exists } n \in \mathbb{N} \text{ such that } y - n\log(\mu) \in \tilde{\mathcal{B}}_0\} \supset \tilde{\mathcal{B}}_0.\end{aligned}$$

Since $\operatorname{mod}(\tilde{S}_0) = \operatorname{mod}(S_0)$, there exist $\widehat{\kappa} > 0$ and biholomorphic mappings

$$\begin{aligned}\psi_0 &: \mathcal{B}_{0,\infty} \to B_{\widehat{\kappa}} \quad \text{and} \\ \tilde{\psi}_0 &: \tilde{\mathcal{B}}_{0,\infty} \to B_{\widehat{\kappa}}\end{aligned}$$

such that for all $y \in B_{\widehat{\kappa}}$,

$$\psi_0(\psi_0^{-1}(y) - \sigma_r) = y - 1 \quad \text{and} \quad \tilde{\psi}_0(\tilde{\psi}_0^{-1}(y) - \sigma_r) = y - 1.$$

Both of them can be extended continuously to the boundaries. By Corollary 3.10, the map

$$]-\infty, c] \to \mathbb{R},\ t \mapsto \tilde{\psi}_0 \circ \widehat{g} \circ \psi_0^{-1}$$

is near translation for a real number c. By Lemma 3.7, the maps

$$\begin{array}{lcl}]-\infty, d] \to \mathbb{R} & : & \tilde{\psi} \circ \tilde{\psi}_0^{-1}(t) - i\tilde{\kappa} \quad \text{and} \\ ]-\infty, e] \to \mathbb{R} & : & \psi_0 \circ \psi^{-1}(t + i\kappa) \end{array}$$

are also near translation for suitable $d, e \in \mathbb{R}$; then this is also true for

$$]-\infty, b] \to \mathbb{R},\ t \mapsto (\tilde{\psi} \circ \tilde{\psi}_0^{-1}) \circ (\tilde{\psi}_0 \circ \widehat{g} \circ \psi_0^{-1}) \circ (\psi_0 \circ \psi^{-1})(t + i\kappa) - i\tilde{\kappa}$$

$$= h(t + i\kappa) - i\tilde{\kappa},$$

which proves the claim.

It remains to prove the continuity statement: since the multiplier of Q_c^n at the point $z = z(c)$ and the linearizing map depend analytically on the parameter c, it is sufficient to examine the dependence of the map $\widehat{g} : \mathcal{B} \to \tilde{\mathcal{B}}$ on the parameter c. By Theorem 2.17, the maps $\tilde{\psi}$ and ψ depend continuously on c, and by Lemma 3.9, the map h depends continuously on c as well. □

4.3 Polynomial-like mappings

The theory of polynomial-like mappings has been investigated by A. Douady and J. Hubbard in [DH2]. This theory is the basic ingredient for the renormalization theory of C.T. McMullen (compare Section 4.4) to prove the universality of the Multibrot sets. This concept of polynomial-like mappings (or more generally: quasi-polynomial-like mappings) will be necessary to construct homeomorphisms between certain subsets of the Multibrot sets (compare [BD], [EY]). One of the most important statements in this context is the Straightening Theorem (Theorem 4.24) which shows the connection between polynomials and (quasi-)polynomial-like mappings. In this section we will introduce this concept of polynomial-like mappings and state some important theorems.

Definition 4.20 (Polynomial-like and Quasi-Polynomial-like Mappings)
Let U, $V \subset \mathbb{C}$ be bounded, simply connected open domains. A mapping $f : U \to V$ of degree d is called a

(1) **polynomial-like mapping of degree** d *if $\overline{U} \subset V$ and $f : U \to V$ is holomorphic.*

(2) **quasi-polynomial-like mapping of degree** d *if $\overline{U} \subset V$ and if there exists an almost complex f-invariant structure σ on V, i.e. if there exists a quasiconformal homeomorphism $\psi : V \to V'$ such that $\psi \circ f \circ \psi^{-1}\ :\ \psi(U) \to V'$ is a polynomial-like mapping of degree d.*

The **filled-in Julia set** K_f *of a (quasi-)polynomial-like mapping f is defined as the set of all points $z \in U$ which do not leave U iterating f.*

Lemma and Definition 4.21 (Critical Points)
Let $f : U \to V$ be a (quasi-)polynomial-like mapping of degree d. A point $z \in U$ is called **critical point of** f *if the restriction of f to any neighborhood of z is not injective. Then f has $d-1$ critical points (counted with multiplicity). If f has only one critical point of multiplicity $d-1$, then f is called* **unicritical**. □

Definition 4.22 (Hybrid Equivalence)
A (quasi-)polynomial-like mapping $f : U \to V$ is said to be **hybrid equivalent** *to a polynomial P if there exists a quasiconformal homeomorphism φ from an open neighborhood of K_f to an open neighborhood of K_P such that*

(1) $\varphi \circ f \circ \varphi^{-1}(z) = P(z)$ *for all z in a neighborhood of K_P and*

(2) $\overline{\partial}\varphi(z) = 0$ *for Lebesgue almost all $z \in K_f$.*

Definition 4.23 (Fixed Points)
Let $f : U \to V$ be a unicritical (quasi-)polynomial-like mapping of degree d with connected filled-in Julia set; let f be hybrid equivalent to the polynomial P_c. Then the α-fixed point of P_c corresponds to a fixed point $\alpha_f \in U$ of f; then α_f is called the **α-fixed point of** f. *The β-fixed points of P_c correspond to fixed points $\beta_f^{(0)}, \ldots, \beta_f^{(d-2)}$ of f; these are called the* **β-fixed points of** f.

Theorem 4.24 (Straightening Theorem)

(a) Every polynomial-like mapping $f : U \to V$ of degree d is hybrid equivalent to a polynomial P of the same degree.

(b) If K_f is connected then P is unique up to conjugation by an affine map.

This theorem and the following corollary were first stated and proved in [DH2, Theorem I and Corollary I.2].

Corollary 4.25
Let P and Q be two polynomials with connected filled-in Julia sets K_P and K_Q. If P and Q are hybrid equivalent, then they are conjugate by an affine map.

From this corollary together with Lemma 4.1 one can conclude

Corollary 4.26
Let $c_1, c_2 \in \mathcal{M}_d$ and $\tilde{\zeta} := e^{2\pi i/(d-1)}$. If the polynomials $z \mapsto z^d + c_1$ and $z \mapsto z^d + c_2$ are hybrid equivalent, then there is an integer $j \in \{0, \ldots, d-2\}$ such that

$$c_2 = \tilde{\zeta}^j c_1$$

Consider the set $Poly_d$ of polynomial-like mappings of degree $d \geq 2$. Then

$$f_1 \sim_{hb} f_2 \quad :\Leftrightarrow \quad f_1 \text{ and } f_2 \text{ are hybrid equivalent}$$

defines an equivalence relation on $Poly_d$. By the Straightening Theorem, every equivalence class contains at least one polynomial. Now we restrict to unicritical polynomial-like mappings with connected Julia set.

Corollary 4.27
Let f be a unicritical polynomial-like mapping of degree d with connected Julia set. Then f is hybrid equivalent to one of the polynomials P_c and the parameter c is uniquely determined up to multiplication with a $(d-1)$-th root of unity.

PROOF. By the Straightening Theorem, there exists a polynomial P which is hybrid equivalent to f. Since the critical points of f correspond to the critical points of P and vice versa, the polynomial P is unicritical. Then the claim follows from Lemma 4.1. □

It turns out to be useful to discuss not only hybrid equivalence but also quasiconformal equivalence: this will for example be important in the proof of continuity of our mapping χ in Lemma 5.10. A proof of the following corollary is given in [DH2, Proposition I.7]:

Corollary 4.28 (Quasiconformal Conjugation)
Let $c_1 \in \partial\mathcal{M}_d$, $c_2 \in \mathcal{M}_d$ such that the polynomials P_{c_1} and P_{c_2} are quasiconformally equivalent. Then there is an integer $j \in \{0, \ldots, d-2\}$ such that $c_2 = \tilde{\zeta}^j c_2$.

Lemma 4.29 (Rigidity)
Let $c \in \mathcal{M}_d$ and $K \subset K_c$ be compact and connected such that

$$P_c(K) \subset K \quad \text{and} \quad \{0,\ P_c(0),\ P_c^2(0),\ \ldots\} \subset K.$$

Moreover, let $\varphi : V \to V$ be a quasiconformal homeomorphism from a neighborhood V of K_c to itself such that

(1) $\varphi|_K = \mathrm{id}|_K$,

(2) *the complex dilatation $\overline{\partial}\varphi$ of φ vanishes a.e. on K,*

(3) *there exists a parameter $\tilde{c}$ and a quasiconformal homeomorphism $\psi : V \to \tilde{V}$ from V to a neighborhood $\tilde{V}$ of $K_{\tilde{c}}$ such that*

$$\psi \circ (\varphi \circ P_c) \circ \psi^{-1}|_{\psi(U)} = P_{\tilde{c}}|_{\psi(U)},$$

defining $U := P_c^{-1}(V)$.

Then the angled internal addresses of the parameters c and $\tilde{c}$ are equal except possibly the information about the sectors.

PROOF. The angled internal address of a parameter c can be described in the dynamics of a polynomial P_c as follows (compare the definition of an angled internal address, Lemma 4.11 and Lemma 4.12):

(1) The first entry in the internal address is $n_1 = 1$.

(2) Let $n_2 > 1$ be the smallest integer such that there exists a point z_2 of period n_2 in K_c separating α from the critical value c; we can choose z_2 so that there is no other point with this property separating z_2 and c. Then n_2 is the second entry in the internal address of c and the combinatorial rotation number of $P_c^{n_2}$ at z_2 is equal to the angle placed at this second entry.

(3) Let n_3 be the smallest integer $n_3 > n_2$ such that there exists a point z_3 of period n_3 in K_c separating z_2 from the critical value; we can choose z_3 so that there is no other point separating z_3 and the critical value with this property. Then n_3 is the third entry in the internal address of c and the combinatorial rotation number of $P_c^{n_3}$ at z_3 is equal to the angle placed at the third entry.

(4) ...

By this procedure, we have constructed the angled internal address of c. Applying the same procedure to $\tilde{c}$ we reach the angled internal address of $\tilde{c}$. By the assumptions, the periodic points in K_c separating α from c correspond exactly to the periodic points in $K_{\tilde{c}}$ separating α from $\tilde{c}$. Moreover, the combinatorial rotation numbers are in both cases determined by the iterates of P_c. Therefore the angled internal addresses are equal. □

REMARK. If the information about the sectors in the angled internal address happens to be equal for the parameters c and $\tilde{c}$, then the polynomials P_c and $P_{\tilde{c}}$ have the same combinatorics, i.e. the parameters c and $\tilde{c}$ are contained in the same fiber of $\mathcal{M}_d$ (for the definition of fibers of $\mathcal{M}_d$ see [Schl1] and [Schl2]).

4.4 Renormalization

4.4.1 Definition and Types of Renormalization

Definition 4.30 (Renormalization)
Let $c \in \mathcal{M}_d$ and $l \geq 2$. The polynomial P_c is called ***l*-renormalizable** *if there exist neighborhoods U, V of the critical point such that*

(a) $P_c^l : U \to V$ is polynomial-like of degree d and

(b) the Julia set K_l of this polynomial-like mapping is connected.

Definition 4.31 (Simple and Crossed Renormalization)
Let $c \in \mathcal{M}_d$ such that P_c is l-renormalizable and denote by K_l the small filled-in Julia set of the corresponding polynomial-like mapping P_c^l. If the images K_l, $P_c(K_l)$, ..., $P_c^{l-1}(K_l)$ of the small filled-in Julia set

(a) are pairwise disjoint, then the l-renormalization of P_c is said to be of **disjoint type**.

(b) touch at the orbit of a periodic point but do not disconnect each other, then the l-renormalization of P_c is said to be of β **type**.

(c) cross at the orbit of a periodic point, then the l-renormalization of P_c is said to be of **crossed type**.

In the first two cases the l-renormalization of P_c is called **simple**.

This definition distinguishes between three types of renormalizations. It is not obvious that every renormalization is of one of these types. This is shown in [McM1, Chapter 7]:

Lemma 4.32 (No Other Types of Renormalization)
Every renormalization of a polynomial P_c is either simple or crossed. □

4.4.2 A Dynamic Description of Simple Renormalization

Let $c \in \mathcal{M}_d$ and $l \geq 2$ such that P_c is l-renormalizable of simple type. Then there exist U and V as described in Definition 4.30; we want to construct other neighborhoods $\tilde{U}$ and $\tilde{V}$ of the critical point such that

$$P_c^l \,:\, \tilde{U} \to \tilde{V}$$

is a proper mapping of degree d. This proper map will not be polynomial-like, since the closure of $\tilde{U}$ will not be contained in $\tilde{V}$ but these neighborhoods of the critical point can be enlarged slightly to get a polynomial-like mapping (we will not make this precise here, see [Mi] for the details). The reason why we reconstruct the proper mapping is that $\tilde{U}$ and $\tilde{V}$ can be described easily by an equipotential and by finitely many preperiodic and periodic dynamic rays. For every connected component of the simple l-renormalization locus, the angles of these rays will be the same for the parameter c in such a component. Conversely, one can determine every connected component of the simple l-renormalization locus by looking at the boundary rays of $\tilde{V}$ and their preimages.

Denote by β_l the β-fixed point of the polynomial-like mapping $P_c^l : U \to V$ which separates the points $\alpha, \ldots, \zeta^{d-1}\alpha$; this fixed point is uniquely determined and will be called the **essential β-fixed point of the l-renormalization**. Using the renormalization theory of C.T. McMullen in [McM1] one knows that

(1) the P_c-period of β_l divides l;

(2) this period is exactly equal to l if and only if the considered renormalization is of disjoint type.

Let $q \geq 2$ be the number of dynamic rays landing at β_l and let

$$\tilde{V} := \mathcal{R}(\beta_l, \zeta\beta_l, \ldots, \zeta^{d-1}\beta_l).$$

Then $\tilde{V}$ can be pulled back along the orbit of β_l homeomorphically exactly $l-1$ times to get a certain region R behind $P_c(\beta_l)$ containing the critical value. The preimage of R is bounded by $2d^2$ dynamic rays and it is denoted by $\tilde{U}$. By construction,

$$P_c^l \,:\, \tilde{U} \to \tilde{V}$$

is a proper map of degree d and the P_c^l-orbit of the critical point is contained in $\tilde{U}$. As mentioned above we can construct a polynomial-like mapping $P_c^l : U \to V$ of degree d from this proper map by extending $\tilde{U}$ slightly around the boundary.

This construction can be used to describe the connected components of the locus of simple l-renormalization (compare [RS]).

Lemma 4.33 (Injective Iteration)
Let $c \in \mathcal{M}_d$ be l-renormalizable of simple type and $\tilde{U}$, $\tilde{V}$ as described in Section 4.4.2. Denote by α_l the α-fixed point of the polynomial-like map $P_c^l : \tilde{U} \to \tilde{V}$ and by β_l its essential β-fixed point. Then for every two points w_1, w_2 with at least two dynamic rays landing at each of them and with $0 \notin \mathcal{R}(w_1, w_2) \subset \tilde{U}$, the map P_c^l is injective on $\mathcal{R}(w_1, w_2)$.

Proof. This statement follows easily from the fact that

$$P_c^{l-1} : P_c(\tilde{U}) \to \tilde{V}$$

is biholomorphic. □

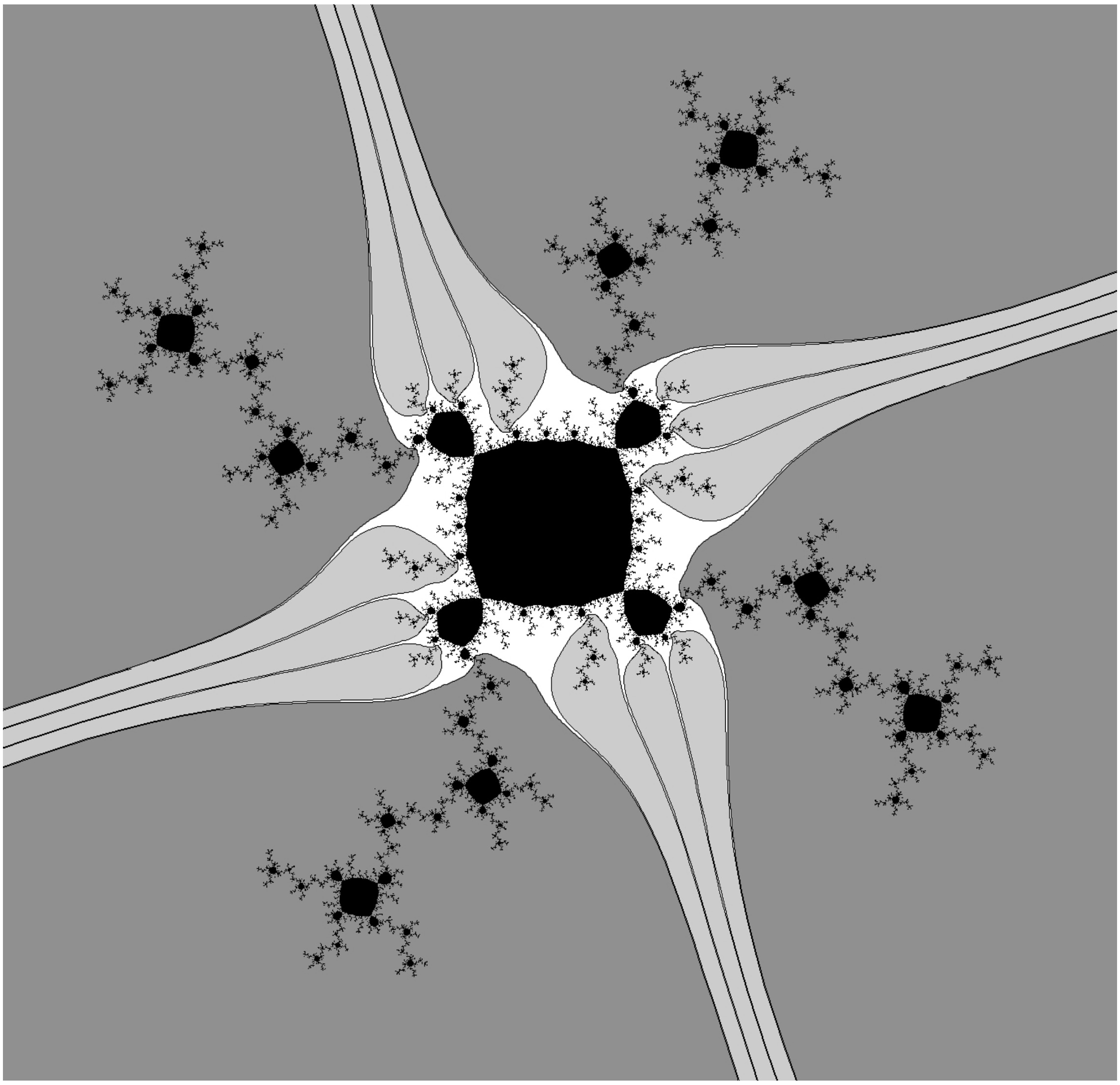

Figure 11: Construction of $\tilde{U}$ and $\tilde{V}$ for a polynomial P_c of degree four. The white set is $\tilde{U}$ and the union of the white and the light grey set is $\tilde{V}$. The set $\tilde{V}$ is bounded by the dynamic rays with angles 12/252, 16/252, 76/252, 79/252, 139/252, 142/252, 202/252, 205/252 and the set $\tilde{U}$ is in addition to that bounded by the dynamic rays with angles 835/16128, 895/16128, 898/16128, 958/16128, 961/16128, 1021/16128, ...

4.4.3 Non-Renormalizable Polynomials

Lemma 4.34 (Preimages of α)
Let $c \in \mathcal{M}_d$ be a Misiurewicz point of preperiod k and period l such that $q \geq 2$ parameter rays land at c. If the polynomial P_c is not renormalizable, then for every periodic point $z \in \,]\alpha, 0[$, there exists a preimage of α separating z from the critical point.

Proof. Assume that there is no preimage of α separating the critical point from z. Then the polynomial P_c is l'-renormalizable for an integer $l' > 1$: choose $\eta > 0$ and let $X_c^\eta := \{z \in \mathbb{C} \;:\; G_c(z) < \eta\}$; for every integer $n \geq 1$, let $\mathcal{P}_n$ be the closure of the connected component of

$$X_c^\eta \setminus \bigcup_{w \in P_c^{-n}(\alpha)} \mathrm{rays}(w)$$

containing the critical point. This yields a nested sequence $(\mathcal{P}_n)_n$ of connected and compact sets; thus

$$\mathcal{P}_\infty := \bigcap_{n \geq 1} \mathcal{P}_n$$

is connected and compact as well. By assumption,

$$\{0, z\} \subset \mathcal{P}_\infty.$$

Now we argue that

$$\alpha \notin \mathcal{P}_\infty.$$

Denoting by $\tilde{q}$ the number of dynamic rays which land at α, the map

$$P_c^{\tilde{q}} : \mathrm{star}(\alpha) \to \mathcal{R}_0(\alpha)$$

is proper of degree d and by thickening the considered regions slightly along the boundary rays as usual, we get a unicritical polynomial-like mapping. Since P_c is not $\tilde{q}$-renormalizable, the $P_c^{\tilde{q}}$-orbit of the critical point cannot be contained completely in $\mathrm{star}(\alpha)$ and there exists $n > 1$ such that $P_c^{n\tilde{q}}(0) \notin \mathrm{star}(\alpha)$. Since c is a Misiurewicz point, the filled-in Julia set K_c is locally connected and thus path connected. Then every path in $\mathrm{star}(\alpha) \cap K_c$ connecting α with the critical point is mapped by $P_c^{n\tilde{q}}$ (not necessarily injective) to a path in K_c containing one of the points $\zeta\alpha, \ldots, \zeta^{d-1}\alpha$. Thus the image path contains a preimage of α. Pulling this preimage back by the dynamics of $P_c^{n\tilde{q}}$ we find a preimage $\alpha_{-n\tilde{q}}$ of α separating α from the critical point. Since the sets $\mathcal{P}_n$ are symmetric w.r.t. the origin, we have

$$\mathcal{P}_\infty \subset \mathrm{star}(\alpha_{-n\tilde{q}}) \not\ni \alpha.$$

Since the point z is periodic and $\alpha \notin \mathcal{P}_\infty$, there exists a minimal integer $l' > 1$ such that $P_c^{l'}(\mathcal{P}_\infty) \cap \mathcal{P}_\infty \neq \emptyset$. Then the sets

$$\mathcal{P}_\infty,\; P_c(\mathcal{P}_\infty),\; \ldots,\; P_c^{l'-1}(\mathcal{P}_\infty)$$

are pairwise disjoint by construction and

$$P_c^{l'}(\mathcal{P}_\infty) = \mathcal{P}_\infty.$$

More precisely, $\mathcal{P}_\infty$ is mapped d to one to itself by $P_c^{l'}$. Thus there exists an integer $N > l'$ such that

$$\mathcal{P}_N, P_c(\mathcal{P}_N), \ \ldots, \ P_c^{l'}(\mathcal{P}_N)$$

are pairwise distinct and

$$P_c^{l'} : \mathcal{P}_N \to \mathcal{P}_{N-l'}$$

is a proper mapping of degree d and the $P_c^{l'}$-orbit of the critical point is contained within $\mathcal{P}_\infty \subset \mathcal{P}_N$. By the usual thickening procedure along the dynamic rays bounding $\mathcal{P}_N$, we get a unicritical polynomial-like mapping $P_c^{l'}$ in a neighborhood of the critical point the small filled-in Julia set of which is connected, i.e. P_c is l'-renormalizable. Contradiction. Therefore z is disconnected from the critical point by a preimage of α. □

4.5 Piecewise Dynamic Homeomorphisms

In this section we introduce the concept of piecewise dynamic homeomorphisms. These are piecewise holomorphic mappings and will be used to change the dynamics of a polynomial P_c to get a new unicritical mapping which is defined outside a finite number of dynamic rays. To finally construct a unicritical quasi-polynomial-like mapping one has to smoothen the mapping as it is shown in Lemma 4.37.

Definition 4.35 (Piecewise Dynamic Homeomorphisms)
Let $c \in \mathcal{M}_d$ and consider finitely many dynamic rays landing on the Julia set. These rays together with their landing points divide the complex plane into several open connected components. Two such components R_1 and R_2 are called **piecewise dynamically homeomorphic** *if*

(1) there exists a finite collection of dynamic rays dividing R_1 and R_2 into open components $(R_1^{(j)})_{j\in J}$ and $(R_2^{(j)})_{j\in J}$ such that every $R_1^{(j)}$ is mapped biholomorphically to $R_2^{(j)}$ by a composition f_j of the mappings P_c, certain branches of P_c^{-1} and $z \mapsto \zeta z$ (every mapping f_j extends homeomorphically to the closure of $R_1^{(j)}$);

(2) every dynamic ray within R_1 which is part of the boundaries of $R_1^{(j_1)}$ and $R_1^{(j_2)}$ is mapped as a set to the same ray by the extensions of f_{j_1} and f_{j_2} (but a point on this ray may be mapped to two points of different potentials by these two mappings).

In this case we say that $f : R_1 \to R_2$ defined by f_j on $R_1^{(j)}$ is a piecewise dynamic homeomorphism.
For every point $z \in \bigcup_{j\in J} R_1^{(j)}$ we denote by $\operatorname{iter}(f,z)$ the number of used maps P_c minus the number of used maps P_c^{-1} to define $f(z)$ (here the number of used maps $z \mapsto \zeta z$ does not matter). The function $\operatorname{iter}(f,.)$ is constant on every component $R_1^{(j)}$ and we denote this number by $\operatorname{iter}(f, R_1^{(j)})$. Moreover let

$$\begin{aligned}\operatorname{iter}_{max}(f) &:= \max\left\{\operatorname{iter}(f, R_1^{(j)}) \in \mathbb{Z} \ : \ j \in J\right\}\\ \operatorname{iter}_{min}(f) &:= \min\left\{\operatorname{iter}(f, R_1^{(j)}) \in \mathbb{Z} \ : \ j \in J\right\}.\end{aligned}$$

A piecewise dynamic homeomorphism $f : R_1 \to R_2$ *is called a* **piecewise dynamic homeomorphism with expansion relative to the set** $E \subset \mathbb{C}$ *if*

(3) for every boundary ray ρ *of* $R_1^{(j)}$*, we have* $\bigcup_{n\geq 0} P_c^n(\rho) \cap \overline{E} = \emptyset$ *and*

(4) every point $z \in \bigcup_{j\in J} R_1^{(j)}$ *with* $P_c^n(f(z)) \in E$ *has* $\mathrm{iter}(f,z) + n \geq 0$.

REMARKS.
By this definition, every piecewise dynamic homeomorphism maps the filled-in Julia set within R_1 to the filled-in Julia set within R_2, since P_c, P_c^{-1} and $z \mapsto \zeta z$ have this property. The mapping

$$f : \bigcup_{j\in J} R_1^{(j)} \to \bigcup_{j\in J} R_2^{(j)}$$

cannot be extended to a homeomorphism between R_1 and R_2 in general; but its restriction to $R_1 \cap K_c$ gives a homeomorphism to $R_2 \cap K_c$.
Since $P_c(\zeta z) = P_c(z)$ and since $\zeta P_c^{-1}(z)$ can be written as P_c^{-1} with a suitably chosen branch of the inverse, for every $j \in J$, there exist e, m_1, $m_2 \in \mathbb{N}_0$ such that the restriction of f to R_1^j can be written in the form

$$f(z) = P_c^{-m_2}(\zeta^e P_c^{m_1}(z)).$$

Corollary 4.36 (Composition and Inverse)
Let $f : R_1 \to R_2$ *and* $g : R_2 \to R_3$ *be piecewise dynamic homeomorphisms. Then* $g \circ f : R_1 \to R_3$ *and* $f^{-1} : R_2 \to R_1$ *are piecewise dynamic homeomorphisms as well.* □

Lemma 4.37 (Quasiconformal Gluing)
Let $c \in \mathcal{M}_d$*,* $f : R_1 \to R_2$ *be a piecewise dynamic homeomorphism between two regions* R_1 *and* R_2 *in the dynamic plane of* P_c*,* $N := \mathrm{iter}_{max}(f)$ *and* $\Theta \subset \mathbb{R}/\mathbb{Z}$ *the finite set of angles of all rays in* $R_1 \setminus \bigcup_{j\in J} R_1^{(j)}$*. For every potential* $\eta > 0$*, let* $U^\eta := R_1 \cap \{z \in \mathbb{C} \ : \ G_c(z) < \eta\}$*. Then there exists a width* $w_0 = w_0(\eta) > 0$ *such that for every* $w \in \,]0, w_0]$*, there is a quasiconformal homeomorphism*

$$\tilde{f} : U^\eta \longrightarrow V \subset R_2 \cap \{z \in \mathbb{C}\,;\, G_c(z) < d^N \eta\}$$

such that $\tilde{f}(z) = f(z)$ *for all* $z \in U^\eta \setminus \bigcup_{\vartheta\in\Theta} S_w^\eta(\vartheta)$*.*

PROOF. Let $n := \mathrm{iter}_{min}(f)$ and $\eta > 0$. We choose $w_0 > 0$ such that the sectors

$$\left(S_{w_0}^{d^N\eta}(\vartheta)\right)_{\vartheta\in\Theta}$$

are pairwise disjoint. Let $w \in \,]0, w_0[$, $\vartheta \in \Theta$ and $\tilde{\vartheta}$ be the angle of the f-image of the dynamic ray with angle ϑ. Then there are exactly two components $R_1^{(j_1)}$ and $R_1^{(j_2)}$ such that the dynamic ray with angle ϑ is part of their boundaries.

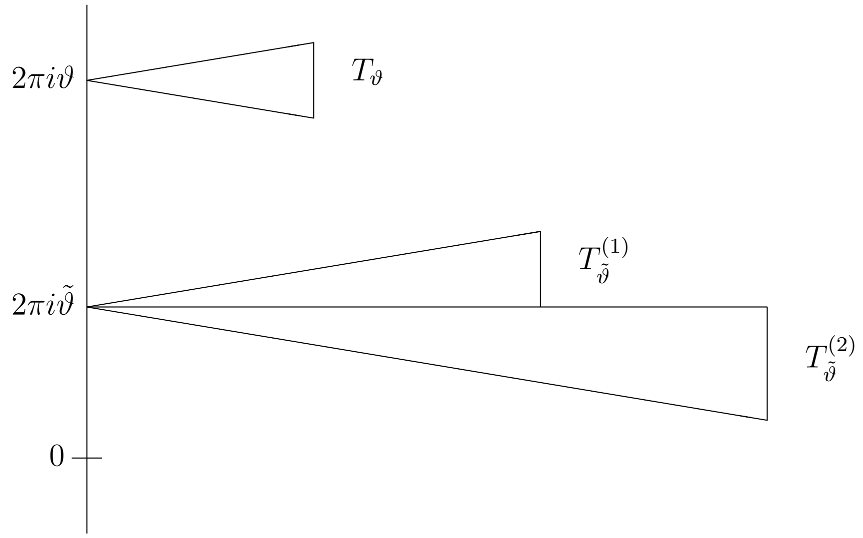

Figure 12: The triangles T_ϑ, $T_{\tilde\vartheta}^{(1)}$ and $T_{\tilde\vartheta}^{(2)}$

One of them is on the right side of the dynamic ray with angle ϑ and the other one on the left side (this orientation is defined by the parametrization of the ray). Assume that $R_1^{(j_1)}$ is on the left side and $R_1^{(j_2)}$ on the right side. Every point $z \in R_1^{(j)} \setminus K_c$ with potential ν is mapped by f to a point with potential $d^{\mathrm{iter}(f,z)} \cdot \nu$. Let

$$n_1 := \mathrm{iter}(f, R_1^{(j_1)}), \quad n_2 := \mathrm{iter}(f, R_1^{(j_2)})$$

and

$$\begin{aligned} T_\vartheta &:= 2\pi i\vartheta + \{y \in \mathbb{C} \;:\; \mathrm{Re}(y) \in \,]0, \eta[,\ \mathrm{Im}(y) \in \,]-w\mathrm{Re}(y), w\mathrm{Re}(y)[\,\}, \\ T_{\tilde\vartheta}^{(1)} &:= 2\pi i\tilde\vartheta + \{y \in \mathbb{C} \;:\; \mathrm{Re}(y) \in \,]0, d^{n_1}\eta[,\ \mathrm{Im}(y) \in \,]0, w\mathrm{Re}(y)[\,\}, \\ T_{\tilde\vartheta}^{(2)} &:= 2\pi i\tilde\vartheta + \{y \in \mathbb{C} \;:\; \mathrm{Re}(y) \in \,]0, d^{n_2}\eta[,\ \mathrm{Im}(y) \in \,]-w\mathrm{Re}(y), 0[\,\}. \end{aligned}$$

Conjugating f with $\log \circ \Phi_c$ on $S_w^\eta(\vartheta)$ yields

$$g_\vartheta : T_\vartheta \setminus (\mathbb{R} + 2\pi i\vartheta) \to T_{\tilde\vartheta}^{(1)} \cup T_{\tilde\vartheta}^{(2)},$$

$$y \mapsto (\log \circ \Phi_c) \circ f \circ (\log \circ \Phi_c)^{-1}(y) = \begin{cases} (y - 2\pi i\vartheta)d^{n_1} + 2\pi i\tilde\vartheta & : \quad \mathrm{Im}(y) > 2\pi\vartheta \\ (y - 2\pi i\vartheta)d^{n_2} + 2\pi i\tilde\vartheta & : \quad \mathrm{Im}(y) < 2\pi\vartheta \end{cases}.$$

Conjugating

$$T_\vartheta - 2\pi i\vartheta \longrightarrow (T_{\tilde\vartheta}^{(1)} \cup T_{\tilde\vartheta}^{(2)}) - 2\pi i\tilde\vartheta,\ y \mapsto g_\vartheta(y + 2\pi i\vartheta) - 2\pi i\tilde\vartheta$$

with $z \mapsto \log(z)$ yields a restriction of the map

$$\tilde g_\vartheta \;:\; S \to S\,,\ x \mapsto \begin{cases} x + n_1\log(d) & : \quad \mathrm{Im}(x) > 0 \\ x + n_2\log(d) & : \quad \mathrm{Im}(x) < 0 \end{cases}$$

where S is the straight strip

$$S := \{x \in \mathbb{C} \, : \, \mathrm{Im}(x) \in \,]-\arctan(w), \arctan(w)[\,\} \setminus \mathbb{R}.$$

Then

$$\widehat{g}_\vartheta \, : \, S \cup \mathbb{R} \to S \cup \mathbb{R}\,, \; x \mapsto \left[\frac{1}{2}(n_1+n_2)\left(1 - \frac{\mathrm{Im}(x)}{\arctan(w)}\right) + n_1 \frac{\mathrm{Im}(x)}{\arctan(w)}\right] x$$

is a quasiconformal homeomorphism which coincides with $\tilde{g}_\vartheta$ on $i\arctan(w) + \mathbb{R}$ and on $-i\arctan(w) + \mathbb{R}$. Transferring this map back into the dynamic plane of P_c we get the required quasiconformal homeomorphism

$$\tilde{f}|_{S^n_w(\vartheta)}(z) = (\log \circ \Phi_c)^{-1}\left((\log^{-1} \circ \widehat{g}_\vartheta \circ \log)(\log \circ \Phi_c(z) - 2\pi i \vartheta) + 2\pi i \tilde{\vartheta})\right).$$

Repeating this procedure for every angle $\vartheta \in \Theta$ proves the claim. □

Lemma 4.38 (Pullback of Non-Critical Regions)
Let $c \in \mathcal{M}_d$ be a Misiurewicz point of preperiod k and period l such that $q \geq 3$ parameter rays land at c. Denote by $z_0, \ldots, z_{l-1}$ the periodic orbit of $P_c^k(c)$. Then there exists a sequence $(w_n)_{n\geq 1} \subset \,]z_0, 0[$ such that

$$P_c^{nlq} : \mathcal{R}(z_0, w_n) \to \mathcal{R}(z_0, 0)$$

is biholomorphic for $n \geq 1$.

Proof. The region $\mathcal{R}(z_0, 0)$ is bounded by two dynamic rays with angles $\vartheta' < \vartheta$ landing at z_0 and two dynamic rays with angles $\varphi' < \varphi$ landing at 0. Therefore the critical value is the landing point of the two dynamic rays with angles $d\varphi$ and $d\varphi'$. The region $\mathcal{R}(z_0, 0)$ can be pulled back along the orbit of z_0 as long as the preimage does not contain the critical value. The m-th pull back is a region bounded by two ray pairs: one ray pair with angles $\vartheta'_m < \vartheta_m$ landing at a point of the orbit of z_0 and another ray pair with angles $\varphi'_m < \varphi_m$ landing at a preimage of the critical point:

$$\begin{aligned} |\vartheta_m - \varphi_m| &= (\vartheta - \varphi)/d^m \quad \text{and} \\ |\varphi'_m - \vartheta'_m| &= (\varphi' - \vartheta')/d^m. \end{aligned}$$

Thus the critical value is never contained in these preimages of $\mathcal{R}(z_0, 0)$. Considering the pull backs of order nlq for $n \geq 1$ we have constructed regions $\mathcal{R}(z_0, w_n)$ which are mapped biholomorphically to $\mathcal{R}(z_0, 0)$ by P_c^{nlq}; for $n \geq 1$, the point w_n is the landing point of the dynamic rays with angles φ_{nlq} and φ'_{nlq}. This defines a sequence $(w_n)_n$; it remains to prove that the points w_n separate z_0 from the critical point. Since the number of used pull backs is a multiple of lq, the points w_n are contained in $\mathcal{R}(z_0, 0)$. If $w_n \notin \,]z_0, 0[$, then $0 \in \mathcal{R}(z_0, w_n)$, in contradiction to the injectivity of P_c^{nlq} on $\mathcal{R}(z_0, w_n)$. □

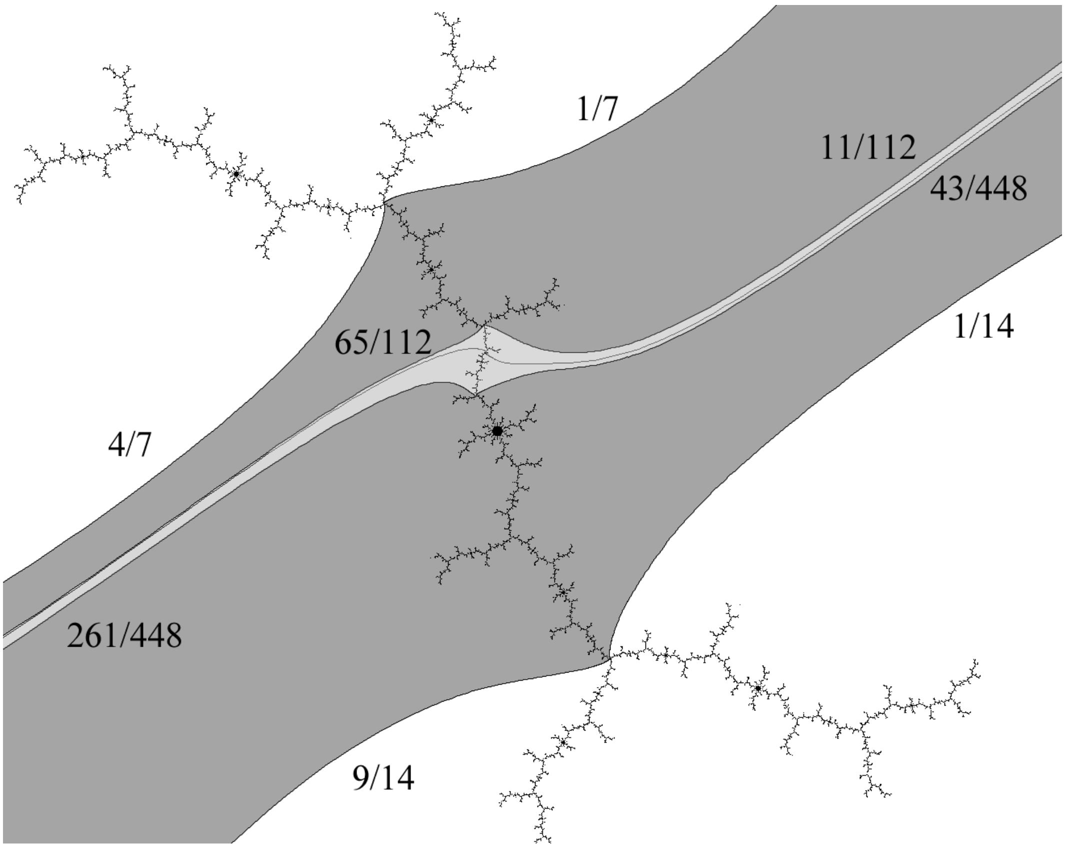

Figure 13: The dynamic plane of the quadratic polynomial which corresponds to the center of the hyperbolic component with internal address $1 \to 3 \to 5 \to 6$. The iterate P_c^5 maps the region bounded by the dynamic rays with angles 43/448, 11/112, 65/112 and 261/448 (marked in light grey) biholomorphically to the region bounded by the dynamic rays with angles 1/14, 1/7, 4/7 and 9/14 (union of the regions marked in light and dark grey). Lemma 4.39 proves the existence of a fixed point; in this example, the fixed point is the common landing point of the dynamic rays with angles 3/31 and 18/31 within the region marked in light grey.

Lemma 4.39 (Finding Fixed Points)
Let $c \in \mathcal{M}_d$ be a parameter such that the Julia set of P_c is locally connected and let z_1, z_1', z_2, $z_2' \in J_c$ be landing points of at least two dynamic rays each such that

$$z_1' \in]z_1, z_2[\quad \textit{and} \quad z_2' \in]z_1', z_2[.$$

Then for every piecewise dynamic homeomorphism

$$f : \mathcal{R}(z_1', z_2') \to \mathcal{R}(z_1, z_2)$$

there exists a fixed point $z \in]z_1', z_2'[$ of f in J_c. In particular, z is the landing point of at least two dynamic rays.

PROOF. For every integer $n \geq 0$ let

$$z_1^{(n)} := f^{-n}(z_1) \quad \text{and} \quad z_2^{(n)} := f^{-n}(z_2).$$

Then

$$z_1^{(n+1)} \in \left]z_1^{(n)}, z_2^{(n+1)}\right[\quad \text{and} \quad z_2^{(n+1)} \in \left]z_1^{(n+1)}, z_2^{(n)}\right[$$

and all the mappings

$$f : \mathcal{R}(z_1^{(n+1)}, z_2^{(n+1)}) \longrightarrow \mathcal{R}(z_1^{(n)}, z_2^{(n)}),$$

are piecewise dynamic homeomorphisms.

By definition, a piecewise dynamic homeomorphism maps dynamic rays to dynamic rays. Since the sequence of angles of the external rays landing at $z_1^{(n)}$ from above (below) is strictly decreasing (increasing) and the sequence of angles of the rays landing at $z_2^{(n)}$ from above (below) is strictly increasing (decreasing), all these sequences of angles converge. By the Theorem of Carathéodory, the inverse Böttcher mapping $\Phi_c^{-1} : \mathbb{P} \setminus \overline{\mathbb{D}} \to \mathbb{P} \setminus K_c$ extends continuously to the boundaries. Thus the landing points $z_1^{(n)}$ (respectively $z_2^{(n)}$) of these rays converge to points $z_1^{\infty} \in J_f$ (respectively $z_2^{\infty} \in J_f$). In any case these are fixed points of f (it may happen that $z_1^{\infty} = z_2^{\infty}$) and every dynamic ray landing at $z_1^{(n)}$ ($z_2^{(n)}$) yields in the limit $n \to \infty$ a dynamic ray landing at z_1^{∞} (z_2^{∞}). □

4.6 Periodic Points

Lemma 4.40 (Regions behind Periodic Points)
Let $c \in \mathcal{M}_d$ and $\{z_0, \dots, z_{l-1}\}$ be a repelling periodic orbit of P_c. Then all the regions behind the characteristic point z_1 of this orbit are pairwise piecewise dynamically homeomorphic with expansion relative to $\operatorname{star}(z_0)$*. In particular, at most two of the regions around each point z_j contain some of the points $z_0, \dots, z_{l-1}$.*

PROOF. Let q be the number of dynamic rays landing at z_1. If $q \leq 2$ then there is nothing to prove and we can assume $q \geq 3$. If $l = 1$, then the dynamic rays landing at $z_0 = \alpha$ disconnect the dynamic plane into q connected components around α. Since

$$0 \notin P_c^m(\mathcal{R}_c(\alpha))$$

for $m \in \{0, \dots, q-2\}$, the maps P_c, P_c^2, …, P_c^{q-2} are injective on $\mathcal{R}_c(\alpha)$ and the images are the regions behind α. Thus for every region $\mathcal{R}_i$ behind α, there exists a piecewise dynamic homeomorphism $f_i : \mathcal{R}_i \to \mathcal{R}_0(\alpha)$ and

$$f_i^{-1} \circ f_j : \mathcal{R}_j \to \mathcal{R}_i$$

is a piecewise dynamic homeomorphism between $\mathcal{R}_j$ and $\mathcal{R}_i$ with expansion relative to $\operatorname{star}(\alpha)$: by construction, none of the considered regions is subdivided to define the piecewise dynamic homeomorphisms; this makes condition three in Definition 4.35 obvious. The expansion property follows immediately from the fact that

$$P_c^m(z) \notin \operatorname{star}(\alpha)$$

for every $z \in \mathcal{R}_j$ and for every $m \in \{0, \dots, \operatorname{iter}_{max}(f_j) - 1\}$.

If $l > 1$, we prove

$$0 \notin P_c^m(\mathcal{R}_c(z_1))$$

for all $m \in \{0, \ldots, (q-2)l\}$; then we continue as before to find piecewise dynamic homeomorphisms with extension relative to $\mathrm{star}(z_0)$ between each pair of regions behind z_1.

Assuming that there exists $m \in \{0, \ldots, (q-2)l\}$ such that $0 \in P_c^m(\mathcal{R}_c(z_1))$, we yield a contradiction as follows: let

$$\tilde{m} := l(\lfloor m/l \rfloor + 1) \ \in \ \{l, 2l, \ldots, (q-1)l\}.$$

Since there land at least three dynamic rays at z_1, these dynamic rays are permuted transitively by the dynamics of P_c^l. Thus

$$P_c^m(z_1) \neq z_0$$

(otherwise P_c^{m+1} would map each of the two rays bounding $R_c(z_1)$ to itself and the dynamic rays landing at z_1 are not permuted transitively by the dynamics). Therefore, one of the points $z_0, \zeta z_0, \ldots, \zeta^{d-1} z_0$ separates $P_c^m(z_1)$ from the critical point and defining $m_1 := m + 1$, there exists a neighborhood U_1 of z_1 such that

$$P_c^{m_1} : U_1 \cap \mathcal{R}_c(z_1) \to \mathcal{R}(P_c^{m_1}(z_1), z_1)$$

is biholomorphic. Then

- either $P_c^{l(\lfloor m_1/l \rfloor + 1)}$ is injective on $U_1 \cap \mathcal{R}_c(z_1)$
- or there exists $m_2 \in \{m_1 + 1, \ldots, l(\lfloor m/l \rfloor + 1)\}$ such that $0 \in P_c^{m_2}(U_1 \cap \mathcal{R}_c(z_1))$.

But the first case cannot occur, since $P_c^{\tilde{m}-(m_1+1)}(z_1) \in \{z_2, \ldots, z_{q-1}\}$ cannot be contained in a region behind the characteristic point z_1. In the second case,

$$P_c^{m_2}(z_1) \neq z_0$$

by the same reason as above. Then we find a neighborhood U_2 of z_1 such that $P_c^{m_2+1}$ is injective on $U_2 \cap \mathcal{R}_c(z_1)$ and the $P_c^{m_2+1}$-image of this region contains at least two points of the orbit $z_0, z_1, \ldots, z_{l-1}$. By similar arguments as before, $P_c^{\tilde{m}}$ cannot be injective on $U_2 \cap \mathcal{R}_c(z_1)$ and we find $m_3 \in \{m_2+1, m_2+2, \ldots, \tilde{m}-1\}$ such that $0 \in P_c^{m_3}(U_2 \cap \mathcal{R}_c(z_1))$. Continuing with these arguments, we construct an infinite strictly increasing sequence of integers $m_j < \tilde{m}$; this is of course impossible and proves the injectivity of the mappings $P_c, P_c^2, \ldots, P_c^{(q-2)l}$ on $\mathcal{R}_c(z_1)$. □

4.7 Misiurewicz Points

Now we start to discuss Misiurewicz points and prove some statements which will be needed in Section 5.1 to construct many homeomorphisms between certain subsets of $\mathcal{M}_d$. There are Misiurewicz points for which the branches behind them are of similar shape and size; these narrow Misiurewicz points are the preperiodic analogs to the narrow hyperbolic components as defined in [LS] and they are much easier to treat than other Misiurewicz points.

4.7.1 Misiurewicz Points of α-type

Definition 4.41 (Narrow Points of α-type)

(1) Let $c \in \mathcal{M}_d$ be a parameter such that the α-fixed point of P_c is repelling and let $z \in K_c$ be a point of preperiod $k \geq 1$ with $P_c^k(z) = \alpha$. If P_c^{k-1} maps every region behind z one to one to a region behind one of the points $\zeta\alpha, \ldots, \zeta^{d-1}\alpha$ then z is called a **narrow preperiodic point of α-type**.

(2) A parameter $c \in \mathcal{M}_d$ is called a **narrow Misiurewicz point of α-type** *if the critical value of P_c is a narrow preperiodic point of α-type. In this case let q denote the number of dynamic rays landing at α; then $\mathcal{M}_d \setminus \{c\}$ has exactly q connected components. These components are labeled by $\mathcal{M}_d(c, 0)$ (the one which contains the parameter 0) and counterclockwise $\mathcal{M}_d(c, 1), \ldots, \mathcal{M}_d(c, q-1)$.*

Remark. The fact that a Misiurewicz point c of α-type is narrow of preperiod k is determined by the angles $\vartheta_1 < \ldots < \vartheta_q$ of the parameter rays landing at this point: c is narrow of preperiod $k \geq 1$ if and only if

(1) all angles ϑ_j have preperiod k (i.e. the parameter c is a Misiurewicz point of preperiod k),

(2) $q \geq 2$ and two of the parameter rays with angles in $\{d^n\vartheta_1 \ : \ n \geq 1\}$ land on the boundary of the main cardioid of $\mathcal{M}_d$ (i.e. the Misiurewicz point is of α-type) and

(3) $d^{k-1}(\vartheta_q - \vartheta_1) < 1$ (i.e. the α-type Misiurewicz point is narrow).

Lemma 4.42 (Existence of Narrow Misiurewicz Points)
Let $c \in \mathcal{M}_d$ be the landing point of $q \geq 2$ parameter rays. Then every branch of $\mathcal{M}_d$ behind c contains exactly one or exactly $d-1$ narrow Misiurewicz points of α-type and minimal preperiod. These preperiods are different on different branches behind c.

Proof. Let $K_c(c, 0), \ldots, K_c(c, q-1)$ denote the connected components of $K_c \setminus \{c\}$ ordered counterclockwise around c such that $0 \in K_c(c, 0)$. For $j \in \{1, \ldots, q-1\}$, denote the angles of the dynamic rays disconnecting $K_c(c, j)$ from the rest of K_c by $\vartheta_j < \vartheta_{j+1}$ and the angles of the rays landing at α by $\varphi_1, \ldots, \varphi_q$. There is a minimal integer $\tilde{n} \geq 1$ such that

$$0 \in P_c^{\tilde{n}}\left(K_c(c, j)\right).$$

Thus $P_c^{\tilde{n}}$ is injective on $K_c(c, j)$ and

$$\{\alpha, \ldots, \zeta^{d-1}\alpha\} \cap P_c^{\tilde{n}}(K_c(c, j)) \neq \emptyset.$$

Since $\alpha \notin K_c(c, j)$, there is a minimal integer $n \in \{\tilde{n}-1, \tilde{n}\}$ such that

$$\{\zeta\alpha, \ldots, \zeta^{d-1}\alpha\} \cap P_c^n(K_c(c, j)) \neq \emptyset \quad \text{and} \quad \alpha \notin P_c^n(K_c(c, j)).$$

Therefore,

$$P_c^n(c) \in \,]\alpha, 0[\, \cup \ldots \cup \,]0, \zeta^{d-1}\alpha[$$

and no angle in $[\vartheta_j, \vartheta_{j+1}]$ is mapped to one of the angles $\varphi_1, \ldots, \varphi_q$ by

$$\mathbb{S}^1 \to \mathbb{S}^1, \; \vartheta \mapsto d^n \vartheta.$$

Transferring this to the parameter space $\mathcal{M}_d$ there is no α-type Misiurewicz point of preperiod less than $n+1$ within $M_d(c, j)$.

(1) If $P_c^n(c) \in \,]\alpha, 0[$, then $\{\zeta\alpha, \ldots, \zeta^{d-1}\alpha\} \subset P_c^n(K_c(c, j))$ and for every integer $s \in \{1, \ldots, q\}$, there are exactly $d-1$ angles

$$\varphi_s^{(1)}, \ldots, \varphi_s^{(d-1)} \;\in\; [\vartheta_j, \vartheta_{j+1}]$$

which are mapped to the angle φ_s by

$$\mathbb{S}^1 \to \mathbb{S}^1, \; \vartheta \mapsto d^{n+1} \vartheta.$$

In the dynamic plane of P_c the dynamic rays with angles $\varphi_1^{(t)}, \ldots, \varphi_q^{(t)}$ land together at an $(n+1)$-th preimage $\alpha_{-(n+1)}^{(t)}$ of α for $t \in \{1, \ldots, d-1\}$. The points

$$\alpha_{-(n+1)}^{(1)}, \ldots, \alpha_{-(n+1)}^{(d-1)}$$

are pairwise disjoint. Since there exists no preimage of α with preperiod less than $n+1$ in $P_c^n(K_c(c, j))$, the parameter rays with angles $\varphi_1^{(t)}, \ldots, \varphi_q^{(t)}$ land for every considered integer t at a Misiurewicz point of α-type (Lemma 4.12). By construction, all these Misiurewicz points are narrow and of minimal preperiod $m_j := n+1$.

(2) If $P_c^n(c) \in \,]0, \zeta\alpha[\, \cup \ldots \cup \,]0, \zeta^{d-1}\alpha[$, then there is $t \in \{1, \ldots, d-1\}$ such that

$$P_c^n(K_c(c, j)) \,\cap\, \{\zeta\alpha, \ldots, \zeta^{d-1}\alpha\} = \{\zeta^t \alpha\}.$$

Thus there are exactly q angles

$$\varphi_1^{(t)}, \ldots, \varphi_q^{(t)} \;\in\; [\vartheta_j, \vartheta_{j+1}]$$

which are mapped to the angles $\varphi_1, \ldots, \varphi_q$ by

$$\mathbb{S}^1 \to \mathbb{S}^1, \; \vartheta \mapsto d^{n+1} \vartheta.$$

Thus the parameter rays with these angles land at a narrow Misiurewicz point of α-type and has minimal preperiod $m_j = n+1$.

It remains to prove that the minimal preperiods are different for different branches: As we have seen above the preperiod m_j is exactly the number of iterations needed to map the j-th region behind the critical value c biholomorphically to a region containing one of the points $\zeta\alpha, \ldots, \zeta^{d-1}\alpha$. Obviously, it cannot happen that two such regions are mapped at the same time to regions containing one of the points $\zeta\alpha, \ldots, \zeta^{d-1}\alpha$. This proves the claim. □

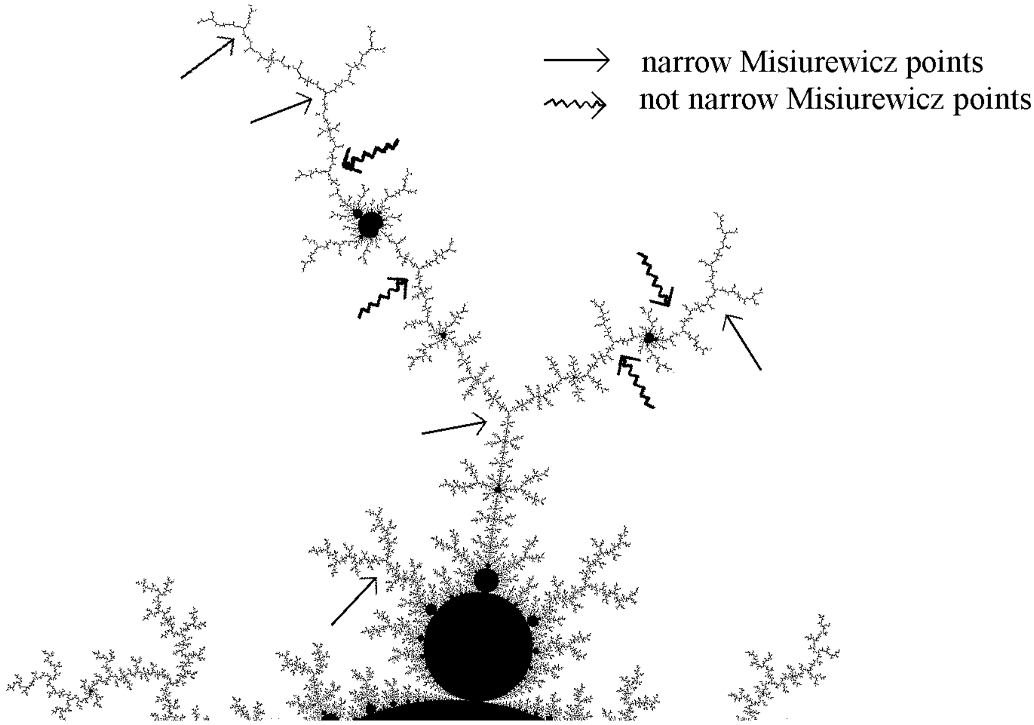

Figure 14: Some narrow and not narrow Misiurewicz points are marked in the 1/3-Limb of the Mandelbrot Set by different types of arrows.

Lemma 4.43 (Regions behind Narrow Preperiodic Points are Nice)
Let $c \in \mathcal{M}_d$ such that all fixed points of P_c are repelling and let z_{-k} be a narrow preperiodic point of α-type and preperiod k. Then the regions $\mathcal{R}_1, \ldots, \mathcal{R}_{q-1}$ behind z_{-k} are pairwise piecewise dynamically homeomorphic.

Proof. By assumption,

$$P_c^{k-1}(z_{-k}) \in \{\zeta\alpha, \ldots, \zeta^{d-1}\alpha\}$$

and P_c^{k-1} is injective on $\mathcal{R}_j$ for $1 \leq j \leq q-1$. Thus for every j, there exists $m_j \in \{1, \ldots, q\}$ such that

$$P_c^{m_j} \circ P_c^{k-1} \,:\, \mathcal{R}_j \longrightarrow \mathcal{R}_0(\alpha)$$

is biholomorphic. For $i \in \{1, \ldots, q-1\}$, the mapping

$$P_c^{-(k-1+m_j)} \circ P_c^{k-1+m_i} \,:\, \mathcal{R}_i \longrightarrow \mathcal{R}_j$$

is a piecewise dynamic homeomorphism if the branch of the inverse is chosen correctly. □

Lemma 4.44 (Pullback of Critical Regions for α-Type)
Let $c \in \mathcal{M}_d$ be an α-type Misiurewicz point of preperiod k such that $q \geq 3$ parameter rays land at c. Then there exists a sequence of points $(w_n)_n \subset \,]\alpha, 0]$ converging to α and a point

$$w \in \,]0, \zeta\alpha] \,\cup \ldots \cup \,]0, \zeta^{d-1}\alpha]$$

such that $P_c^{nq}(w_n) = w$ and

$$P_c^{nq} : \mathcal{R}(\alpha, w_n) \longrightarrow \mathcal{R}(\alpha, w)$$

is biholomorphic for all $n \geq 1$.

Proof. Consider the biholomorphic mapping

$$P_c^q : \mathcal{R}(\alpha, 0) \longrightarrow \mathcal{R}(\alpha, P_c^q(0)).$$

Since no point of the critical orbit is contained in a region behind the critical value, we have

$$P_c^q(0) \in \mathcal{R}_{\zeta\alpha}(0) \cup \ldots \cup \mathcal{R}_{\zeta^{d-1}\alpha}(0).$$

(1) If $P_c^q(0) \notin \mathcal{R}(0, \zeta\alpha) \cup \ldots \cup \mathcal{R}(0, \zeta^{d-1}\alpha)$, then

$$\zeta^j\alpha \in \,]0, P_c^q(0)]$$

for a unique integer $j \in \{1, \ldots, d-1\}$. Then there exists $w_1 \in \,]\alpha, 0[$ such that

$$P_c^{-q}(\zeta^j\alpha) \cap \,]\alpha, 0[\,= \{w_1\};$$

Defining $w := \zeta^j\alpha$, the map

$$P_c^q \;:\; \mathcal{R}(\alpha, w_1) \rightarrow \mathcal{R}(\alpha, w)$$

is biholomorphic.

(2) If $P_c^q(0) \in \mathcal{R}(0, \zeta\alpha) \cup \ldots \cup \mathcal{R}(0, \zeta^{d-1}\alpha)$, then there exists a unique integer $j \in \{1, \ldots, d-1\}$ and a unique point $w_1 \in \,]\alpha, 0]$ such that

(a) $w := P_c^q(w_1) \in \,]0, \zeta^j\alpha[$ and

(b) for every point $w' \in \,]w_1, 0]$, the image point $P_c^q(w')$ does not separate the points α, $\zeta\alpha$, …, $\zeta^{d-1}\alpha$.

Then the map

$$P_c^q : \mathcal{R}(\alpha, w_1) \rightarrow \mathcal{R}(\alpha, w)$$

is biholomorphic.

To find the whole sequence $(w_n)_{n\geq 1}$ we pull w_1 back by P_c^q: if w_n is already defined then

$$f_n := P_c^q \;:\; \mathcal{R}(\alpha, w_n) \rightarrow \mathcal{R}(\alpha, w_{n-1})$$

is biholomorphic ($w_0 := w$) and $w_n \in \mathcal{R}(\alpha, w_{n-1})$. Let

$$w_{n+1} := f_n^{-1}(w_n).$$

Since the α-fixed point is repelling, the sequence $(w_n)_n$ converges to α. □

4.7.2 Misiurewicz Points in General

Lemma 4.45 (Misiurewicz Points are contained in Certain Wakes)
Let c be a Misiurewicz point of preperiod k and period l such that $q \geq 3$ parameter rays land at c. Let $\vartheta_1 < \ldots < \vartheta_q$ denote the angles of these rays. Then two parameter rays with angles in

$$\{d^{k+j}\vartheta_m \ : \ j \in \{0, \ldots, ql-1\}\}$$

land at the essential root of a hyperbolic component H_{ql} of period ql; this component bifurcates immediately from a hyperbolic component H_l of period l. Thus the Misiurewicz point c is contained in the p/q-sublimb of a sector of H_l, where $p/q \neq 1/2$.

Proof. All parameter rays with angles $\vartheta_1, \ldots, \vartheta_q$ have the same preperiod k and the same period ql. Since the orbit of the critical value does not contain the critical point, for every integer $n \geq k$, the dynamic rays with angles $d^n\vartheta_1, \ldots, d^n\vartheta_q$ are pairwise distinct, periodic and they land at a common periodic point of period l. Denote the orbit of this periodic point by $a_0, \ldots, a_{l-1}$ such that a_1 is the characteristical point of this orbit.

Since at least three dynamic rays land at every point of this orbit, they are permuted transitively by the dynamics. The characteristic sector behind a_1 (this is the region behind a_1 containing the critical value) is bounded by two dynamic rays with angles in

$$\{d^{k+j}\vartheta_m \ : \ j \in \{0, \ldots, ql-1\}\}.$$

By Lemma 4.12 the parameter rays with these two angles land together at the essential root of a hyperbolic component H_{ql} of period ql. Since the period of a_1 is equal to l, that component H_{ql} is attached to a hyperbolic component H_l of period l. The parameter c is then contained in the p/q-sublimb of a sector of H_l where $p \in \{1, \ldots, q-1\}$ such that $\gcd(p, q) = 1$. Since $q \geq 3$, $p/q \neq 1/2$. □

The following statement generalizes Lemma 4.44:

Lemma 4.46 (Pullback of Critical Regions in General)
Let $c \in \mathcal{M}_d$ be a Misiurewicz point of preperiod k and period l such that $q \geq 3$ parameter rays land at c. Denote the orbit of $P_c^k(c)$ by $z_0, \ldots, z_{l-1}$ such that z_1 is the characteristic point of this orbit. Then there exists a sequence of points $(w_n)_n \subset]z_0, 0]$ converging to z_0 and a point

$$w \in]0, \zeta z_0] \cup \ldots \cup]0, \zeta^{d-1} z_0]$$

such that $P_c^{nlq}(w_n) = w$ and

$$P_c^{nlq} : \mathcal{R}(z_0, w_n) \to \mathcal{R}(z_0, w)$$

is biholomorphic for all $n \geq 1$.

Proof. By Lemma 4.45, the parameter c is contained in the wake of a hyperbolic component H_l of period l. Since at least three parameter rays land at c, this parameter cannot be contained in the $1/2$-sublimb of any sector of H_l and we can construct simply connected neighborhoods $\tilde{U}$, $\tilde{V}$ of the critical point as shown in Section 4.4.2 to get a proper mapping

$$P_c^l : \tilde{U} \to \tilde{V}$$

of degree d. Using Lemma 4.33 we can proceed as in the proof of Lemma 4.44 to find the sequence $(w_n)_n$ replacing P_c^q by P_c^{lq}. □

4.7.3 Construction of Piecewise Dynamic Homeomorphisms

Lemma 4.47 (Piecewise Dynamic Homeomorphisms of α-type)
Let $c_0 \in \mathcal{M}_d$ be an α-type Misiurewicz point of preperiod k such that $q \geq 3$ parameter rays with angles $\vartheta_1, \ldots, \vartheta_q$ land at c_0. Let

$$E_{c_0} := P_{c_0}^{-1}(\overline{R_b(c_0)}).$$

Then the regions behind the critical value are pairwise piecewise dynamically homeomorphic with expansion relative to E_{c_0}. Every such piecewise dynamic homeomorphism $\mathcal{R}_1 \to \mathcal{R}_2$ is described by

(1) a set of (pre-)periodic angles of dynamic rays cutting $\mathcal{R}_1$ into finitely many pieces and

(2) a composition of the maps P_{c_0}, certain branches of $P_{c_0}^{-1}$ and $z \mapsto \zeta z$ defined on every such piece.

For every parameter c behind c_0, we denote the landing point of the dynamic rays with angles $\vartheta_1, \ldots, \vartheta_q$ by α_{-k}. The same sets of angles as used for the parameter c_0, cut the regions behind α_{-k} in the dynamic plane of P_c into certain pieces; the relative positions of these pieces do not change for all parameters behind the Misiurewicz point c_0. Then for all parameters c behind c_0, the piecewise dynamic homeomorphisms between the regions behind c_0 yield piecewise dynamic homeomorphisms between the regions behind α_{-k} with expansion relative to $E_c := P_c^{-1}(\overline{R_b(\alpha_{-k})})$.

Proof. In the following we will make use of the fact that the Julia set of a Misiurewicz polynomial is locally connected so that all dynamic rays land and Lemma 4.39 applies. Then the dynamic rays needed to define the piecewise dynamic homeomorphism in the dynamic plane of P_{c_0} will not change their landing properties if we replace the parameter c_0 by a parameter c behind c_0. This will be made precise later.

Step 1: *Every region $\mathcal{R}$ behind the critical value c_0 can be mapped biholomorphically to a region $\mathcal{R}'$ containing exactly one or exactly $d-1$ of the points $\alpha, \ldots, \zeta^{d-1}\alpha$*
Since $0 \notin \mathcal{R}$ and none of the points $\alpha, \ldots, \zeta^{d-1}\alpha$ is contained in $\mathcal{R}$, there exists a largest integer $m' \geq 1$ such that

$$P_{c_0}^{m'} : \mathcal{R} \to \mathcal{R}_0\left(P_{c_0}^{m'}(c_0)\right)$$

is biholomorphic. At least one of the immediate preimages of α is contained in this region and we can find $m'' \in \{m', m'-1\}$ such that

$$\mathcal{R}' := P_{c_0}^{m''}(\mathcal{R})$$

contains at least one of the points $\zeta\alpha, \ldots, \zeta^{d-1}\alpha$ but not α itself. Therefore there is $j \in \{0, \ldots, d-1\}$ such that

$$P_{c_0}^{m''}(c_0) \in \,]0, \zeta^j\alpha].$$

Applying a rotation it is sufficient to treat the case

$$\alpha_{-m} := P_{c_0}^{m''}(c_0) \in \,[\alpha, 0[,$$

where $m := k - m''$; then the preperiod of α_{-m} is $\max\{m, 0\}$.

Step 2: *The region $\mathcal{R}_0(\alpha_{-m})$ is piecewise dynamically homeomorphic to the region $\mathcal{R}_0(\alpha)$.* The case $\alpha_{-m} \in \{\alpha, \ldots, \zeta^{d-1}\alpha\}$ is trivial. Otherwise by Lemma 4.38, there is a sequence of points $(w_n)_n \subset]\alpha, 0[$ converging to α such that

$$P_{c_0}^{nq} : \mathcal{R}(\alpha, w_n) \to \mathcal{R}(\alpha, 0)$$

is biholomorphic. Since $P_{c_0}^m$ maps a small neighborhood of α_{-m} biholomorphically to a small neighborhood of α and since the dynamic rays landing at α are permuted transitively by P_{c_0}, there exist integers $i \in \{0, \ldots, q-1\}$, $N \geq 1$ and a point $w'_N \in]\alpha_{-m}, 0[$ such that

$$P_{c_0}^{m+i} : \mathcal{R}(\alpha_{-m}, w'_N) \to \mathcal{R}(\alpha, w_N)$$

and thus

$$P_{c_0}^{m+i+Nq} : \mathcal{R}(\alpha_{-m}, w'_N) \to \mathcal{R}(\alpha, 0)$$

are biholomorphic. Since the closure of $\mathcal{R}(\alpha_{-m}, w'_N)$ is contained in $\mathcal{R}(\alpha, 0)$, by Lemma 4.39, we find a fixed point $w \in]\alpha_{-m}, w'_N[$ of $P_{c_0}^{m+i+Nq}$. Then

$$\mathcal{R}_0(\alpha_{-m}) \longrightarrow \mathcal{R}_0(\alpha)\,,\; z \mapsto \begin{cases} P_{c_0}^{m+i+Nq}(z) & : \quad z \in \mathcal{R}(\alpha_{-m}, w) \\ z & : \quad z \in \mathcal{R}_0(\alpha_{-m}) \setminus \overline{\mathcal{R}(\alpha_{-m}, w)} \end{cases}$$

is a piecewise dynamic homeomorphism (compare Figure 15).

We argue now that this piecewise dynamic homeomorphism for the parameter c_0 also yields a piecewise dynamic homeomorphism for the parameters c behind c_0.

- In the dynamic plane of P_{c_0} the dynamic rays with angles in

  $$\{d^{m''}\vartheta_1, \ldots, d^{m''}\vartheta_q\} \subset \mathbb{S}^1$$

  land at $\alpha_{-m}(c_0) := P_{c_0}^{m''}(c_0)$. By Lemma 4.12, the dynamic rays with angles $\vartheta_1, \ldots, \vartheta_q$ land together at an k-th preimage $\alpha_{-k}(c)$ of α for all parameters c behind c_0. Thus for all these parameters, the dynamic rays with angles in $\Theta(\alpha_{-m}(c_0))$ land at an m-th preimage $\alpha_{-m}(c)$ of α; this preimage depends continuously on c.
- By Lemma 4.12, the dynamic rays with angles in $\Theta(w)$ land for all parameters c behind c_0 at a common repelling periodic point $w = w(c)$ which depends continuously on c.

Thus the dynamic rays with angles in $\Theta(\alpha_{-m}) \cup \Theta(w)$ have the same landing properties for all parameters c behind c_0. This proves the existence of a piecewise dynamic homeomorphism

$$\mathcal{R}_0(\alpha_{-m}(c)) \to \mathcal{R}_0(\alpha)$$

for all parameters c behind c_0.

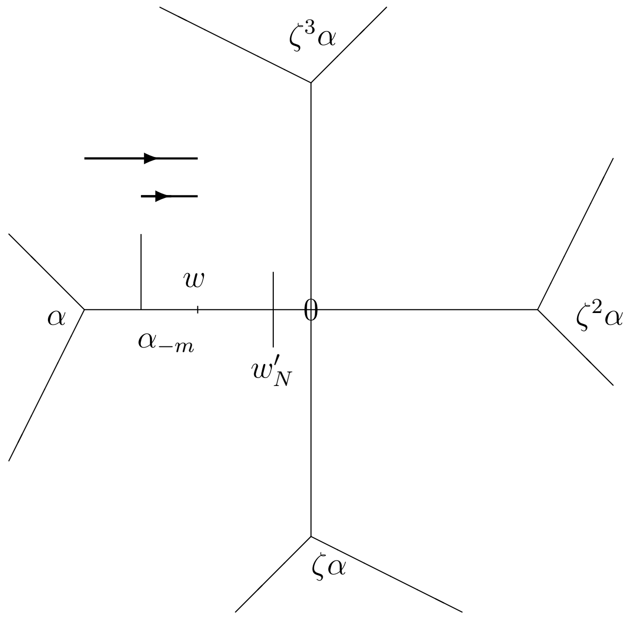

Figure 15: The Construction of a piecewise dynamic homeomorphism $\mathcal{R}_0(\alpha_{-m}) \to \mathcal{R}_0(\alpha)$ in the case $\alpha_{-m} \in]\alpha, 0[$. The region between α_{-m} and w (marked by the short arrow) is mapped to the region between α and w (marked by the long arrow) by an iterate of the polynomial P_{c_0}.

Step 3: *The region* $\mathcal{R}_\alpha(\alpha_{-m})$ *is piecewise dynamically homeomorphic to the region* $\mathcal{R}_0(\alpha)$. By Lemma 4.44, there exists a sequence of points $(w_n)_n \subset]\alpha, \alpha_{-m}[$ converging to α such that

(1) for all $n \geq 1$, the map

$$P_{c_0}^{nq} \;:\; \mathcal{R}(\alpha, w_n) \longrightarrow \mathcal{R}(\alpha, w)$$

is biholomorphic and

(2) $w := P_{c_0}^q(w_1) \in]0, \zeta^j\alpha]$ for some $j \in \{1, \ldots, d-1\}$.

Since $P_{c_0}^m$ maps a small neighborhood of α_{-m} biholomorphically to a small neighborhood of α and since P_{c_0} permutes the dynamic rays landing at α transitively, there exist integers $N \geq 1$, $s \in \{0, \ldots, q-1\}$ and a point $w'_N \in P_{c_0}^{-(m+s)}(w_N) \cap]\alpha, \alpha_{-m}[$ such that the mappings

$$\mathcal{R}(\alpha_{-m}, w'_N) \xrightarrow{P_{c_0}^{m+s}} \mathcal{R}(\alpha, w_N) \xrightarrow{P_{c_0}^{Nq}} \mathcal{R}(\alpha, w)$$

is biholomorphic; this yields the piecewise dynamic homeomorphism

$$f : \mathcal{R}(\alpha_{-m}, w'_N) \longrightarrow \mathcal{R}(\zeta^{-j}\alpha, \zeta^{-j}w) \;:\; z \mapsto \zeta^{-j} P_{c_0}^{m+s+Nq}(z).$$

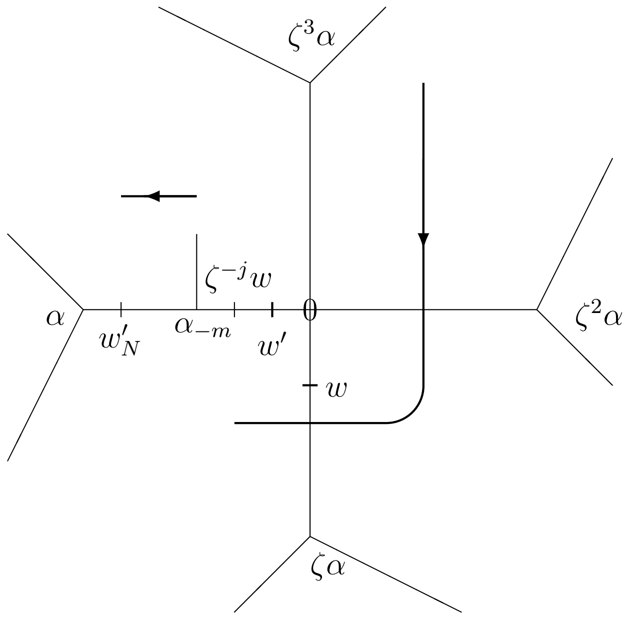

Figure 16: The first step to construct a piecewise dynamic homeomorphism from $\mathcal{R}_\alpha(\alpha_{-m})$ to $\mathcal{R}_0(\alpha)$: f maps $\mathcal{R}(\alpha_{-m}, w'_N)$ biholomorphically to $\mathcal{R}(\zeta^{-j}\alpha, \zeta^{-j}w)$ (again both regions are marked by arrows).

Similarly, since $\alpha = P_{c_0}^{k+1}(0)$, there exists an integer $N' \geq k+1$ and a point $w' \in \,]\zeta^{-j}w, 0[$ such that

$$P_{c_0}^{N'} \;:\; \mathcal{R}(w', 0) \longrightarrow \mathcal{R}(w, \alpha)$$

is biholomorphic. Let

$$\alpha_{-(m+N')} \in \,]w', 0[\quad \text{such that } P_{c_0}^{N'}(\alpha_{-(m+N')}) = \alpha_{-m}.$$

Similarly, since $\alpha = P_{c_0}^{m+N'}(\alpha_{-(m+N')})$, there exists an integer $N'' \geq m + N'$ and a point

$$w'' \in \,]\alpha_{-(m+N')}, 0[$$

such that

$$P_{c_0}^{N''} \;:\; \mathcal{R}(\alpha_{-(m+N')}, w'') \longrightarrow \mathcal{R}(\alpha, w)$$

is biholomorphic. By Lemma 4.39, there exists a fixed point $v' \in \,]\alpha_{-(m+N')}, w''[$ of $P_{c_0}^{N''}$. This yields the piecewise dynamic homeomorphism

$$g : \mathcal{R}(f^{-1}(\alpha_{-(m+N')}), \alpha_{-m}) \longrightarrow \mathcal{R}(\alpha, \zeta^{-j}\alpha)\;,$$

$$z \mapsto \begin{cases} P_{c_0}^{N''}(f(z)) & : \; z \in \mathcal{R}(f^{-1}(\alpha_{-(m+N')}), f^{-1}(v')) \\ f(z) & : \; z \in \mathcal{R}(f^{-1}(\alpha_{-(m+N')}), \alpha_{-m}) \setminus \overline{\mathcal{R}(f^{-1}(\alpha_{-(m+N')}), f^{-1}(v'))} \end{cases} .$$

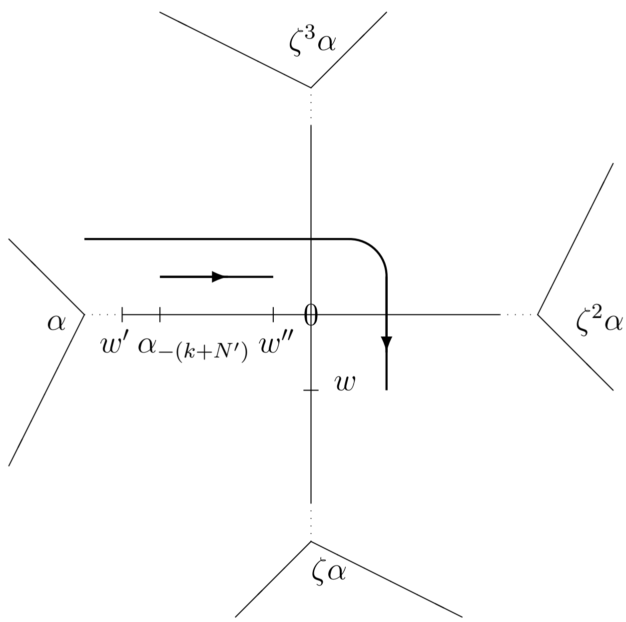

Figure 17: The second step to construct a piecewise dynamic homeomorphism between $\mathcal{R}_\alpha(\alpha_{-m})$ and $\mathcal{R}_0(\alpha)$: $P_{c_0}^{N''}$ maps $\mathcal{R}(\alpha_{-(k+N')}, w'')$ biholomorphically to $\mathcal{R}(\alpha, w)$ (again these regions are marked by arrows).

By Lemma 4.39, there exists a fixed point $v \in \,]f^{-1}(\alpha_{-(m+N')}), \alpha_{-m}[$ of g. This defines the piecewise dynamic homeomorphism

$$h : \mathcal{R}_\alpha(\alpha_{-m}) \longrightarrow \mathcal{R}_0(\alpha),\ z \mapsto \begin{cases} \zeta^j g(z) & : \quad z \in \mathcal{R}(v, \alpha_{-m}) \\ \zeta^j z & : \quad z \in \mathcal{R}_\alpha(\alpha_{-m}) \setminus \overline{\mathcal{R}(v, \alpha_{-m})} \end{cases} .$$

Now we have to prove that this construction gives a piecewise dynamic homeomorphism $\mathcal{R}_\alpha(\alpha_{-m}) \to \mathcal{R}_0(\alpha)$ not only for the parameter c_0 but also for all parameters c behind c_0. To do so we have to assure that the dynamic rays with angles in $\Theta(\alpha_{-m})$, $\Theta(v)$ and $\Theta(f^{-1}(v'))$ have the same landing properties for all parameters behind c_0, i.e. for all these parameters, the dynamic rays

(1) with angles in $\Theta(\alpha_{-m})$ land together at a point $\alpha_{-m}(c)$,

(2) with angles in $\Theta(v)$ land together at a point $v(c)$,

(3) with angles in $\Theta(f^{-1}(v'))$ land together at a point $f^{-1}(v')(c)$.

We use Lemma 4.12 and Lemma 4.11 as follows:

(1) By Lemma 4.12 part (3), the dynamic rays with angles $\vartheta_1, \ldots, \vartheta_q$ land at a k-th preimage $\alpha_{-k}(c)$ of α for all parameters c behind c_0. By the continuity of P_c, the dynamic rays with angles $d^n\vartheta_1, \ldots, d^n\vartheta_q$ land together at $P_c^n(\alpha_{-k})$ for all $n \geq 0$; in particular, the dynamic rays with angles in $\Theta(\alpha_{-m})$ land at an m-th preimage $\alpha_{-m}(c)$ of α.

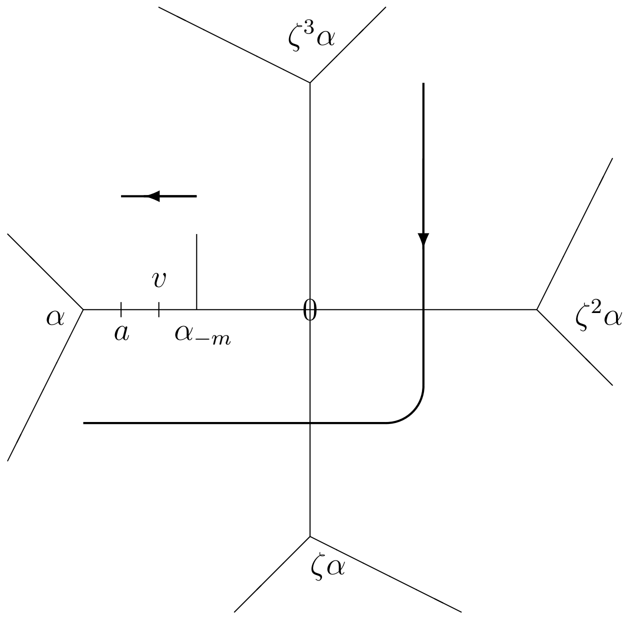

Figure 18: The third step to construct a piecewise dynamic homeomorphism between $\mathcal{R}_\alpha(\alpha_{-m})$ and $\mathcal{R}_0(\alpha)$: g maps $\mathcal{R}(\alpha_{-k}, a)$ biholomorphically to $\mathcal{R}(\alpha, \zeta^{-j}\alpha)$; here $a := f^{-1}(\alpha_{-(m+N')})$.

(2) By construction, either v or $\zeta^j v$ is a P_{c_0}-periodic point. Thus for all parameters c behind c_0, the dynamic rays with angles in $\Theta(v)$ land together (compare Lemma 4.12 part (1) and (2)).

(3) Since v' is P_{c_0}-periodic, the dynamic rays with angles in $\Theta(v')$ land together for all parameters c behind c_0. By symmetry, this is true also for the dynamic rays with angles in $\Theta(\zeta^j v')$. The map f^{-1} is defined by a certain branch of $z \mapsto P_{c_0}^{-(m+s+Nq)}(\zeta^j z)$. By construction, no point of the P_{c_0}-orbit of $f^{-1}(v')$ is contained in E_{c_0}: we never continue iterating P_{c_0} when we reach E_{c_0} in the construction of the piecewise dynamic homeomorphism. This is also true for the parameters c behind c_0. Thus for all parameters c behind c_0, the landing point of the dynamic rays with angles in $\Theta(f^{-1}(v'))$ is never contained in the backward orbit of the critical point and this proves that the dynamic rays with angles in $\Theta(f^{-1}(v'))$ land together for all these parameters (compare Lemma 4.11).

Step 4: *Construction of a piecewise dynamic homeomorphism between two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ behind the critical value c_0*

In the last three steps we have proved that there are piecewise dynamic homeomorphisms

$$f_1 : \mathcal{R}_1 \to \mathcal{R}_0(\alpha)$$
$$f_2 : \mathcal{R}_2 \to \mathcal{R}_0(\alpha).$$

By Lemma 4.36,

$$f_2^{-1} \circ f_1 : \mathcal{R}_1 \to \mathcal{R}_2$$

is a piecewise dynamic homeomorphism and it remains to prove that it is expanding relative to E_{c_0}. We have shown in Step 2 and Step 3 before that condition three in Definition 4.35 is satisfied. Thus it remains to check condition four: in the construction of f_1 and f_2 we use a composition of certain iterates of P_{c_0} and $z \mapsto \zeta z$ but we have never used some branches of $P_{c_0}^{-1}$, i.e. for every $w \in \mathcal{R}_1 \cup \mathcal{R}_2$ there exist integers $n_1(w), n_2(w) \in \mathbb{N}_0$ and $j_1(w), j_2(w) \in \{0, \dots, d-1\}$ such that

$$f_1(w) = \zeta^{j_1(w)} P_{c_0}^{n_1(w)}(w) \quad \text{and} \quad f_2(w) = \zeta^{j_2(w)} P_{c_0}^{n_2(w)}(w).$$

Moreover we never continue iterating when we have reached E_{c_0}, i.e.

$$P_{c_0}^m(w) \notin E_{c_0} \quad \text{for} \quad w \in \mathcal{R}_2,\ m \in \{0, \dots, n_2(w) - 1\}.$$

Thus we have

$$\Big\{ P_{c_0}^m \left((f_2^{-1} \circ f_1)(z) \right) \ : \ z \in \mathcal{R}_1,\ 0 \le m \le n_2\Big((f_2^{-1} \circ f_1)(z) \Big) - 1 \Big\} \cap E_{c_0} = \emptyset.$$

We conclude

$$n_2(w) = \mathrm{iter}(f_2, w) \ge \mathrm{iter}(f_2, w) - \mathrm{iter}(f_1, z) = -\mathrm{iter}(f_2^{-1} \circ f_1, z)$$

for all $z \in \mathcal{R}_1$ and $w := (f_2^{-1} \circ f_1)(z)$. This proves the expansion property relative to E_{c_0} for the parameter c_0. Since the piecewise dynamic homeomorphism for every parameter c behind c_0 is simply given by replacing P_{c_0} by P_c, the constructed piecewise dynamic homeomorphisms are expanding relative to E_c.

□

Lemma 4.48 (Piecewise Dynamic Homeomorphisms in General I)
Let $c_0 \in \mathcal{M}_d$ be a Misiurewicz point of preperiod k and period l such that $q \ge 3$ parameter rays with angles $\vartheta_1, \dots, \vartheta_q$ land at c_0. Let $E_{c_0} := P_{c_0}^{-1}(\overline{R_b(c_0)})$ and denote by $z_0, \dots, z_{l-1}$ the repelling periodic orbit of $P_{c_0}^k(0)$ such that z_1 is the characteristic point of this orbit. Then for every integer $m > 0$ and for every m-th preimage

$$z_{-m} \in \,]z_0, 0[$$

of z_0 with $P_{c_0}^j(z_{-m}) \notin E_{c_0}$ ($j \in \mathbb{N}$), there exist piecewise dynamic homeomorphisms

$$\mathcal{R}_0(z_{-m}) \longrightarrow \mathcal{R}_0(z_0) \quad \textit{and} \quad \mathcal{R}_{z_0}(z_{-m}) \longrightarrow \mathcal{R}_0(z_0)$$

with expansion relative to E_{c_0}.
For every parameter c behind c_0, we denote the landing point of the dynamic rays with angles in $\Theta(z_{-m})$ by $z_{-m}(c)$. The same sets of angles used for the parameter c_0 cut the regions $\mathcal{R}_0(z_{-m}(c))$ and $\mathcal{R}_{z_0(c)}(z_{-m}(c))$ in the dynamic plane of P_c into certain pieces; the relative positions of these pieces do not change for all parameters behind the Misiurewicz point c_0. Then for all these parameters c, we find piecewise dynamic homeomorphisms

$$\mathcal{R}_0(z_{-m}(c)) \longrightarrow \mathcal{R}_0(z_0(c)) \quad \textit{and} \quad \mathcal{R}_{z_0(c)}(z_{-m}(c)) \longrightarrow \mathcal{R}_0(z_0(c))$$

with expansion relative to $E_c := P_c^{-1}(\overline{\mathcal{R}_b(z_{-k})})$ replacing P_{c_0} by P_c in the definition of the piecewise dynamic homeomorphisms for c_0.

PROOF. By Lemma 4.45, the parameter c_0 is contained in the p/q-sublimb of a certain hyperbolic component H_l: two parameter rays with angles

$$\vartheta', \vartheta \in \{d^{k+j}\vartheta_1 \ : \ j \in \{0, \ldots, ql-1\}\}$$

land at the essential root of a hyperbolic component H_{ql} of period ql which bifurcates from the hyperbolic component H_l at an internal angle $p/q \neq 1/2$. Denote the angles of the parameter rays landing at the essential root of H_l by $\varphi' < \varphi$. In the dynamic plane of P_{c_0} the dynamic rays with angles φ and φ' land at a point y_1 of period l' dividing l. Its P_{c_0}-orbit is denoted by $y_0, \ldots, y_{l'-1}$. The number of dynamic rays landing at the orbit of y_0 is either l or $2l$, depending on whether H_l is primitive or not. Since $ql \geq 3l$ dynamic rays land at the orbit of z_0, $y_0 \neq z_0$. As described in Section 4.4.2 we find simply connected neighborhoods $\tilde{U}$ and $\tilde{V}$ of the critical point such that

$$P^l_{c_0} : \tilde{U} \to \tilde{V}$$

is a proper mapping of degree d; the domain $\tilde{V}$ is bounded by the dynamic rays landing at the points $y_0, \ldots, \zeta^{d-1}y_0$. The regions $\tilde{U}$ and $\tilde{V}$ can be extended to get a polynomial-like mapping

$$P^l_{c_0} : U \to V$$

of degree d. If c_0 is contained in the small Multibrot set centered at H_l then z_0 is the α-fixed point of the polynomial-like mapping $P^l_{c_0} : U \to V$ and y_0 is one of its β-fixed points. In any case the dynamic rays landing at the critical point separate the points y_0 and z_0 from each other (compare [McM1, Theorem 7.9]).

We have to transfer the construction of Lemma 4.47 for the case $l = 1$ to the general case $l > 1$. To do so we use Lemma 4.33. This lemma guarantees that for any two pinching points $w_1, w_2 \in \mathcal{R}(y_0, \zeta y_0, \ldots, \zeta^{d-1}y_0) \cap K_{c_0}$ the mapping $P^l_{c_0}$ is injective on $\mathcal{R}(w_1, w_2)$ if this region contains neither the critical point nor one of the points $y_0, \ldots, \zeta^{d-1}y_0$ in its interior.

Step 1: Construction of a piecewise dynamic homeomorphism $\mathcal{R}_0(z_{-m}) \longrightarrow \mathcal{R}_0(z_0)$

By Lemma 4.38, there exist a point $w' \in \,]z_{-m}, 0[$ and an integer $n \geq m$ such that

$$P^n_{c_0} : \mathcal{R}(z_{-m}, w') \to \mathcal{R}(z_0, 0)$$

is biholomorphic. By Lemma 4.39, we find a fixed point $w \in \,]z_{-m}, w'[$ of $P^n_{c_0}$ and a piecewise dynamic homeomorphism is given by

$$f_{c_0} : \mathcal{R}_0(z_{-m}) \to \mathcal{R}_0(z_0), \ z \mapsto \begin{cases} P^n_{c_0}(z) & : \quad z \in \mathcal{R}(z_{-m}, w) \\ z & : \quad z \in \mathcal{R}_0(z_{-m}) \setminus \overline{\mathcal{R}(z_{-m}, w)} \end{cases} .$$

Step 2: Construction of a piecewise dynamic homeomorphism $\mathcal{R}_{z_0}(z_{-m}) \longrightarrow \mathcal{R}_0(z_0)$

By Lemma 4.33, the analogous construction can be used as in the case of an α-type Misiurewicz point; we simply have to replace the map P_{c_0} by $P^l_{c_0}$ in Lemma 4.47 and we have to use Lemma 4.46 instead of Lemma 4.44.

Similarly as in Lemma 4.47, one can see from the construction and from the assumption $\{z_{-m}, P_{c_0}(z_{-m}), \ldots\} \cap E_{c_0} = \emptyset$ that the dynamic rays used in the construction of f_{c_0} have the same landing property for all parameters c behind c_0. Thus for all these parameters, we have piecewise dynamic homeomorphisms

$$\mathcal{R}_0(z_{-m}(c)) \to \mathcal{R}_0(z_0(c)) \quad \text{and} \quad \mathcal{R}_{z_0(c)}(z_{-m}(c)) \to \mathcal{R}_0(z_0(c))$$

with expansion relative to E_c. □

The following Lemma is only stated for quadratic polynomials. We would be interested in such a statement for degree $d > 2$ as well to construct homeomorphisms between the trees behind all Misiurewicz points in $\mathcal{M}_d$. The reason why our proof does not work in general is the following: for a simple r-renormalizable quadratic polynomial and a region $\mathcal{R}$ behind a pinching point z of the small Julia set with $\mathcal{R} \cap \{\alpha, \ldots, \zeta^{d-1}\alpha\} = \emptyset$, let $n(z, \mathcal{R})$ denote the number of iterates needed to map $\mathcal{R}$ to a region which contains one of the points $\zeta\alpha, \ldots, \zeta^{d-1}\alpha$. Then

$$n(z, \mathcal{R}) \bmod r$$

does not depend on z; this is in general not true for $d > 2$. We will mark the places in the proof of the next lemma where the assumption $d = 2$ is needed.

Lemma 4.49 (Piecewise Dynamic Homeomorphisms in General II)
Let $d = 2$ and $c_0 \in \mathcal{M}_2$ be a Misiurewicz point of preperiod k and period l such that $q \geq 3$ parameter rays with angles $\vartheta_1, \ldots, \vartheta_q$ land at c_0. Denote by $z_0, \ldots, z_{l-1}$ the orbit of the repelling periodic orbit of $P_{c_0}^k(c_0)$ such that z_1 is its characteristic point. Let

$$E_{c_0} := P_{c_0}^{-1}(\overline{R_b(c_0)}).$$

Then the regions behind the critical value are pairwise piecewise dynamically homeomorphic with expansion relative to E_{c_0}.
For every parameter c behind c_0 we denote by $z_{-k}(c)$ the landing point of the dynamic rays with angles $\vartheta_1, \ldots, \vartheta_q$. The same sets of angles as used in the construction of the piecewise dynamic homeomorphisms for the parameter c_0 give a piecewise dynamic homeomorphism between the regions behind $z_{-k}(c)$ with expansion relative to $E_c := P_c^{-1}(\overline{R_b(z_{-k}(c))})$ for all parameters c behind c_0.

Proof. Since the critical value is eventually mapped to a point of period l, the polynomial P_{c_0} is n-renormalizable of simple type for only finitely many integers $n \geq 1$ and these integers are ordered by division (compare [McM1, Section 7]). The polynomial P_{c_0} can be n-renormalizable of simple type only if n divides l or $n = ql$. Let $r \leq l$ be the largest integer such that P_{c_0} is r-renormalizable of simple type. As described in Section 4.4.2 we construct simply connected neighborhoods $\tilde{U}_r$ and $\tilde{V}_r$ of the critical point such that

$$f : \tilde{U}_r \to \tilde{V}_r, \; z \mapsto P_{c_0}^r(z)$$

is a proper mapping of degree 2.

By the usual thickening procedure along rays and around repelling (pre-)periodic points, we can extend $\tilde{U}_r$ and $\tilde{V}_r$ slightly to get a polynomial-like mapping $P_{c_0}^r : U_r \to V_r$; its filled-in Julia set is denoted by K_r. Denote by α_r (β_r) the α- (β-) fixed point of f. The region $\tilde{V}_r$ is bounded by the dynamic rays landing at β_r and $-\beta_r$. For $j \in \{0, \dots, r-1\}$ let

$$K_r(j) := P_{c_0}^j(K_r)$$

be the j-th small filled-in Julia set; for $j \in \{1, \dots, r-1\}$ let

$$\beta_r(j),\ \beta_r^-(j),\ \alpha_r(j),\ \alpha_r^-(j)$$

be the points in $K_r(j)$ which are mapped to β_r, $-\beta_r$, α_r and $-\alpha_r$ by $P_{c_0}^{r-j}$.

Step 1: *Every region $\mathcal{R}$ behind the critical value c_0 is piecewise dynamically homeomorphic to $\mathcal{R}_{\beta_r^-(1)}(z_{l-(r-1)})$*

The point $z_{l-(r-1)}$ is the point in $K_r(1)$ which maps under $P_{c_0}^{r-1}$ to the point z_0 in $K_r(0)$; thus $z_{l-(r-1)}$ is the pullback of the precharacteristic point z_0 to $K_r(1)$ by $P_{c_0}^{-(r-1)}$. Since $\beta_r(1)$ separates $K_r(1)$ from the critical point, we have

$$\beta_r(1) \notin \mathcal{R}.$$

Part 1: *The region $\mathcal{R}$ can be mapped biholomorphically by an iterate of $P_{c_0}^r$ to a region $\mathcal{R}'$ which contains $\beta_r^-(1)$ but neither $\beta_r(1)$ nor $\alpha_r^-(1)$*

Let $\tilde{r}$ be the first number in $1, 2, \dots, r-1, 0$ such that $K_r(\tilde{r})$ separates α from $-\alpha$. Since $\mathcal{R}\Big(\beta_r(\tilde{r}), \beta_r^-(\tilde{r})\Big)$ contains neither α nor $-\alpha$,

$$\beta_r(\tilde{r}),\ \beta_r^-(\tilde{r}) \in \,]\alpha, -\alpha[.$$

There exists a unique integer $m \geq 0$ such that $P_{c_0}^m$ is injective on $\mathcal{R}$ and that $P_{c_0}^m(\mathcal{R})$ contains $-\alpha$ but not α. Therefore $P_{c_0}^m(c_0) \in \,]\alpha, -\alpha[$; since

$$K_r(j) \cap \,]\alpha, -\alpha[\,= \begin{cases} \emptyset & : \quad j \in \{1, \dots, \tilde{r}-1\} \\ ]\beta_r(j), \beta_r^-(j)[& : \quad j = \tilde{r} \end{cases}$$

and since P_{c_0} is r-renormalizable, the integer $m - (\tilde{r}-1)$ is a multiple of r (this cannot be transferred to the case $d > 2$) and

$$P_{c_0}^m(c_0) \ \in \,]\beta_r(\tilde{r}), \beta_r^-(\tilde{r})[\,\subset \,]\alpha, -\alpha[.$$

Moreover $P_{c_0}^m(c_0) \ \notin \ [\beta_r(\tilde{r}), \alpha_r^-(\tilde{r})]$ since otherwise the point $P_{c_0}^{m-r}(c_0)$ has separated $\beta_r(\tilde{r})$ from $\beta_r^-(\tilde{r})$ and thus α from $-\alpha$ in contradiction to the choice of m. Therefore

$$P_{c_0}^m(c_0) \in \,]\alpha_r^-(\tilde{r}), \beta_r^-(\tilde{r})[\quad \Longrightarrow \quad P_{c_0}^{m-(\tilde{r}-1)}(c_0) \ \in \,]\alpha_r^-(1), \beta_r^-(1)[$$

and $\mathcal{R}' := P_{c_0}^{m-(\tilde{r}-1)}(\mathcal{R})$ contains $\beta_r^-(1)$ but neither $\beta_r(1)$ nor $\alpha_r^-(1)$.

Part 2: *Finding a preimage $z_{-\hat{m}} \in \ [\alpha_r(1), \alpha_r^-(1)[$ of z_0 such that the region $\mathcal{R}'$ is piecewise dynamically homeomorphic to $\mathcal{R}_{\beta_r^-(1)}(z_{-\hat{m}})$*

If

$$z_{-\tilde{m}} := P_{c_0}^{m-(\tilde{r}-1)}(c_0) \in \,]\alpha_r^-(1), \alpha_r(1)],$$

then we are done; otherwise $z_{-\widetilde{m}} \in \,]\alpha_r(1), \beta_r^-(1)[$ and there is a minimal integer $j \geq 0$ such that

$$z_{-\widehat{m}} := P_{c_0}^{-(r-1)}\left(-P_{c_0}^{jr}(P_{c_0}^{r-1}(z_{-\widetilde{m}}))\right) \in \,]\alpha_r^-(1), \alpha_r(1)].$$

Since we want to map the whole region $\mathcal{R}(\beta_r^-(1), \beta_r(1))$ biholomorphically to $\mathcal{R}(-\beta_r, \beta_r)$ we need to know that $d = 2$ for this construction. By Lemma 4.33, the mapping

$$\mathcal{R}_{\beta_r^-(1)}(z_{-\widetilde{m}}) \to \mathcal{R}_{\beta_r^-(1)}(z_{-\widehat{m}}),$$

$$z \mapsto \begin{cases} P_{c_0}^{-(r-1)}\left(-P_{c_0}^{jr}(P_{c_0}^{r-1}(z))\right) & : \quad z \in \mathcal{R}(z_{-\widetilde{m}}, \beta_r^-(1)) \\ z & : \quad z \in \mathcal{R}' \setminus \overline{\mathcal{R}(z_{-\widetilde{m}}, \beta_r^-(1))} \end{cases}$$

is a piecewise dynamic homeomorphism.

Part 3: *Construction of a piecewise dynamic homeomorphism from* $\mathcal{R}_{\beta_r^-(1)}(z_{-\widehat{m}})$ *to* $\mathcal{R}_{\beta_r^-(1)}(z_{l-(r-1)}))$

Obviously, it is enough to construct a piecewise dynamic homeomorphism

$$\mathcal{R}\Big(\beta_r^-(1), z_{-\widehat{m}}\Big) \to \mathcal{R}\Big(\beta_r^-(1), z_{l-(r-1)}\Big).$$

Since

$$P_{c_0}^{r-1} : \mathcal{R}\Big(\beta_r^-(1), \beta_r(1)\Big) \to \mathcal{R}(-\beta_r, \beta_r)$$

is biholomorphic, we can find this piecewise dynamic homeomorphism by constructing a piecewise dynamic homeomorphism

$$\mathcal{R}(-\beta_r, z_{-m'}) \to \mathcal{R}(-\beta_r, z_0)$$

where

$$z_{-m'} := P_{c_0}^{r-1}(z_{-\widehat{m}}).$$

To find this piecewise dynamic homeomorphism we have to distinguish between $r = l$ and $r < l$:

<u>**Case 1:**</u> $r = l$

Since either l or $2l$ dynamic rays land at β_l, we have $z_0 = \alpha_l$.
If $z_{-m'} = \alpha_r = z_0$ then there is nothing to prove.
If $z_{-m'} = -\alpha_r$ then the inverse of the biholomorphic mapping

$$-P_{c_0}^l \ : \ \mathcal{R}(-\beta_l, \alpha_l) \to \mathcal{R}(-\beta_l, -\alpha_l)$$

gives the piecewise dynamic homeomorphism we are looking for.
If $z_{-m'} \in \,]-\alpha_r, \alpha_r[$, then by Lemma 4.48, the region $\mathcal{R}(-\beta_l, z_{-m'})$ is piecewise dynamically homeomorphic to $\mathcal{R}(\beta_l, \alpha_l)$ which again is piecewise dynamically homeomorphic to $\mathcal{R}(-\beta_l, \alpha_l) = -\mathcal{R}(\beta_l, -\alpha_l)$ as we have seen shortly before.

Case 2: $r < l$

Similarly to the definition of the proper mapping f at the beginning of our proof we construct a proper mapping

$$g : \tilde{U}_l \to \tilde{V}_l,\ z \mapsto P_{c_0}^l(z)$$

of degree d: let $\alpha_l := z_0$. By Lemma 4.45, the parameter c_0 is contained in the p/q-sublimb of a certain hyperbolic component H_l of period l where $p/q \neq 1/2$. Denote by φ, φ' the angles of the parameter rays landing at the essential root of H_l. Then the dynamic rays with angles φ, φ' land together at a point of period l' dividing l ($l = l'$ if and only if H_l is primitive). Denote the precharacteristic point of this repelling periodic orbit by β_l and let

$$\beta_l(j) := P_{c_0}^j(\beta_l) \quad \text{for} \quad j \in \{0, \ldots, l-1\}.$$

Then we pull back the region

$$\tilde{V}_l := \mathcal{R}(\beta_l, -\beta_l)$$

along the orbit of β_l to get the region $\tilde{U}_l$ which defines the proper mapping g of degree 2 as above. Both sets $\tilde{U}_l$ and $\tilde{V}_l$ are simply connected neighborhoods of the critical point. Since P_{c_0} is not l-renormalizable, the g-orbit of the critical point eventually leaves $\tilde{U}_l$. For $i \in \{0, \ldots, l-1\}$, let

$$\alpha_l(i),\ \alpha_l^-(i),\ \beta_l(i) \text{ and } \beta_l^-(i)$$

be the points in $P_c^i(\tilde{U}_l)$ mapped by P_c^{l-i} to z_0, $-z_0$, β_l and $-\beta_l$.

We argue now that the critical point separates the α-fixed point α_l of g (compare Definition 4.23) from its β-fixed point β_l: by Lemma 4.46, there exists a sequence of points $(w_n)_{n\geq 1} \subset]\beta_l, 0]$, $w_1 = 0$, converging to β_l such that $P_{c_0}^l : \mathcal{R}(\beta_l, w_{n+1}) \to \mathcal{R}(\beta_l, w_n)$ is biholomorphic for $n \geq 1$. If there was a fixed point $z \in \mathcal{R}(\beta_l, 0)$ of $P_{c_0}^l$, then

$$z,\ \left(P_{c_0}^l|_{\mathcal{R}(\beta_l,0)}\right)^{-1}(z),\ -\left(P_{c_0}^l|_{\mathcal{R}(\beta_l,0)}\right)^{-1}(z)$$

would be three distinct g-preimages of z in $\tilde{U}_l$, in contradiction to the fact that g has degree 2.

Thus there are only two possibilities for the relative positions of the small α- and β-fixed points of f and g (see Figure 19). In the following we will describe how to find a piecewise dynamic homeomorphism between $\mathcal{R}(-\beta_r, z_{-m'})$ and $\mathcal{R}(-\beta_r, z_0)$ only in the situation given on the upper part of Figure 19. The construction of the corresponding piecewise dynamic homeomorphisms in the other case is completely analogous.

We have to distinguish between the following four subcases (remember $z_{-m'} \in]-\alpha_r, \alpha_r]$):

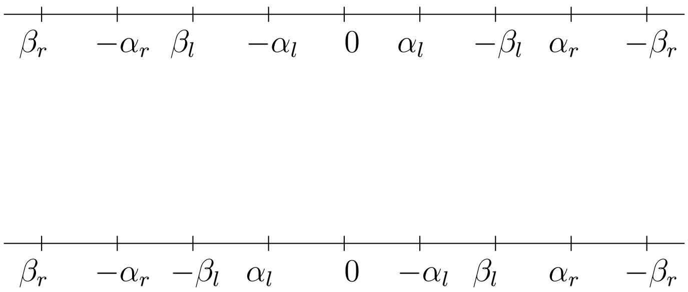

Figure 19: The two possibilities for the relative positions of the small α- and β-fixed points. In both cases the point $z_{-m'}$ separates β_r from $-\beta_r$ and the region we have to consider points to the right.

Case 2.1: $z_{-m'} \in [-\alpha_l, \alpha_l]$

The piecewise dynamic homeomorphism can be constructed analogous to Case 1.

Case 2.2: $z_{-m'} \in \,]\alpha_l, -\beta_l]$

By Lemma 4.33, the mapping

$$\mathcal{R}(\alpha_l, -\beta_l) \longrightarrow \mathcal{R}(-\alpha_l, -\beta_l),\ z \mapsto -P_{c_0}^l(z)$$

is biholomorphic. Then there exists an integer $j \geq 1$ such that $\left(-P_{c_0}^l\right)^j = -P_{c_0}^{jl}$ is injective on $\mathcal{R}(z_{-m'}, -\beta_l)$ and

$$-P_{c_0}^{jl}(z_{-m'}) \in \,]-\alpha_l, \alpha_l].$$

This yields the piecewise dynamic homeomorphism

$$\mathcal{R}_{-\beta_r}(z_{-m'}) \to \mathcal{R}_{-\beta_r}\left(-P_{c_0}^{jl}(z_{-m'})\right),$$

$$z \mapsto \begin{cases} -P_{c_0}^{jl}(z) & : \quad z \in \mathcal{R}(z_{-m'}, -\beta_l) \\ z & : \quad z \in \mathcal{R}_{-\beta_r}(z_{-m'}) \setminus \overline{\mathcal{R}(z_{-m'}, -\beta_l)} \end{cases}$$

By Case 1, there is a piecewise dynamic homeomorphism

$$\mathcal{R}_{-\beta_r}\left(-P_{c_0}^{jl}(z_{-m'})\right) \longrightarrow \mathcal{R}_{-\beta_r}(\alpha_l).$$

The composition of both gives the piecewise dynamic homeomorphism we are looking for.

Case 2.3: $z_{-m'} \in \,]-\alpha_r, -\alpha_l[$

Here we can reduce the problem to Case 1 as well: by Lemma 4.34, there exists a preimage $w \in \,]-\alpha_l, 0[$ of α_r. By the same arguments as used in the proof of Lemma 4.46, there exists $n \geq 1$ and a preimage $w' \in \,]w, 0[$ of the critical point such that we have the biholomorphic mapping

$$-P_{c_0}^n : \mathcal{R}(w, w') \longrightarrow \mathcal{R}(-\alpha_r, 0).$$

Therefore by Lemma 4.39, we find a fixed point $w'' \in\ ]w, w'[$ of $-P_{c_0}^n$ which yields the piecewise dynamic homeomorphism

$$f : \mathcal{R}_0(w) \to \mathcal{R}_0(-\alpha_r),\ z \mapsto \begin{cases} -P_{c_0}^n(z) & : \quad z \in \mathcal{R}(w, w'') \\ z & : \quad z \in \mathcal{R}_0(w) \setminus \overline{\mathcal{R}(w, w'')} \end{cases} .$$

Then $\mathcal{R}_0(z_{-m'}) \subset \mathcal{R}_0(-\alpha_r)$ and

$$f^{-1}\left(\mathcal{R}_0(z_{-m'})\right) = \mathcal{R}_0\left(f^{-1}(z_{-m'})\right).$$

Moreover,

$$f^{-1}(z_{-m'}) \in\ ]w, w''[\ \subset\ ]-\alpha_l, \alpha_l[$$

and we can continue as in Case 1 to find a piecewise dynamic homeomorphism g between $f^{-1}\left(\mathcal{R}_0(z_{-m'})\right)$ and $\mathcal{R}_{-\beta_r}(\alpha_l)$. Then the composition

$$g \circ f^{-1} : \mathcal{R}(z_{-m'}, -\beta_r) \to \mathcal{R}(\alpha_l, -\beta_r)$$

defines a piecewise dynamic homeomorphism.

Case 2.4: $z_{-m'} \in\ ]-\beta_l, \alpha_r[$

In this case we repeat essentially the arguments of the proof of Step 2 in Lemma 4.47. By Lemma 4.34 and Lemma 4.38, there exists a sequence

$$(w_n)_{n\geq 1} \subset\ ]\alpha_l, \alpha_r[$$

of preimages of α_r converging to α_l. Since an iterate of P_{c_0} maps a small neighborhood of $z_{-m'}$ injectively to a neighborhood of α_l and since the dynamic rays landing at α_l are permuted transitively, there exists an integer $N \geq 1$ and a sequence

$$(w'_n)_{n\geq N} \subset\ ]z_{-m'}, \alpha_r[$$

converging to $z_{-m'}$ such that

$$P_{c_0}^{s_n} : \mathcal{R}(z_{-m'}, w'_n) \to \mathcal{R}(\alpha_l, w_n)$$

is biholomorphic for $n \geq N$ and some $s_n \in \mathbb{N}$. By a similar argument as in the proof of Lemma 4.46 there exists a preimage $w \in\ ]\alpha_l, w_N[$ of the critical point and an integer $t \in \mathbb{N}$ such that

$$P_{c_0}^t : \mathcal{R}(w, w_N) \to \mathcal{R}(0, \alpha_r)$$

is biholomorphic. By Lemma 4.38, there is a fixed point $w' \in\ ]w, w_N[$ of $P_{c_0}^t$ which yields the piecewise dynamic homeomorphism

$$f : \mathcal{R}(\alpha_l, w_N) \to \mathcal{R}(\alpha_l, \alpha_r),$$

$$z \mapsto \begin{cases} z & : \quad z \in \mathcal{R}(\alpha_l, w_N) \setminus \overline{\mathcal{R}(w', w_N)} \\ P_{c_0}^t(z) & : \quad z \in \mathcal{R}(w', w_N) \end{cases} .$$

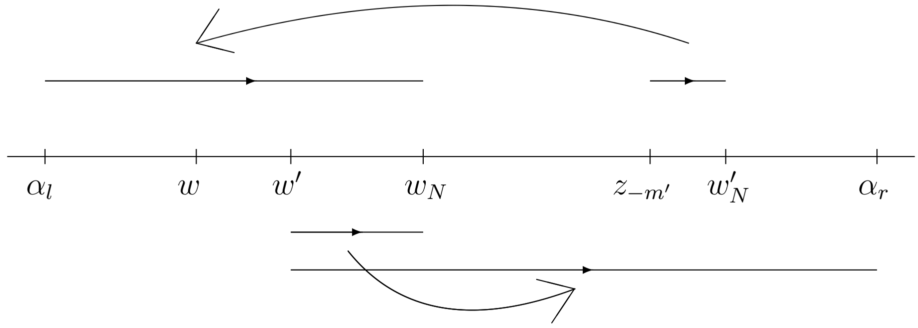

Figure 20: The construction of a piecewise dynamic homeomorphism $\mathcal{R}(z_{-m'}, \alpha_r) \to \mathcal{R}(\alpha_l, \alpha_r)$ as described in Case 2.4 .

Thus the piecewise dynamic homeomorphism

$$f \circ P_{c_0}^{s_N} : \mathcal{R}(z_{-m'}, w'_N) \to \mathcal{R}(\alpha_l, \alpha_r)$$

has a fixed point $w'' \in\,]z_{-m'}, w'_N[$ such that

$$\mathcal{R}(z_{-m'}, \alpha_r) \to \mathcal{R}(\alpha_l, \alpha_r),$$

$$z \mapsto \begin{cases} f(P_{c_0}^{s_N}(z)) & : \quad z \in \mathcal{R}(z_{-m'}, w'') \\ z & : \quad z \in \mathcal{R}(z_{-m'}, \alpha_r) \setminus \overline{\mathcal{R}(z_{-m'}, w'')} \end{cases}$$

is a piecewise dynamic homeomorphism as well and we are done.

Step 2: *Any two regions $\mathcal{R}_1$, $\mathcal{R}_2$ behind the critical value are piecewise dynamically homeomorphic with expansion relative to E_{c_0}.*
In Step 1 we have constructed piecewise dynamic homeomorphisms

$$\begin{aligned} f_1 &: \mathcal{R}_1 \to \mathcal{R}_{\beta_r^-(1)}(z_{l-(r-1)}) \quad \text{and} \\ f_2 &: \mathcal{R}_2 \to \mathcal{R}_{\beta_r^-(1)}(z_{l-(r-1)}). \end{aligned}$$

Thus we get the piecewise dynamic homeomorphism

$$f := f_2^{-1} \circ f_1 : \mathcal{R}_1 \to \mathcal{R}_2$$

which we have to prove to be expanding relative to E_{c_0}. This can be checked in the same way as for an α-type Misiurewicz point (compare proof of Lemma 4.47).

Step 3: *For all parameters c behind c_0, we can transfer the piecewise dynamic homeomorphisms from above to get one between the regions $\mathcal{R}_1(c)$ and $\mathcal{R}_2(c)$ behind $z_{-k}(c)$*
Denote by Θ the set of angles such that the corresponding dynamic rays are used to cut $\mathcal{R}_1$ in several pieces in the construction of the piecewise dynamic homeomorphism in Step 1. Similarly as in the proof of Lemma 4.47, we can check that the landing properties of the dynamic rays with angles in Θ do not change for all parameters c behind c_0 (Lemma 4.11 and Lemma 4.12). Then the constructed piecewise dynamic homeomorphism $\mathcal{R}_1 \to \mathcal{R}_2$ easily gives a piecewise dynamic homeomorphism $\mathcal{R}_1(c) \to \mathcal{R}_2(c)$ replacing P_{c_0} by P_c. □

4.7.4 Partial Trees behind Misiurewicz Points

Definition 4.50 (Partial Trees)
Let $c_0 \in \mathcal{M}_d$ be a Misiurewicz point which is the landing point of $q \geq 3$ parameter rays with angles $\vartheta_1 < \ldots < \vartheta_q$. The **$j$-th partial tree $T_{c_0}^{(j)}$ at** *c_0 is the j-th branch $\mathcal{M}_d(c_0, j)$ behind c_0 between the rays with angles ϑ_j and ϑ_{j+1} together with the Misiurewicz point c_0 itself with the following sub-branches removed:*

Let $c \in \mathcal{M}_d(c_0, j)$ be a Misiurewicz point which is the landing point of q parameter rays with angles $\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_q$ such that there is an integer $m > 0$ such that for all $n \in \{1, \ldots, q\}$, we have $d^m\tilde{\vartheta}_n = \vartheta_n \mod 1$. Then we remove all branches behind c between one of the following parameter ray pairs: $(\tilde{\vartheta}_1, \tilde{\vartheta}_2), \ldots, (\tilde{\vartheta}_{j-1}, \tilde{\vartheta}_j), (\tilde{\vartheta}_{j+1}, \tilde{\vartheta}_{j+2}), \ldots, (\tilde{\vartheta}_{q-1}, \tilde{\vartheta}_q)$.

Lemma 4.51 (Properties of Partial Trees)
Every partial tree $T_{c_0}^{(j)}$ is a connected compact subset of $\overline{\mathcal{M}_d(c_0, j)}$.

Proof. Since $\mathcal{M}_d$ has countably many Misiurewicz points, the partial tree $T_{c_0}^{(j)}$ can be written as a countable nested intersection of compact connected subsets of $\mathcal{M}_d(c_0, j)$. □

Remarks. The j-th partial tree at c_0 is by construction exactly the set of parameters c within $\mathcal{M}_d(c_0, j)$ for which the critical orbit of P_c does never visit the sectors $\Sigma(\vartheta_1, \vartheta_{j-1})$ and $\Sigma(\vartheta_{j+1}, \vartheta_q)$.

We will see in Lemma 5.14 that every partial tree contains the dyadic Misiurewicz point of lowest preperiod within $\mathcal{M}_d(c_0, j)$.

Lemma 4.52 (Subtrees)
Let $c_0 \in \mathcal{M}_d$ be a Misiurewicz point and $c \in T_{c_0}^{(j)}$ another Misiurewicz point at which we had to cut off certain sub-branches during the construction of $T_{c_0}^{(j)}$. Then for all $m \in \{1, \ldots, q-1\}$,

$$\mathcal{M}_d(c, m) \cap T_{c_0}^{(j)} \subset T_c^{(m)}.$$

Proof. Denote the angles of the parameter rays landing at c_0 by $\vartheta_1 < \ldots < \vartheta_q$. Let $\vartheta' < \vartheta$ be the angles of the parameter rays landing at c such that $T_c^{(m)}$ is contained between these parameter rays. Then there exists an integer $n > 0$ such that

$$(d^n\vartheta' = \vartheta_j \quad \text{and} \quad d^n\vartheta = \vartheta_{j+1}) \qquad \textit{or} \qquad (d^n\vartheta' = \vartheta_q \quad \text{and} \quad d^n\vartheta = \vartheta_1).$$

In the first case let $\tilde{\vartheta}$, $\tilde{\vartheta}'$ be the angles of the parameter rays landing at c such that $d^n\tilde{\vartheta}' = \vartheta_q$ and $d^n\tilde{\vartheta} = \vartheta_1$; in the second case let $\tilde{\vartheta}$, $\tilde{\vartheta}'$ be the angles of the parameter rays landing at c such that $d^n\tilde{\vartheta}' = \vartheta_j$ and $d^n\tilde{\vartheta} = \vartheta_{j+1}$.

If $T_c^{(m)}$ belongs to a branch behind c which has to be cut off during the construction of $T_{c_0}^{(j)}$, then $\mathcal{M}_d(c, m) \cap T_{c_0}^{(j)} = \{c\} \subset T_c^{(m)}$. Otherwise, for every Misiurewicz point $\tilde{c}$ behind c where we have to cut off all but one branches to construct $T_c^{(m)}$, $\tilde{c}$ is a Misiurewicz point behind c_0 where we have to cut off all but one branches to construct $T_{c_0}^{(j)}$ as well. It remains to argue that in both cases the same branches behind $\tilde{c}$ are removed: let φ', φ, $\tilde{\varphi}'$ and $\tilde{\varphi}$ denote the angles of the parameter rays landing at $\tilde{c}$ such that for an integer $\tilde{n} \geq 1$,

$$d^{\tilde{n}}\varphi' = \vartheta', \quad d^{\tilde{n}}\varphi = \vartheta, \quad d^{\tilde{n}}\tilde{\varphi}' = \tilde{\vartheta}', \quad d^{\tilde{n}}\tilde{\varphi} = \tilde{\vartheta}.$$

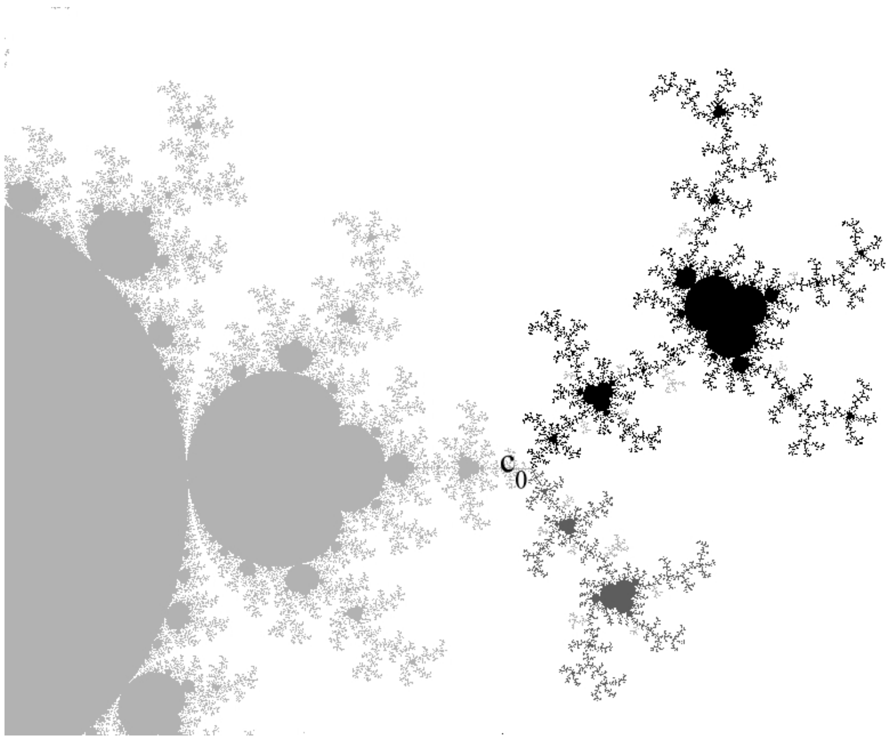

Figure 21: Two partial Trees in $\mathcal{M}_4$ behind the Misiurewicz point c_0 are marked in black and dark grey.

To construct $T_c^{(m)}$ we have to remove all branches around $\tilde{c}$ but not the one between the rays with angles φ' and φ and not the one between the rays with angles $\tilde{\varphi}'$ and $\tilde{\varphi}$. Since

$$\{d^{n+\tilde{n}}\varphi', d^{n+\tilde{n}}\varphi, d^{n+\tilde{n}}\tilde{\varphi}', d^{n+\tilde{n}}\tilde{\varphi}\} = \{\vartheta_1, \vartheta_j, \vartheta_{j+1}, \vartheta_q\},$$

the partial tree $T_{c_0}^{(j)}$ is constructed in the same way near the parameter $\tilde{c}$. This proves

$$\mathcal{M}_d(c, m) \cap T_{c_0}^{(j)} \subset T_c^{(m)}$$

in any case. □

5 Homeomorphisms within the Multibrot Sets

In this section we will construct three different types of homeomorphisms between certain subsets of $\mathcal{M}_d$ (Theorem 5.1, Theorem 5.17 and Theorem 5.24).
The homeomorphisms between parts of the branches behind Misiurewicz points (which we will call partial trees, Definition 4.50) are new and therefore we will describe these homeomorphisms and their properties in much more detail than the other homeomorphisms; this is done in Section 5.1. The most important ingredients for the proofs are given in [DH2] and [BD].
In Section 5.2 we construct homeomorphisms between a part of the p/q- and a part of the p'/q-sublimb in every sector of a hyperbolic component of $\mathcal{M}_d$. Homeomorphisms of this type have been investigated by D. Schleicher (unpublished) and B. Branner/N. Fagella [BF]: they proved that there is a homeomorphism between the p/q- and the p'/q-limb of $\mathcal{M}_2$ with different methods. B. Branner and N. Fagella construct a homeomorphism from every p/q-limb ($1 \leq p \leq q-1$, $\gcd(p,q) = 1$) of $\mathcal{M}_2$ to a certain part of the parameter space of a family of degree q polynomials. Our construction will work for arbitrary mapping degree $d \geq 2$ and for the sublimbs in a sector of every hyperbolic component; it is based on an idea of D. Schleicher.
In Section 5.3 we present a proof for the existence of a homeomorphic embedding of the $1/q$-sublimb into the $1/(q+1)$-sublimb within any sector of a hyperbolic component of $\mathcal{M}_d$. The corresponding homeomorphic embedding of the $1/2$-limb into the $1/3$-limb of $\mathcal{M}_2$ has been found by B. Branner and A. Douady. In the general case the proof is essentially the same as in the special case discussed in [BD].

5.1 Homeomorphic Trees at Misiurewicz Points

Theorem 5.1 (Homeomorphic Trees at α-type Misiurewicz Points)
Let $c_0 \in \mathcal{M}_d$ be an α-type Misiurewicz point with preperiod k such that $q \geq 3$ parameter rays land at c_0. Then for every integer $j \in \{1, \ldots, q-2\}$, there exists a mapping

$$\chi : T_{c_0}^{(j)} \to T_{c_0}^{(j+1)}$$

which is continuous on the boundary and analytic on hyperbolic components.
In the case $d = 2$ the map χ is a homeomorphism.

Theorem 5.2 (Homeomorphic Trees at general Misiurewicz Points)
For $d = 2$, let $c_0 \in \mathcal{M}_2$ be a Misiurewicz point with preperiod k and period l such that $q \geq 3$ parameter rays land at c_0. Then for every integer $j \in \{1, \ldots, q-2\}$, there exists a homeomorphism

$$\chi : T_{c_0}^{(j)} \to T_{c_0}^{(j+1)}.$$

In the general case, i.e. if c_0 is a Misiurewicz point but not of α-type, we have to restrict our statement to degree $d = 2$. The reason why we do not have such a continuous mapping for arbitrary degree and Misiurewicz points of arbitrary period is described shortly before Lemma 4.49.

The second theorem can be treated by using Lemma 4.49 instead of Lemma 4.47 and repeating the arguments below. Altogether we prove that the partial trees behind all Misiurewicz points of the Mandelbrot set are pairwise homeomorphic. Moreover for $d \geq 2$, the partial trees behind all α-type Misiurewicz points in $\mathcal{M}_d$ can be mapped continuously onto each other. We will prove now Theorem 5.1 in several steps:

5.1.1 Construction

In the following we will use sectors at preperiodic points as introduced in Section 4.2.

Lemma 5.3 (Changing the Dynamics)
Let $c_0 \in \mathcal{M}_d$ be an α-type Misiurewicz point with preperiod k such that $q \geq 3$ parameter rays with angles $\vartheta_1 < \ldots < \vartheta_q$ land at c_0. Then for every integer $j \in \{1, \ldots, q-2\}$ and for every parameter $c \in \mathcal{M}_d(c_0, j)$, one can find a potential $\eta > 0$, a width $w > 0$ and a quasiconformal homeomorphism $\varphi : X_c^\eta \to X_c^\eta$ such that

(a) $\varphi(\Sigma_w^\eta(\vartheta_j, \vartheta_{j+1}) \cap K_c) = \Sigma_w^\eta(\vartheta_{j+1}, \vartheta_{j+2}) \cap K_c$,

(b) $\varphi(z) = z$ *for all* $z \in X_c^\eta \setminus \left(\overline{S_w^\eta(\vartheta_j) \cup \Sigma_w^\eta(\vartheta_j, \vartheta_{j+1}) \cup S_w^\eta(\vartheta_{j+1}) \cup \Sigma_w^\eta(\vartheta_{j+1}, \vartheta_{j+2}) \cup S_w^\eta(\vartheta_{j+2})}\right)$, *here and in the following the closure is considered relative to* X_c^η.

(c) $\overline{\partial}\varphi = 0$ *a.e. on* $K_{\varphi \circ P_c} \cup K_c$,

(d) every orbit of $(\varphi \circ P_c) : X_c^{\eta/d} \to X_c^\eta$ *contains at most one point at which the complex dilatation of* $\varphi \circ P_c$ *does not vanish.*

The map $\varphi \circ P_c$ is quasi-polynomial-like with connected filled-in Julia set if $c \in T_{c_0}^{(j)}$. In this case $\varphi \circ P_c$ is hybrid equivalent to a unique polynomial $P_{c'}$ with $c' \in T_{c_0}^{(j+1)}$. This defines a mapping

$$\chi : T_{c_0}^{(j)} \to T_{c_0}^{(j+1)}.$$

Remark. It is stated in this lemma that for $c \in \mathcal{M}_d(c_0, j)$, the filled-in Julia set of $\varphi \circ P_c$ is connected if $c \in T_{c_0}^{(j)}$. In the case $q \geq 4$ these are not the only parameters in $\mathcal{M}_d(c_0, j)$ for which this filled-in Julia set is connected. But for the statement that all partial trees behind c_0 are pairwise homeomorphic we need to restrict to the partial trees as considered in Definition 4.50.

Proof. Choose $c \in \mathcal{M}_d(c_0, j)$; the following construction will be done in the dynamic plane of P_c. By Lemma 4.11, all dynamic rays with angles $\vartheta_1, \ldots, \vartheta_q$ land at a common point. Let α_{-k} denote this landing point and let $\alpha_{-(k+1)}, \ldots, \zeta^{d-1}\alpha_{-(k+1)}$ denote its immediate preimages. These points are mapped to α by P_c^{k+1}. By Lemma 4.38, there exists a sequence of pinching points $(z_n)_n \subset]\alpha, 0[$ converging to α for the parameter c_0; the dynamic rays with angles in $\Theta(z_n)$ give such a sequence of pinching points for the parameter c as well. According to Lemma 4.18, a width $w > 0$ can be chosen such that

$$\mathrm{mod}(S_{w/2}(\vartheta_j)) = \mathrm{mod}(\Sigma_w(\vartheta_j, \vartheta_{j+1})).$$

The potential η will be defined within Step 1.

Step 1: $\Sigma_w^\eta(\vartheta_j, \vartheta_{j+1}) \to \Sigma_w^\eta(\vartheta_{j+1}, \vartheta_{j+2})$

By Lemma 4.47, there exists a piecewise dynamic homeomorphism

$$\tilde{\varphi} : \Sigma(\vartheta_j, \vartheta_{j+1}) \to \Sigma(\vartheta_{j+1}, \vartheta_{j+2})$$

with expansion relative to E_c. Let

$$N := \max\{\mathrm{iter}_{max}(\tilde{\varphi}), 0\}$$

and $\Theta \subset \mathbb{S}^1$ be the set of angles used in the construction of $\tilde{\varphi}$.

We choose the potential $\eta > 0$ to be small enough that

(1) the closures of the sectors

$$P_c^m(S_w^{\eta d^N}(\vartheta_j)), \quad P_c^m(S_w^{\eta d^N}(\vartheta_{j+1})) \quad \text{and} \quad P_c^m(S_w^{\eta d^N}(\vartheta_{j+2}))$$

do not intersect for $m \in \{0, \ldots, k+q\}$,

(2) all the dynamic rays with angles $\vartheta \in \Theta$ do not intersect the boundaries of $\Sigma_w^{2\eta}(\vartheta_j, \vartheta_{j+1})$ and $\Sigma_w^{2\eta}(\vartheta_{j+1}, \vartheta_{j+2})$ at a potential less than 2η.

Using Lemma 4.37 with the potential η just defined, we can redefine $\tilde{\varphi}$ in a sectorial neighborhood of every dynamic ray with angle in Θ to get a quasiconformal homeomorphism

$$\Sigma_w^{\eta}(\vartheta_j, \vartheta_{j+1}) \longrightarrow \Sigma \;\subset\; \Sigma_w^{d^N \eta}(\vartheta_{j+1}, \vartheta_{j+2}).$$

Obviously, we can assume that these sectorial neighborhoods are chosen so small that these sectors and all their P_c-orbits are pairwise disjoint (since every $\vartheta \in \Theta$ is preperiodic, we only have to care about finitely many such sectors). Let $\nu > 0$ be as large as possible such that

$$\Sigma_w^{\nu}(\vartheta_{j+1}, \vartheta_{j+2}) \;\subset\; \Sigma.$$

Then there is a quasiconformal homeomorphism Q between the quadrilaterals

$$Q : \Sigma \setminus \Sigma_w^{\nu/1.5}(\vartheta_{j+1}, \vartheta_{j+2}) \longrightarrow \Sigma_w^{\eta}(\vartheta_{j+1}, \vartheta_{j+2}) \setminus \Sigma_w^{\nu/1.5}(\vartheta_{j+1}, \vartheta_{j+2})$$

such that Q is tangent to the identity at the equipotential $\nu/1.5$ and

$$\begin{array}{lll}
\Phi_c^{-1}\left(\exp\left(\frac{\nu}{1.5} + i(2\pi\vartheta_{j+1} + w\frac{\nu}{1.5})\right)\right) & & \text{is fixed by } Q \\
\Phi_c^{-1}\left(\exp\left(\frac{\nu}{1.5} + i(2\pi\vartheta_{j+2} - w\frac{\nu}{1.5})\right)\right) & & \text{is fixed by } Q \\
(\tilde{\varphi} \circ \Phi_c^{-1})\Big(\exp\left(\eta + i(2\pi\vartheta_{j+1} + w\eta)\right)\Big) & \xrightarrow{Q} & \Phi_c^{-1}\Big(\exp\left(\eta + i(2\pi\vartheta_{j+1} + w\eta)\right)\Big) \\
(\tilde{\varphi} \circ \Phi_c^{-1})\Big(\exp(\eta + i(2\pi\vartheta_{j+2} - w\eta))\Big) & \xrightarrow{Q} & \Phi_c^{-1}\Big(\exp(\eta + i(2\pi\vartheta_{j+2} - w\eta))\Big).
\end{array}$$

Then the restriction of φ to $\Sigma_w^{\eta}(\vartheta_j, \vartheta_{j+1})$ is defined by

$$\varphi \;:\; \Sigma_w^{\eta}(\vartheta_j, \vartheta_{j+1}) \longrightarrow \Sigma_w^{\eta}(\vartheta_{j+1}, \vartheta_{j+2}) \;,$$

$$z \mapsto \begin{cases} \tilde{\varphi}(z) & : \quad \tilde{\varphi}(z) \in \Sigma_w^{\nu/1.5}(\vartheta_j, \vartheta_{j+1}) \\ Q(\tilde{\varphi}(z)) & : \quad \tilde{\varphi}(z) \in \Sigma \setminus \Sigma_w^{\nu/1.5}(\vartheta_j, \vartheta_{j+1}) \end{cases} .$$

Step 2:

On the closure of

$$X_c^{\eta} \setminus \left(\overline{S_w^{\eta}(\vartheta_j) \cup \Sigma_w^{\eta}(\vartheta_j, \vartheta_{j+1}) \cup S_w^{\eta}(\vartheta_{j+1}) \cup \Sigma_w^{\eta}(\vartheta_{j+1}, \vartheta_{j+2}) \cup S_w^{\eta}(\vartheta_{j+2})}\right)$$

we define $\varphi(z) = z$.

Step 3: $S_w^\eta(\vartheta_j) \longrightarrow S_w^\eta(\vartheta_j) \cup \overline{\Sigma_w^\eta(\vartheta_j, \vartheta_{j+1})} \cup S_w^\eta(\vartheta_{j+1})$

Let

$$\psi : \overline{S_{w/2}^\eta(\vartheta_j)} \longrightarrow \overline{\Sigma_w^\eta(\vartheta_j, \vartheta_{j+1})}$$

be the Riemann mapping (extended to the boundary) such that α_{-k} is fixed and that the other two vertices of $S_{w/2}^\eta(\vartheta_j)$ are mapped to the corresponding two vertices of $\Sigma_w^\eta(\vartheta_j, \vartheta_{j+1})$:

$$\begin{aligned} \Phi_c^{-1}\Big(\exp(\eta + i(2\pi\vartheta_j - \eta w/2))\Big) &\overset{\psi}{\to} \Phi_c^{-1}\Big(\exp(\eta + i(2\pi\vartheta_j + \eta w))\Big), \\ \Phi_c^{-1}\Big(\exp(\eta + i(2\pi\vartheta_j + \eta w/2))\Big) &\overset{\psi}{\to} \Phi_c^{-1}\Big(\exp(\eta + i(2\pi\vartheta_{j+1} - \eta w))\Big). \end{aligned}$$

For $z \in \overline{S_{w/2}^\eta(\vartheta_j)}$, we define

$$\varphi(z) = \psi(z).$$

It remains to define φ in the two sectors $S_w^\eta(\vartheta_j) \setminus \overline{S_{w/2}^\eta(\vartheta_j)}$. By Lemma 4.19, on both sectors the map φ as defined so far can be extended to a quasiconformal homeomorphism

$$\overline{S_w^\eta(\vartheta_j)} \longrightarrow \overline{S_w^\eta(\vartheta_j) \cup \Sigma_w^\eta(\vartheta_j, \vartheta_{j+1}) \cup S_w^\eta(\vartheta_{j+1})}$$

and this ends Step 3.

Step 4: $S_w^\eta(\vartheta_{j+1}) \cup \overline{\Sigma_w^\eta(\vartheta_{j+1}, \vartheta_{j+2})} \cup S_w^\eta(\vartheta_{j+2}) \longrightarrow S_w^\eta(\vartheta_{j+2})$

Since the modulus of an annulus does not change under a biholomorphic mapping, we have

$$\begin{aligned} \mathrm{mod}(S_{w/2}^\eta(\vartheta_{j+2})) &= \mathrm{mod}(S_{w/2}^\eta(\vartheta_j)) \\ &= \mathrm{mod}(\Sigma_w^\eta(\vartheta_j, \vartheta_{j+1})) \\ &= \mathrm{mod}(\Sigma_w^\eta(\vartheta_{j+1}, \vartheta_{j+2})). \end{aligned}$$

Therefore the same construction as in Step 2 can be used to define φ on the remaining two sectors $S_w^\eta(\vartheta_{j+2}) \cup \overline{\Sigma_w^\eta(\vartheta_{j+1}, \vartheta_{j+2})} \cup S_w^\eta(\vartheta_{j+1})$:

$$\varphi : \Sigma_w^\eta(\vartheta_{j+1}, \vartheta_{j+2}) \longrightarrow S_{w/2}^\eta(\vartheta_{j+2})$$

is defined by the Riemann mapping between these two sectors such that vertices are again mapped to vertices. Using Lemma 4.19 again, we have constructed a quasiconformal homeomorphism

$$\varphi : X_c^\eta \to X_c^\eta.$$

Now we have to prove that the properties (a) to (d) of the claim (Lemma 5.3) hold for φ. The properties (a) and (b) are certainly true by construction. To prove (c) and (d) we first prove an expanding property for

$$f : X_c^{\eta/d} \to X_c^\eta,\ z \mapsto (\varphi \circ P_c)(z).$$

Let $H \subset X_c^{\eta/d}$ be the closure of the set on which f is defined by a composition of the maps P_c, certain branches of P_c^{-1} and $z \mapsto \zeta z$. In particular, $(K_c \cap K_f) \cap E_c \subset H$. For every $z \in H$, the following cases occur:

(1) either there is a smallest integer $n \geq 1$ such that $f^n(z) \in E_c$,

(2) or $\{f^m(z) \ : \ m \geq 1\} \ \subset \ X^{\eta/d} \setminus E_c$,

(3) or there is a smallest integer $n \geq 1$ such that

$$\{f^m(z) \ : \ 1 \leq m \leq n-1\} \ \subset \ X_c^{\eta/d} \setminus E_c \qquad \text{and} \qquad f^n(z) \in X_c^{\eta} \setminus X_c^{\eta/d}.$$

In the first case (and this is the one of interest) there is a positive integer $\tilde{n} > 0$ such that $f^n(z) = P_c^{\tilde{n}}(z)$: if $z \notin E_c$ then $\tilde{n} = n$ and if $z \in E_c$ then, by the expansion property of φ,

$$\tilde{n} = n + \text{iter}(f,z) = 1 + \underbrace{n - 1 + \text{iter}(f,z)}_{\geq 0} > 0.$$

<u>Connectivity of K_f:</u>

The only critical point of f is $z = 0$ and its P_c-orbit is contained in $K_c \subset X_c^{\eta/d}$. Moreover,

$$f\Big(K_c \setminus P_c^{-1}(\Sigma_w^{\eta}(\vartheta_{j+1}, \vartheta_{j+2}))\Big) \subset K_c.$$

Therefore the critical point of f can only escape from K_c if there is an integer $m \geq 1$ such that

$$f^m(0) \in P_c^{-1}(\Sigma_w^{\eta}(\vartheta_{j+1}, \vartheta_{j+2})) \subset E_c.$$

If m was chosen as small as possible, the region $P_c^{-1}(\Sigma_w^{\eta}(\vartheta_{j+1}, \vartheta_{j+2}))$ can be pulled back by f exactly $(m-1)$ - times along the orbit of the critical value. We get a region $\mathcal{R} \ni f(0)$ such that

$$f^{m-1} = P_c^{m-1} \ : \ \mathcal{R} \to P_c^{-1}(\Sigma_w^{\eta}(\vartheta_{j+1}, \vartheta_{j+2}))$$

is biholomorphic. Therefore there is a point $\alpha_{-(k+m)}$ mapped by P_c^{m-1} to one of the d points in $P_c^{-1}(\{\alpha_{-k}\})$ such that $f(0)$ is contained in a region behind $\alpha_{-(k+m)}$. Thus the critical value

$$c = P_c(0) = \varphi^{-1}(f(0))$$

of P_c is contained between two dynamic rays with angles $\tilde{\vartheta}_{j+1} < \tilde{\vartheta}_{j+2}$ landing at

$$\alpha_{-(k+m+m')} := \varphi^{-1}(\alpha_{-(k+m)})$$

for an integer $m' \geq -m+1$ (this inequality follows from the expansion property of φ). Since $m + m' \geq 1$, the point $\alpha_{-(k+m+m')}$ is mapped to α_{-k} by an iterate of P_c and the two dynamic rays with angles $\tilde{\vartheta}_{j+1}$ and $\tilde{\vartheta}_{j+2}$ land at $\alpha_{-(k+m+m')}$ but they are not mapped to the dynamic rays with angles ϑ_j and ϑ_{j+1}. Therefore there is a Misiurewicz point $\tilde{c}_0 \in \mathcal{M}_d(c_0, i)$ such that c is contained between the parameter rays with angles $\tilde{\vartheta}_{j+1}$ and $\tilde{\vartheta}_{j+2}$ landing at $\tilde{c}_0$. Then

$$c \notin T_{c_0}^{(j)}.$$

This proves that K_f is connected if $c \in T_{c_0}^{(j)}$.

<u>$\overline{\partial}(\varphi \circ P_c) = 0$ a.e. on $K_{\varphi \circ P_c} \cup K_c$:</u>

For $z \in X_c^{\eta/d} \setminus E_c$, we have

$$\overline{\partial}(\varphi \circ P_c)(z) = \overline{\partial} P_c(z) = 0.$$

If $z \in E_c$ then there are the following possibilities:

(1) either $P_c(z) \in \overline{S_w^\eta(\vartheta_{j+2}) \cup \Sigma_w^\eta(\vartheta_{j+1}, \vartheta_{j+2}) \cup S_w^\eta(\vartheta_{j+1})}$ and

$$(\varphi \circ P_c)(z) \in S_w^\eta(\vartheta_{j+2}).$$

By assumption, there exist $m \geq 1$ and $\vartheta \in \Theta(\alpha)$ with

$$f^m((\varphi \circ P_c)(z)) = P_c^m((\varphi \circ P_c)(z)) \ \in \ S_w^\eta(\vartheta) \setminus S_w^{\eta/d}(\vartheta);$$

this means

$$f^{m+1}(z) \notin K_f \quad \text{and} \quad z \notin K_f.$$

Moreover,

$$\overline{\partial}(\varphi \circ P_c)(z) = 0$$

for all $z \in \Sigma_w^{\eta/d}(\vartheta_{j+1}, \vartheta_{j+2})$ by definition.

(2) or $P_c(z) \in \Sigma_w^{\eta/d}(\vartheta_j, \vartheta_{j+1})$. If z is contained in the region where φ is defined by a composition of the maps P_c, certain branches of P_c^{-1} and $z \mapsto \zeta z$, then

$$\overline{\partial} f(z) = 0.$$

If $P_c(z)$ is contained in a sector around a dynamic ray where φ is not holomorphic, then $f(z) \notin K_c$ and by the expansion property and the choice of the potential η, we have

$$f^m(f(z)) = P_c^m(f(z))$$

for all suitable $m \geq 0$ and there exists an integer $m \geq 0$ such that

$$f^m(f(z)) \in X_c^\eta \setminus X_c^{\eta/d};$$

thus

$$z \notin K_f.$$

(3) or $P_c(z) \in S_{w/2}^\eta(\vartheta_j)$. Then φ is holomorphic at $P_c(z)$ and thus

$$\overline{\partial}(\varphi \circ P_c)(z) = 0.$$

(4) or $P_c(z) \in S_w^\eta(\vartheta_j) \setminus S_{w/2}^\eta(\vartheta_j)$. Then by the choice of the potential η,

$$f^m(f(z)) = P_c^m(f(z))$$

for all suitable $m \geq 0$. Therefore there exists an integer $m \geq 0$ such that

$$f^m(f(z)) \in X_c^\eta \setminus X_c^{\eta/d}$$

and thus

$$z \notin K_f.$$

(5) or $z \in E_c \setminus (S_w^\eta(\vartheta_{j+2}) \cup \Sigma_w^\eta(\vartheta_{j+1}, \vartheta_{j+2}) \cup S_w^\eta(\vartheta_{j+1}) \cup \Sigma_w^\eta(\vartheta_{j+1}, \vartheta_j) \cup S_w^\eta(\vartheta_j))$. Then

$$\varphi(P_c(z)) = P_c(z)$$

and

$$\overline{\partial} f(z) = 0.$$

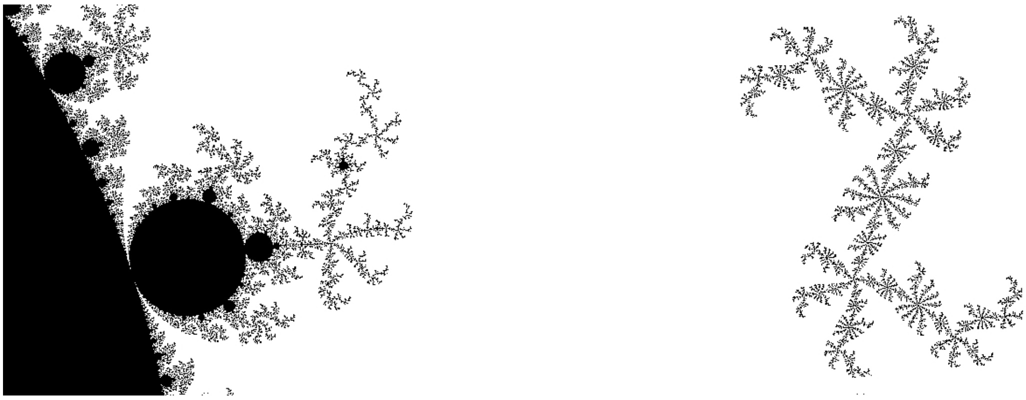

Figure 22: *Left:* The 1/5-Limb of the Mandelbrot Set $\mathcal{M}_2$. *Right:* The filled-in Julia set of the biggest α-type Misiurewicz point in this Limb.

This shows that either z is contained in $X_c^{\eta/d} \setminus (K_f \cup K_c)$ and the f-orbit of z passes a point of not vanishing complex dilatation of f at most once, or $z \in K_f$ and the f-orbit of z never passes a point of not vanishing complex dilatation of f. This proves (c).

Definition of an f-invariant almost complex structure σ on X_c^η:
On K_f, we define the complex structure σ to be the standard complex structure σ_0. On the annulus $X_c^\eta \setminus X_c^{\eta/d}$, we define $\sigma = \sigma_0$ as well and pull this structure back by the dynamics of f. Since every f-orbit passes the non-holomorphy region at most once and since the complex dilatation of φ is bounded, we get an f-invariant almost complex structure σ on X_c^η.

The hybrid conjugation:
By the measurable Riemann mapping theorem (Theorem 3.4), there is a quasiconformal homeomorphism ψ integrating σ. Conjugating f with this quasiconformal map we get a polynomial-like mapping of degree d which itself is hybrid equivalent to a unicritical polynomial $P_{c'}$. Since the parameter c' is uniquely determined up to multiplication by a $(d-1)$-th root of unity, we can choose c' so that it is contained in the same sector of $\mathcal{M}_d$ as the parameter c_0. Then by construction,

$$c' \in T_{c_0}^{(j+1)}.$$

Thus we really have constructed a mapping

$$\chi : T_{c_0}^{(j)} \to T_{c_0}^{(j+1)}.$$

□

5.1.2 Continuity and Bijectivity

Corollary 5.4 (Periodic Points remain Periodic Points)
Suppose we are in the situation of Lemma 5.3. Then every P_c-periodic point $z \in K_f \cap K_c \cap E_c$ is f-periodic as well.

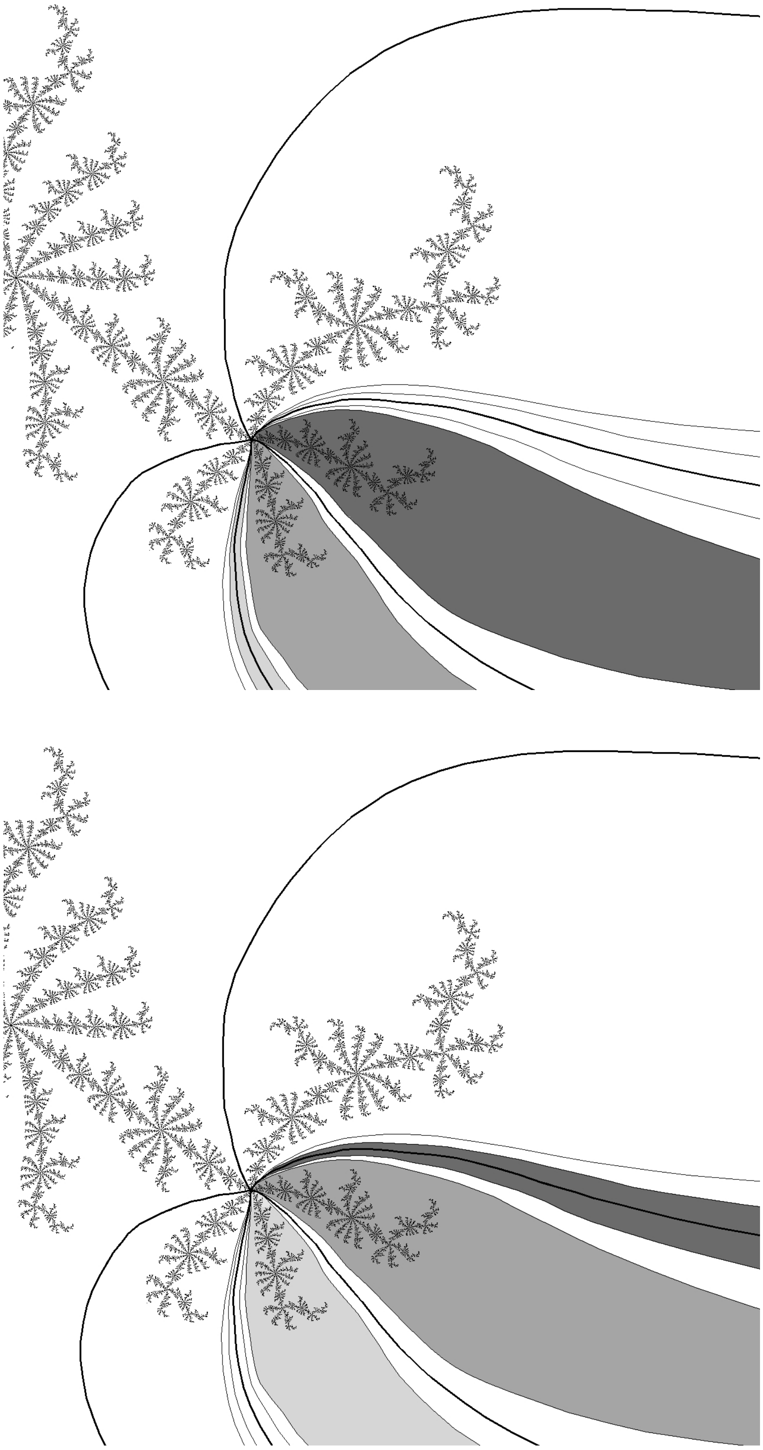

Figure 23: The Construction of φ for the parameter c_0 shown in Figure 22 for $j = 2$: the thick lines are the dynamic rays with angles 33/992, 35/992, 39/992, 47/992 and 63/992 landing at the critical value $z = c_0$; φ is constructed in a way such that the regions with the colors in the upper picture are mapped to the regions with the same colors in the lower one.

Proof. The statement follows from the expanding property of the piecewise dynamic homeomorphism used in the construction of φ: let

$$0 = n_1 < n_2 < n_3 < \dots$$

such that $f^{n_1}(z)$, $f^{n_2}(z)$,... are the only points of the f-orbit of z which are contained in E_c. Then for all $j \geq 1$, the expansion property of φ assures

$$\mathrm{iter}(\varphi \circ P_c, f^{n_j}(z)) + (n_{j+1} - n_j) \geq 1.$$

Denoting the P_c-period of z by m, this shows that

$$\{z, P_c(z), \dots, P_c^{m-1}(z)\} \subset \{f(z), f^2(z), \dots, f^{n_m}(z)\}.$$

Therefore z is f-periodic with period $\leq n_m$. □

Remark. This lemma remains also true, if we replace "periodic" by "strictly preperiodic". But the assumption "$z \in K_c \cap K_f \cap E_c$" cannot be reduced to "$z \in K_c \cap K_f$": a P_c-periodic point outside E_c may be an f-preperiodic point.

Lemma 5.5 (Boundary maps to Boundary)
Let χ be constructed as in Lemma 5.3. Then χ maps

(a) hyperbolic components to hyperbolic components,

(b) non-hyperbolic components to non-hyperbolic components and

(c) boundary points to boundary points.

Proof. Let $c \in T_{c_0}^{(j)}$. Assume that $\chi(c)$ is contained in a hyperbolic component of $\mathcal{M}_d$. Then the critical point $z = 0$ of $f = \varphi \circ P_c$ is contained in the immediate basin B_0 of an attractive f-periodic point $z_0 \in K_f$. Since $0 \in K_c \cap K_f$ and since no dynamic ray pair can separate the critical point from any point in B_0, $B_0 \subset K_c$. Thus the critical point and the point z_0 are contained in the interior of K_c; since z_0 is an attractive f-periodic point, it is also an attractive P_c-periodic point and therefore the parameter c is contained in a hyperbolic component of $\mathcal{M}_d$. The same argument shows that for every parameter c in a hyperbolic component, $\chi(c)$ is also contained in a hyperbolic component.

Assume that $\chi(c)$ is contained in a non-hyperbolic component (a so-called queer component) of $\mathcal{M}_d$. Then there exists a $P_{\chi(c)}$-invariant linefield $\mu_{\chi(c)}$ on $K_{\chi(c)}$. This can be transferred to an invariant linefield μ of $f = \varphi \circ P_c$ on K_f.
Then the support of μ restricted to $K_c \cap K_f$ has positive Lebesgue measure: we can reconstruct K_f from $K_c \cap K_f$ by pulling back the branch of $K_c \cap K_f$ before α by the dynamics of the quasi-regular mapping f, i.e. K_f is the countable union of f-preimages of $K_c \cap K_f$. Pulling the restriction $\mu|_{K_f \cap K_c}$ back on these preimages as well, we reconstruct μ on the whole set K_f and thus its support has Lebesgue measure zero if the support of the restriction of μ to $K_c \cap K_f$ has Lebesgue measure zero, in contradiction to the assumption. Therefore the support of $\mu|_{K_c \cap K_f}$ has positive Lebesgue measure.

Since f is defined on $K_c \cap K_f$ by a composition of the maps P_c, certain branches of P_c^{-1} and $z \mapsto \zeta z$, this restriction of μ to $K_c \cap K_f$ is not only f-invariant but also P_c-invariant. Pulling this restricted μ back by the dynamics of P_c we get a P_c-invariant linefield μ_c on K_c and the parameter c is contained in a non-hyperbolic component of $\mathcal{M}_d$.
Again the same argument shows that every parameter c in a non-hyperbolic component of $\mathcal{M}_d$ is mapped to a parameter $\chi(c)$ in a non-hyperbolic component.

We have shown above that for every point $c \in \partial T_{c_0}^{(j)} = \partial\mathcal{M}_d \cap T_{c_0}^{(j)}$, the image $\chi(c)$ is neither contained in a hyperbolic nor in a non-hyperbolic component of $\mathcal{M}_d$. Thus $\chi(c)$ has to be contained in $\partial T_{c_0}^{(j+1)}$. On the other hand, if $\chi(c) \in \partial T_{c_0}^{(j+1)}$, then $c \in \partial T_{c_0}^{(j)}$ by the same reason. □

Lemma 5.6 (Preservation of the Multiplier)
Let $H \subset T_{c_0}^{(j)}$ be a hyperbolic component. Then for every parameter $c \in H$ the mapping χ preserves the multiplier μ of the attractive periodic orbit, i.e.

$$\mu(c) = \mu(\chi(c)).$$

Proof. By Lemma 5.5, the image $\chi(c)$ is contained in a hyperbolic component of $\mathcal{M}_d$. Therefore the immediate attractive basin of the corresponding f-periodic orbit is mapped $d:1$ over itself by an iterate f^m. Denote the attractive periodic point within this basin by z_0. By the expansion property of the piecewise dynamic homeomorphism in the definition of φ, there exists an integer $\widetilde{m} > 0$ such that $z_0 = f^m(z_0) = P_c^{\widetilde{m}}(z_0)$ and therefore $\widetilde{m}$ has to be a multiple of the P_c-period m' of z_0:

$$\widetilde{m} = am'.$$

Obviously, the integer $a \geq 1$ does not depend on the parameter c within H and

$$\frac{\partial}{\partial z}(f^m)(z_0) = \frac{\partial}{\partial z}(P_c^{\widetilde{m}})(z_0) = \left(\frac{\partial}{\partial z}(P_c^{m'})(z_0)\right)^a.$$

Thus χ can be written in terms of the multiplier map. To prove $a = 1$ we argue as follows: f^m maps the immediate basin B_0 of attraction of z_0 as a $d:1$ covering onto itself. By the Remark after Definition 4.35, the restriction of f^m to this basin can be written in the form

$$f^m(z) = P_c^{-m_1}(\zeta^e P_c^{m_2}(z)).$$

Since $m_2 \geq 1$ and since $P_c|_{B_0}$ has degree d, the map

$$P_c^{-m_1}(\zeta^e P_c^{m_2-1}) \ : \ P_c(B_0) \longrightarrow B_0$$

is biholomorphic and thus equal to $P_c^{m'-1}$. This proves

$$e = 0, \ m_1 = 0 \quad \text{and} \quad m = m_2 = m' = \widetilde{m}.$$

□

Lemma 5.7 (Compatibility with Simple Renormalization)
Let $H_n \subset T^{(j)}_{c_0}$ be a hyperbolic component of period n, $H_{\tilde{n}} := \chi(H_n)$, τ the tuning map sending the main cardioid of $\mathcal{M}_d$ to H_n and $\tilde{\tau}$ the tuning map sending the main cardioid of $\mathcal{M}_d$ to $H_{\tilde{n}}$. Then $\tau(\mathcal{M}_d) \subset T^{(j)}_{c_0}$ and there holds on $\mathcal{M}_d$

$$\tilde{\tau}^{-1} \circ \chi \circ \tau = \mathrm{id}.$$

In particular,

$$\chi|_{\tau(\mathcal{M}_d)} = \tilde{\tau} \circ \tau^{-1} \ : \ \tau(\mathcal{M}_d) \longrightarrow \tilde{\tau}(\mathcal{M}_d)$$

is bijective. Moreover, χ maps $\tau(\mathrm{int}(\mathcal{M}_d))$ biholomorphically to $\tilde{\tau}(\mathrm{int}(\mathcal{M}_d))$.

Proof. As usual we denote the angles of the parameter rays landing at c_0 by $\vartheta_1, \ldots, \vartheta_q$. For every parameter of the small Multibrot set $\tau(\mathcal{M}_d)$, the corresponding small filled-in Julia set is not disconnected by a preimage of α. If a parameter c is separated from H_n by a pair of parameter rays the angles of which are mapped into $\{\vartheta_1, \ldots, \vartheta_q\}$ by an iterate of $\vartheta \mapsto d\vartheta$, then the corresponding dynamic rays land at a preimage of α and these never disconnect the small filled-in Julia set. Thus $\tau(H_n) \subset T^{(j)}_{c_0}$.

For every parameter $c \in \tau(\mathcal{M}_d)$, the small filled-in Julia set is separated from the rest of K_c by countably many dynamic ray pairs the angles of which do not depend on the parameter c. In particular, the small filled-in Julia set corresponding to the center $c \in H_n$ is a disc D_c centered at the critical point $z = 0$. By Lemma 5.6, we have for all $z \in D_c$,

$$f_c^{\tilde{n}}(z) = P_c^n(z).$$

This transfers to the other parameters $c \in \tau(\mathcal{M}_d)$ as well: we can construct a proper mapping $P_c^n : \tilde{U}_c \to \tilde{V}_c$ of degree d such that the regions $\tilde{U}_c$ and $\tilde{V}_c$ are bounded by certain dynamic rays and equipotentials and that $f_c^{\tilde{n}}(z) = P_c^n(z)$ for all $z \in \tilde{U}_c$ (by the usual thickening procedure, we get polynomial-like mappings from the proper mappings). This proves the claim. □

Corollary 5.8 (Analyticity on Non-Hyperbolic Components)
For $d = 2$, the map χ is analytic on every non-hyperbolic component in $T^{(j)}_{c_0}$.

Proof. By a theorem of Yoccoz, every non-hyperbolic component of $\mathcal{M}_2$ is contained in a tuned copy of the Mandelbrot set $\mathcal{M}_2$. Since by Lemma 5.7 the map χ is biholomorphic on the interior of every small Mandelbrot set, we are done. □

Lemma 5.9 (Independence of χ on the Details)
Let $c \in T^{(j)}_{c_0}$ and φ_1, φ_2 be two mappings which have the properties of Lemma 5.3. Denote by c_1 (c_2) the parameter such that $\varphi_1 \circ P_c$ ($\varphi_2 \circ P_c$) is quasiconformally equivalent to P_{c_1} (P_{c_2}). Then the parameters c_1 and c_2 have the same angled internal addresses.

Proof. Let ψ_1 and ψ_2 be quasiconformal homeomorphisms such that

$$P_{c_1} = \psi_1 \circ \varphi_1 \circ P_c \circ \psi_1^{-1},$$

$$P_{c_2} = \psi_2 \circ \varphi_2 \circ P_c \circ \psi_2^{-1}.$$

Then

$$\varphi_1^{-1} \circ \psi_1^{-1} \circ P_{c_1} \circ \psi_1 = P_c = \varphi_2^{-1} \circ \psi_2^{-1} \circ P_{c_2} \circ \psi_2$$

and

$$P_{c_1} = \psi_1 \circ \psi_2^{-1} \circ \underbrace{(\psi_2 \circ \varphi_1 \circ \varphi_2^{-1} \circ \psi_2^{-1})}_{=:\varphi} \circ P_{c_2} \circ \psi_2 \circ \psi_1^{-1}.$$

By the definition of φ_1 and φ_2,

$$\varphi(z) = z$$

for all $z \in X_c^\eta \setminus \left(\overline{S_w^\eta(\vartheta_j) \cup \bigcup_{\vartheta\in\Theta} S_w^\eta(\vartheta) \cup S_w^\eta(\vartheta_{j+1}) \cup \Sigma_w^\eta(\vartheta_{j+1},\vartheta_{j+2}) \cup S_w^\eta(\vartheta_{j+2})}\right)$; here $\Theta \subset \mathbb{S}^1$ is the set of angles such that the corresponding dynamic rays are used to define the piecewise dynamic homeomorphism $\Sigma(\vartheta_j,\vartheta_{j+1}) \to \Sigma(\vartheta_{j+1},\vartheta_{j+2})$ in the construction of φ_1 and φ_2. By Lemma 4.29, the internal addresses of c_1 and c_2 with the information about the sectors removed are equal, since the P_{c_2}-orbit of the critical point is disjoint to the region $\Sigma_w^\eta(\vartheta_{j+1},\vartheta_{j+2})$. By Lemma 5.7, the i-th sector of every hyperbolic component H of $T_{c_0}^{(j)}$ is mapped to the i-th sector of $\chi(H)$ by χ. Thus the complete angled internal addresses of c_1 and c_2 coincide. □

Lemma 5.10 (Continuity on the boundary)
The mapping $\chi : T_{c_0}^{(j)} \to T_{c_0}^{(j+1)}$ is continuous on $\partial T_{c_0}^{(j)} = \partial\mathcal{M}_d(c_0,j) \cap T_{c_0}^{(j)}$.

Proof. Let $c \in \partial T_{c_0}^{(j)}$ and $(c_n)_{n\geq 0} \subset T_{c_0}^{(j)}$ such that

$$c_n \to c \quad \text{for} \quad n \to \infty.$$

We have to prove that the sequence $(c_n' := \chi(c_n))_n$ converges to $c' := \chi(c)$, i.e. that every accumulation point c'' of $(c_n')_n$ is equal to c'. By Lemma 5.5, $c' \in \partial\mathcal{M}_d$ and thus by Corollary 4.28, it remains to prove that $P_{c'}$ is quasiconformally conjugated to $P_{c''}$. Of course, we can assume $c_n' \to c''$. The following diagram reflects the construction of χ:

$$P_{c_n} \rightsquigarrow f_n = \varphi_n \circ P_{c_n} \rightsquigarrow P_{c_n'} = \psi_n \circ f_n \circ \psi_n^{-1} \overset{n\to\infty}{\longrightarrow} P_{c''}$$

and

$$P_c \rightsquigarrow f = \varphi \circ P_c \rightsquigarrow P_{c'} = \psi \circ f \circ \psi^{-1}.$$

By Lemma 4.19, the sequence $(\varphi_n)_n$ converges pointwise to φ and the complex dilatation of the quasiconformal homeomorphisms $(\psi_n)_{n\geq 0}$ is bounded by a constant independent on n. Thus the sequence $(\psi_n)_n$ is an equicontinuous family and by the theorem of Arcela-Ascoli, we find a subsequence which converges locally uniformly to a quasiconformal homeomorphism $\tilde{\psi}$. To simplify the notation we assume that $\psi_n \to \tilde{\psi}$ for $n \to \infty$. Then the equality

$$\psi_n \circ \varphi_n \circ P_{c_n} \circ \psi_n^{-1} = P_{c_n'}$$

yields in the limit $n \to \infty$

$$\tilde{\psi} \circ \varphi \circ P_c \circ \tilde{\psi}^{-1} = P_{c''}.$$

Using $P_{c'} = \psi^{-1} \circ \varphi \circ P_c \circ \psi$, we get

$$\tilde{\psi} \circ \psi \circ P_{c'} \circ \psi^{-1} \circ \tilde{\psi}^{-1} = P_{c''}.$$

Thus there exists a quasiconformal conjugation between $P_{c'}$ and $P_{c''}$. This proves the continuity of χ on $\partial T_{c_0}^{(j)}$. □

Lemma 5.11 (Bijectivity of χ)
For $d = 2$, the mapping $\chi : T_{c_0}^{(j)} \to T_{c_0}^{(j+1)}$ is bijective.

Proof. Analogous to the definition of $\chi : T_{c_0}^{(j)} \to T_{c_0}^{(j+1)}$ we can construct a continuous mapping

$$\tilde{\chi} : T_{c_0}^{(j+1)} \to T_{c_0}^{(j)}.$$

For every degree $d \geq 2$, we prove that

(1) for every $c \in T_{c_0}^{(j)}$, c and $\tilde{\chi}(\chi(c))$ have the same angled internal addresses;

(2) for every $\tilde{c} \in T_{c_0}^{(j+1)}$, $\tilde{c}$ and $\chi(\tilde{\chi}(\tilde{c}))$ have the same angled internal addresses.

Let $\vartheta_1 < \ldots < \vartheta_q$ be the angles of the parameter rays landing at c_0. For $c \in T_{c_0}^{(j)}$, the dynamic rays with angles $\vartheta_1, \ldots, \vartheta_q$ land at a common point $\alpha_{-k}(\chi(c))$ in the dynamics of $P_{\chi(c)}$. Then $\chi(c)$ is given by the unique parameter such that

$$\psi \circ \varphi \circ P_c \circ \psi^{-1} = P_{\chi(c)}$$

and that c and $\chi(c)$ are contained in the same sector of the main cardioid of $\mathcal{M}_d$; here φ is the quasiconformal homeomorphism given in Lemma 5.3 and ψ is a hybrid conjugation between the quasi-polynomial-like mapping $\varphi \circ P_c$ and the polynomial $P_{\chi(c)}$. To determine $\tilde{\chi}(\chi(c))$ we have to construct a piecewise dynamic homeomorphism between the $(j+1)$-th and the j-th region behind $\alpha_{-k}(\chi(c))$: we have constructed a piecewise dynamic homeomorphism from the j-th region $\mathcal{R}_j$ to the $(j+1)$-th region $\mathcal{R}_{j+1}$ behind $\alpha_{-k}(c')$ for all parameters c' behind c_0. The inverse of this mapping then gives in the case $c' = \chi(c)$ the piecewise dynamic homeomorphism we are looking for and we construct a mapping

$$\tilde{\varphi} : X_{\chi(c)}^{\eta} \to X_{\chi(c)}^{\eta}$$

as done in Lemma 5.3. By construction,

$$\tilde{\varphi} \circ \psi \circ \varphi \circ \psi^{-1}|_K = \mathrm{id}_K$$

for the connected set $K := K_{\chi(c)} \cap \psi(K_c)$ which contains the whole orbit of the critical point. Denote by $\tilde{\psi}$ the hybrid conjugation between the quasi-polynomial-like mapping $\tilde{\varphi} \circ P_{\chi(c)}$ and the polynomial $P_{\tilde{\chi}(\chi(c))}$, i.e.

$$\tilde{\psi} \circ \tilde{\varphi} \circ P_{\chi(c)} \circ \tilde{\psi}^{-1} = P_{\tilde{\chi}(\chi(c))}.$$

Then we conclude from Lemma 4.29 that the angled internal addresses of c and $\tilde{\chi}(\chi(c))$ are equal.

Similarly, one proves that $\tilde{c}$ and $\chi(\tilde{\chi}(\tilde{c}))$ have the same angled internal address for all parameters $\tilde{c} \in T_{c_0}^{(j+1)}$.

To assure the bijectivity of χ it remains to prove that the parameters in $T_{c_0}^{(j)}$ with a certain angled internal address are mapped bijectively by χ to the parameters in $T_{c_0}^{(j+1)}$ of another fixed angled internal address. To do so we need to restrict to the case $d = 2$: in this case any two parameters with the same angled internal address are contained in a tuned copy of the Mandelbrot set. By Lemma 5.7, the restriction of χ to every tuned copy of $\mathcal{M}_2$ bijective. This proves the bijectivity of χ. □

5.1.3 Description of the Homeomorphism χ

Corollary 5.12 (Images of Hyperbolic Components)
The map χ preserves hyperbolicity and internal coordinates. □

Corollary 5.13 (Images of Misiurewicz points)
Let $c_0 \in \mathcal{M}_d$ be a Misiurewicz point and $\chi : T_{c_0}^{(j)} \to T_{c_0}^{(j+1)}$ as in Theorem 5.1.

(1) χ maps narrow Misiurewicz points of α-type to narrow Misiurewicz points of α-type and not narrow Misiurewicz points to not narrow ones.

(2) Let c_0 be a narrow α-type Misiurewicz point of preperiod k and let N denote the number of iterates needed to map $P_{c_0}^k\Big(\Sigma(\vartheta_j, \vartheta_{j+1})\Big)$ to $P_{c_0}^k\Big(\Sigma(\vartheta_{j+1}, \vartheta_{j+2})\Big)$ (this number may be negative).

(a) For a Misiurewicz point $c \in T_{c_0}^{(j)}$, assume that the P_c-orbit of the critical point passes the region $\operatorname{star}(\alpha_{-(k+1)})$ exactly $n \geq 1$ times before it is mapped to a periodic point. Then the preperiod of $\chi(c)$ is equal to the preperiod of c minus nN.

(b) For the center c of a hyperbolic component $H \subset T_{c_0}^{(j)}$, assume that the P_c-orbit of the critical point passes the region $\operatorname{star}(\alpha_{-(k+1)})$ exactly $n \geq 1$ times before it is mapped to the critical point again. Then the period of $\chi(c)$ is equal to the period of c minus nN.

Remark. If c_0 is an arbitrary Misiurewicz point, but not a narrow one of α-type, the preperiod of the χ-image of Misiurewicz points and the period of the χ-image of hyperbolic components are harder to calculate. The piecewise dynamic homeomorphism being used to map $\Sigma(\vartheta_j, \vartheta_{j+1})$ to $\Sigma(\vartheta_{j+1}, \vartheta_{j+2})$ can no longer be defined in the same way on the whole sectors. Thus we have to examine not only the number how often the P_c-orbit of the critical point passes $\operatorname{star}(\alpha_{-(k+1)})$ but also which regions are these points contained in. But we do not make this precise here.

PROOF.

(1) Let $c \in T_{c_0}^{(j)}$ and $z' \in K_c$ be mapped to α by an iterate of P_c.

- By definition, the P_c-image of a narrow preperiodic point is also narrow.
- A preimage z' of a narrow preperiodic point $z = P_c(z')$ is not narrow if and only if z' disconnects the points $\alpha, \ldots, \zeta^{d-1}\alpha$:
  Suppose that z' disconnects the points $\alpha, \ldots, \zeta^{d-1}\alpha$; then one region around z' contains the critical point and another region contains one of the points $\alpha, \ldots, \zeta^{d-1}\alpha$. The P_c-image of the last one contains the critical point but $P_c(z') \neq -\alpha$, since c is not contained in the main cardioid of $\mathcal{M}_d$. Thus z' is not narrow.
  On the other hand, assume that z' is not narrow but $z := P_c(z')$ is a narrow preperiodic point. Then there is a region behind z' which is mapped to $\mathcal{R}_0(z)$. Therefore P_c does not map the regions behind z' to the regions behind z. This can only happen if z' disconnects the points $\alpha, \ldots, \zeta^{d-1}\alpha$.

Let $c \in T_{c_0}^{(j)}$ be a narrow Misiurewicz point of α-type. Since the critical value $c \in \Sigma_w^\eta(\vartheta_j, \vartheta_{j+1})$ is a narrow preperiodic point, we have to verify that $\varphi(c)$ is narrow as well: but this follows immediately from the construction of φ and the considerations at the beginning of this proof. Moreover if there was a not narrow Misiurewicz point mapped to a narrow one, the image under the inverse map would be narrow as well, which is a contradiction.

(2) If c_0 is a narrow Misiurewicz point of α-type, then for $c \in T_{c_0}^{(j)}$ the piecewise dynamic homeomorphism φ can be written in the form:

$$\varphi(z) = P_c^{-k} \circ P_c^N \circ P_c^k(z).$$

This clearly proves the claim.

□

Lemma 5.14 (Images of Dyadic Misiurewicz Points)
Let $c_0 \in \mathcal{M}_d$ be a Misiurewicz point such that $q \geq 3$ parameter rays land at c_0. Then for every $j \in \{1, \ldots, q-1\}$ the dyadic Misiurewicz point(s) with lowest preperiod in $\mathcal{M}_d(c_0, j)$ is (are) contained in $T_{c_0}^{(j)}$ and it is (they are) mapped by χ to the dyadic Misiurewicz point(s) in the $(j+1)$-th branch with lowest preperiod.

PROOF. Let ϑ_j and ϑ_{j+1} be the angles of the parameter rays separating $\mathcal{M}_d(c_0, j)$ from the rest of $\mathcal{M}_d$. Behind the parameter c_0 either there is exactly one dyadic Misiurewicz point with lowest preperiod or there are exactly $d-1$ of them: all these Misiurewicz points are the landing points of parameter rays with angles $a/d^n(d-1)$ in the j-th region behind c_0 such that n is as small as possible. In the dynamics of P_{c_0} there is either exactly one or there are exactly $d-1$ points which are mapped to the β-fixed point(s) with minimal preperiod: we have to iterate P_{c_0} on the region behind c_0 as long as there is no β-fixed point in the image.

After finitely many iterations we will have a β-fixed point in the image: when the image of the critical point disconnects $\alpha, \ldots, \zeta^{d-1}\alpha$. By the construction of our piecewise dynamic homeomorphisms used in the proof of Theorem 5.1, these points are fixed by the further construction of the considered piecewise dynamic homeomorphism. In particular, every such point belongs to the filled-in Julia set of f.

Moreover the β-fixed points of P_{c_0} remain the β-fixed points of f and therefore the points which are mapped to them in the dynamics of P_{c_0} with lowest preperiod correspond to the points with the lowest preperiod and period one in the dynamics of $P_{\chi(c_0)}$.

If we repeat our construction with a parameter c behind c_0, in particular, if c is a dyadic Misiurewicz point with lowest possible preperiod, then the f-orbit of the critical point will be contained within K_c and thus the parameter c is contained in $T_{c_0}^{(j)}$. Moreover, $f_c(c)$ is (as described above) a point with period one and lowest possible preperiod in the dynamics of f_c compared with all parameters behind c_0. This means that c is mapped to a dyadic Misiurewicz point of lowest possible preperiod. □

5.1.4 Extension of the Homeomorphisms to more Branches

In Sections 5.1.1 and 5.1.2 we have shown that all partial trees behind every Misiurewicz point c_0 are homeomorphic for $d = 2$ (Theorem 5.1 and 5.2). The partial trees are the branches of $\mathcal{M}_2$ behind c_0 with countably many subbranches removed. Now we want to explain, how these homeomorphisms can be extended to "larger" trees.

For a Misiurewicz point $c_0 \in \mathcal{M}_2$, which is the landing point of $q \geq 3$ parameter rays, and for a partial tree $T_{c_0}^{(j)}$ behind c_0 we consider the homeomorphism

$$\chi : T_{c_0}^{(j)} \to T_{c_0}^{(j+1)}.$$

Let $c_1 \in \partial T_{c_0}^{(j)}$ be a Misiurewicz point at which we had to cut off certain branches of $\mathcal{M}_2$ to construct the partial tree $T_{c_0}^{(j)}$ (compare Definition 4.50). By Lemma 4.52,

$$T_{c_0}^{(j)} \cap \overline{\mathcal{M}_2(c_1, m)} \subset T_{c_1}^{(m)}$$

for $m \in \{1, \ldots, q\}$; in particular this is true for the integer m with $|T_{c_0}^{(j)} \cap \overline{\mathcal{M}_2(c_1, m)}| > 1$. Since $K_{\chi(c_1)} \setminus \{\chi(c_1)\}$ has exactly q connected components (this follows directly from the construction), exactly q parameter rays land at the Misiurewicz point $\chi(c_1)$ and there exists an integer $\widetilde{m} \in \{1, \ldots, q\}$ such that

$$|T_{c_0}^{(j+1)} \cap \overline{\mathcal{M}_2(\chi(c_1), \widetilde{m})}| > 1.$$

Applying the Theorems 5.1 and 5.2 to the partial trees around c_1 and $\chi(c_1)$ we find homeomorphisms

$$\chi_{m'} : T_{c_1}^{(m')} \to T_{c_1}^{(m)} \quad \text{and} \quad \tilde{\chi}_{m'} : T_{\chi(c_1)}^{(m')} \to T_{\chi(c_1)}^{(\widetilde{m})} \qquad \text{for all } m' \in \{1, \ldots, q\}.$$

Using Lemma 4.52, this gives the following extension of the homeomorphism χ:

$$T_{c_0}^{(j)} \cup \bigcup_{m'=1}^{q} \left(\chi_{m'}^{-1}\left(T_{c_0}^{(j)} \cap \mathcal{M}_2(c_1, m')\right)\right) \longrightarrow T_{c_0}^{(j+1)} \cup \bigcup_{m'=1}^{q} \left(\tilde{\chi}_{m'}^{-1}(T_{c_0}^{(j+1)} \cap \mathcal{M}_2(\chi(c_1), m')\right),$$

$$\chi^{ext} : c \mapsto \begin{cases} \chi(c) & : \quad c \in T_{c_0}^{(j)} \\ \left(\chi_{m''}^{-1} \circ \chi \circ \chi_{m'}\right)(c) & : \quad c \in \chi_{m'}^{-1}(T_{c_0}^{(j)} \cap \mathcal{M}_2(c_1, m'')) \end{cases} ;$$

here we have to choose $m''(m')$ such that this really is a surjection and thus a homeomorphism. This homeomorphism does no longer preserve the orientation of the branches of $\mathcal{M}_2$ around the Misiurewicz point c_1. The "tree" where we have defined χ^{ext} can be described as before but we must not remove any branches of $\mathcal{M}_2$ at the Misiurewicz point c_1. This procedure can be used again to extend our homeomorphism to the branches behind a Misiurewicz point c_2 and so on. The constructed mapping will be a homeomorphism as long as this extension procedure is done only finitely many times. The aim of doing this procedure again and again could be to have a continuous mapping from $\mathcal{M}_2(c_0, j)$ to $\mathcal{M}_2(c_0, j+1)$ and thus a homeomorphism between these branches; but it is not clear yet that we reach a continuous mapping in the limit.

5.2 Homeomorphic Trees at Hyperbolic Components

Definition 5.15 (Partial Trees behind Hyperbolic Components)
Let $H \subset \mathcal{M}_d$ be a hyperbolic component of period l. For $i \in \{1, \dots, d-1\}$ and $q > p \geq 1$ with $\gcd(p, q) = 1$, the **partial p/q-tree $T_H^{(p/q)}(i)$ in the i-th sector of** *H is the p/q-sublimb in the i-th sector of H with the following branches removed:*
Let $\vartheta_1 < \vartheta_2$ denote the angles of the parameter rays landing at the root of the p/q-sublimb in the i-th sector of H. Let $\Theta \subset \mathbb{S}^1$ be the set of angles eventually mapped to ϑ_1 by $\vartheta \mapsto d\vartheta$. For every Misiurewicz point c which is the landing point of a parameter ray with angle in Θ, consider a branch behind c. Let ϑ', ϑ denote the angles of the parameter rays separating the considered branch from the rest and let m be the smallest integer such that

$$\{d^m\vartheta, d^m\vartheta'\} \,\cap\, \{\vartheta_1, d^l\vartheta_1, d^{2l}\vartheta_1, \dots, d^{(q-1)l}\vartheta_1\} \neq \emptyset.$$

Then we remove this branch, if

$$\{d^m\vartheta, d^m\vartheta'\} \neq \{\vartheta_1, \vartheta_2\}.$$

Lemma 5.16 (Properties of Partial Trees)
Every partial tree $T_H^{(p/q)}(i)$ is connected and compact. □

Remark. For every partial tree $T_c^{(j)}$ behind a Misiurewicz point c, we have proved that $T_c^{(j)}$ contains the dyadic Misiurewicz Point(s) of smallest preperiod. This is also true for the partial p/q-trees of hyperbolic components (Lemma 5.23).

Theorem 5.17 (Homeomorphic Trees at Hyperbolic Components)
Let $H \subset \mathcal{M}_d$ be a hyperbolic component of period l and let p_1, p_2, q be positive integers with $p_1 \neq p_2$, $\gcd(p_1, q) = 1 = \gcd(p_2, q)$. Then for every $i \in \{1, \dots, d-1\}$, there exists a mapping

$$\chi \;:\; T_H^{(p_1/q)}(i) \longrightarrow T_H^{(p_2/q)}(i)$$

which is continuous on the boundary and analytic on hyperbolic components.
In the case $d = 2$, the map χ is a homeomorphism.

REMARK. In the case $l = 1$, the map χ can indeed be defined as a homeomorphism from the whole p_1/q-limb to the whole p_2/q-limb; this is a consequence of the construction of χ below. For $l \geq 2$, the homeomorphism χ can only be defined on part of the p_1/q-sublimb. This part contains the partial p_1/q-tree in the i-th sector of H; but analogous as for the homeomorphisms between the partial trees behind Misiurewicz points, the map χ could be defined on a slightly larger part of the considered sublimb. For simplicity of notation, we will not take care about this.
In the case $d = 2$ it is obviously sufficient to prove

Theorem 5.18 (Homeomorphic Trees at Hyperbolic Components II) *Let $H \subset \mathcal{M}_d$ be a hyperbolic component of period l and let $q \geq 3$, $p < q$ be positive integers with $\gcd(p, q) = 1$. Then for every $i \in \{1, \ldots, d-1\}$, there exists a mapping*

$$\chi \;:\; T_H^{(1/q)}(i) \longrightarrow T_H^{(p/q)}(i)$$

which is continuous on the boundary and analytic on the hyperbolic components.
In the case $d = 2$, this is a homeomorphism.

Since for $d > 2$, the map χ from Theorem 5.18 is not known to be a homeomorphism, the equivalence of Theorem 5.17 and Theorem 5.18 is not obvious. In this section we will only prove Theorem 5.18; with a little bit more notation one could construct the map χ in Theorem 5.17 directly. For the applications in Section 6 it is enough to have Theorem 5.18.

5.2.1 Construction

The proof of this theorem will again be presented in several steps. We first give a short outline of the proof.

Denote the angles of the two parameter rays landing at the root of the $1/q$-sublimb $L_{1/q}$ of the i-th sector of H by $\vartheta_1 < \vartheta_2$. Then the dynamic rays with angles ϑ_1 and ϑ_2 land at a point z_1 of period l for all parameters $c \in L_{1/q}$ (compare Lemma 4.12). Let

$$z_0, z_1, \ldots, z_{l-1}$$

be the corresponding periodic orbit with $P_c(z_j) = z_{j+1}$, $j \in \{0, 1, \ldots, l-1\}$. Exactly q dynamic rays land at every z_j; the angles of the rays landing at z_1 are denoted by $\vartheta_1, \ldots, \vartheta_q$ such that $d^l \vartheta_n = \vartheta_{n+1} \bmod 1$ $(1 \leq n \leq l-1)$. Since $c \in L_{1/q}$, we have

$$\vartheta_1 < \ldots < \vartheta_q.$$

Starting with a parameter $c \in L_{1/q}$ we will construct simply connected Riemann surfaces

$$\mathcal{X}'_c \subset\subset \mathcal{X}_c$$

and a quasi-polynomial-like mapping

$$f_c : \mathcal{X}'_c \to \mathcal{X}_c$$

of degree d. For every parameter $c \in T_H^{(1/q)}(i)$, the Julia set of f_c will be connected and $f_c : \mathcal{X}'_c \to \mathcal{X}_c$ will be hybrid equivalent to a unique polynomial $P_{c'}$ such that c' and c are contained in the same sector of $\mathcal{M}_d$. That defines a mapping

$$\chi : T_H^{(1/q)}(i) \longrightarrow T_H^{(p/q)}(i).$$

Precisely: first we construct Riemann surfaces $\mathfrak{X}'_c$ and $\mathfrak{X}_c$. Let $\eta > 0$ be an arbitrary potential. During the construction of f_c we will consider finitely many sectors $S^\eta_w(\vartheta)$ of a certain width w around preperiodic rays. For every width $w > 0$, we can choose η so small that all these sectors and all their images are pairwise disjoint. It may be necessary to reduce the potential during the construction to assure that this non-intersection property holds for all these sectors. This is needed to assure that every f_c-orbit passes a region at most once in which f_c is not holomorphic (Lemma 5.19).

The dynamic rays landing at the periodic orbit $z_0, z_1, \ldots, z_{l-1}$ divide the region $X^\eta_c = \{z \in \mathbb{C} \ : \ G_c(z) < \eta\}$ into $q + (q-1)(l-1) = ql - l + 1$ connected components. For $j \in \{0, \ldots, l-1\}$, the closures of the components centered at z_j are denoted by

$$Y^\eta_0(z_j),\ Y^\eta_1(z_j),\ \ldots,\ Y^\eta_{q-1}(z_j)$$

in a counterclockwise order such that

$$P^m_c(0) \in Y^\eta_0(z_m)$$

for $m \in \{0, \ldots, l-1\}$.

With this definition some of the components are labeled twice in the case $l \geq 2$, since the exact number of connected components is $ql - l + 1 < ql$.

By Lemma 4.40, at most two of the components around a point z_j contain further points of the orbit

$$z_0,\ z_1,\ \ldots,\ z_{l-1}$$

on their boundaries. Since $\gcd(p, q) = 1$, there is a unique $p' \in \{1, \ldots, q-1\}$ such that

$$p'p \ = \ 1 \quad \mathrm{mod} \quad q.$$

In Step 1 to Step l we change the relative order of the components around every point z_j and glue these components equipotentially together:

Step 1: The components around z_0 are glued together equipotentially as follows:

$$Y^\eta_0(z_0) \longleftrightarrow Y^\eta_{p'}(z_0) \longleftrightarrow Y^\eta_{2p'}(z_0) \longleftrightarrow \ \ldots \ \longleftrightarrow Y^\eta_{(q-1)p'}(z_0) \longleftrightarrow Y^\eta_0(z_0);$$

here the index of the components has to be read modulo q. More precisely, the identification of the points on the boundaries is done as follows: let

$$\tilde{\vartheta}_1,\ \ldots,\ \tilde{\vartheta}_q$$

denote the angles of the dynamic rays landing at z_0 such that for every $r \in \{0, \ldots, q-1\}$, the component $Y^\eta_r(z_0)$ is bounded by part of the two dynamic rays with angles $\tilde{\vartheta}_r$, $\tilde{\vartheta}_{r+1}$ ($\tilde{\vartheta}_0 := \tilde{\vartheta}_q$) and part of the equipotential of potential η. Since $0 \in Y^\eta_0(z_0)$ and since the components $Y^\eta_m(z_0)$ are ordered counterclockwise around z_0, every angle $\tilde{\vartheta}_n$ is uniquely determined. Then

$$Y^\eta_r(z_0) \longleftrightarrow Y^\eta_s(z_0)$$

means that $z_0 \in \partial Y^\eta_r(z_0)$ is identified with $z_0 \in \partial Y^\eta_s(z_0)$; denoting by $\Phi_c : \mathbb{P} \setminus K_c \to \mathbb{P} \setminus \overline{\mathbb{D}}$ the Böttcher mapping for P_c, for every $p \in \,]0, \eta]$, we identify the points

$$\Phi^{-1}_c\left(\exp(p + 2\pi i \tilde{\vartheta}_{r+1})\right) \ \in \ \partial Y^\eta_r(z_0) \qquad \text{and} \qquad \Phi^{-1}_c\left(\exp(p + 2\pi i \tilde{\vartheta}_s)\right) \ \in \ \partial Y^\eta_s(z_0).$$

Step 2: We continue with a point $z_{j_1} \neq z_0$ on the boundary of one of the components glued together so far, say on the boundary of $Y^\eta_{q_1}(z_0)$. Then we glue together the components around z_{j_1} equipotentially as follows

$$Y^\eta_{q_1}(z_{j_1}) \longleftrightarrow Y^\eta_{q_1+p'}(z_{j_1}) \longleftrightarrow Y^\eta_{q_1+2p'}(z_{j_1}) \longleftrightarrow \;\ldots\; \longleftrightarrow Y^\eta_{q_1+(q-1)p'}(z_{j_1}) \longleftrightarrow Y^\eta_{q_1}(z_{j_1}).$$

This gluing procedure is analogous to that of Step 1: we have to replace z_0 by z_{j_1}.
Step 3: We continue with a point $z_{j_2} \notin \{z_0, z_{j_1}\}$ on the boundary of one of the components glued together so far, say

$$Y^\eta_{q_2}(z_{j_2}) \in \left\{Y^\eta_1(z_0), \ldots, Y^\eta_q(z_0),\ Y^\eta_1(z_{j_1}), \ldots, Y^\eta_q(z_{j_1})\right\}.$$

Then we glue together the components around z_{j_2} equipotentially as follows

$$Y^\eta_{q_2}(z_{j_2}) \longleftrightarrow Y^\eta_{q_2+p'}(z_{j_2}) \longleftrightarrow Y^\eta_{q_2+2p'}(z_{j_2}) \longleftrightarrow \;\ldots\; \longleftrightarrow Y^\eta_{q_2+(q-1)p'}(z_{j_2}) \longleftrightarrow Y^\eta_{q_2}(z_{j_2}).$$

$$\vdots$$

Step l: We continue with a point $z_{j_{l-1}} \notin \{z_0, z_{j_1}, \ldots, z_{j_{l-2}}\}$ (which is unique in this last step) on the boundary of one of the components glued together so far, say

$$Y^\eta_{q_{l-1}}(z_{j_{l-1}}) \in \left\{Y^\eta_1(z_0), \ldots, Y^\eta_q(z_0),\ Y^\eta_1(z_{j_1}), \ldots, Y^\eta_q(z_{j_1}),\ \ldots,\ Y^\eta_1(z_{j_{l-2}}), \ldots, Y^\eta_q(z_{j_{l-2}})\right\}.$$

Then we glue together the components around $z_{j_{l-1}}$ equipotentially as follows

$$Y^\eta_{q_{l-1}}(z_{j_{l-1}}) \longleftrightarrow Y^\eta_{q_{l-1}+p'}(z_{j_{l-1}}) \longleftrightarrow \ldots \longleftrightarrow Y^\eta_{q_{l-1}+(q-1)p'}(z_{j_{l-1}}) \longleftrightarrow Y^\eta_{q_{l-1}}(z_{j_{l-1}}).$$

We have constructed a set $\mathcal{X}_c$ which turns out to be a Riemann surface: for a precise description, we have to distinguish between the sector $Y^\eta_j(z) \subset \mathbb{C}$ and the corresponding sector in $\mathcal{X}_c$; but there is a natural bijective mapping between them. For simplicity of notation, we will not consider this bijection any longer and in both situations ($\mathbb{C}$ and $\mathcal{X}_c$) we will denote the corresponding sector by $Y^\eta_j(z)$. To assure that $\mathcal{X}_c$ is a Riemann surface we have to construct complex coordinates locally on $\mathcal{X}_c$:

(1) In the interior of any sector $Y^\eta_j(z) \subset \mathcal{X}_c$ the complex coordinates are given naturally.

(2) In a small neighborhood U of a point $z \in \Big(\partial Y^\eta_r(z_j) \cap \partial Y^\eta_s(z_j)\Big) \setminus \{z_0, \ldots, z_{l-1}\}$ complex coordinates can be defined as follows: assume that the identification at the point z is done in the order $Y^\eta_r(z_j) \longleftrightarrow Y^\eta_s(z_j)$. For suitable chosen U, the set $U \setminus \partial Y^\eta_r(z_j)$ has exactly two connected components which are mapped by Φ_c to open sets in $\mathbb{C} \setminus \overline{\mathbb{D}}$. A part of the boundary of $\Phi_c\Big(U \cap \mathrm{int}(Y^\eta_r(z_j))\Big)$ is contained in the radial line $\{\rho e^{2\pi i \vartheta_{r+1}} \;:\; \rho > 1\}$ and a part of the boundary of $\Phi_c\Big(U \cap \mathrm{int}(Y^\eta_s(z_j))\Big)$ is contained in the radial line $\{\rho e^{2\pi i \vartheta_s} \;:\; \rho > 1\}$. By the identification property above, the map

$$U \longrightarrow \mathbb{C},\ z \mapsto \begin{cases} \Phi_c(z) & : \quad z \in U \cap \mathrm{int}(Y^\eta_r(z_j)) \\ \exp\left(2\pi i(\vartheta_{r+1} - \vartheta_s)\right) \cdot \Phi_c(z) & : \quad z \in U \cap Y^\eta_s(z_j) \end{cases}$$

is continuous and it maps U to a connected open set in $\mathbb{C}$. This yields complex coordinates on U.

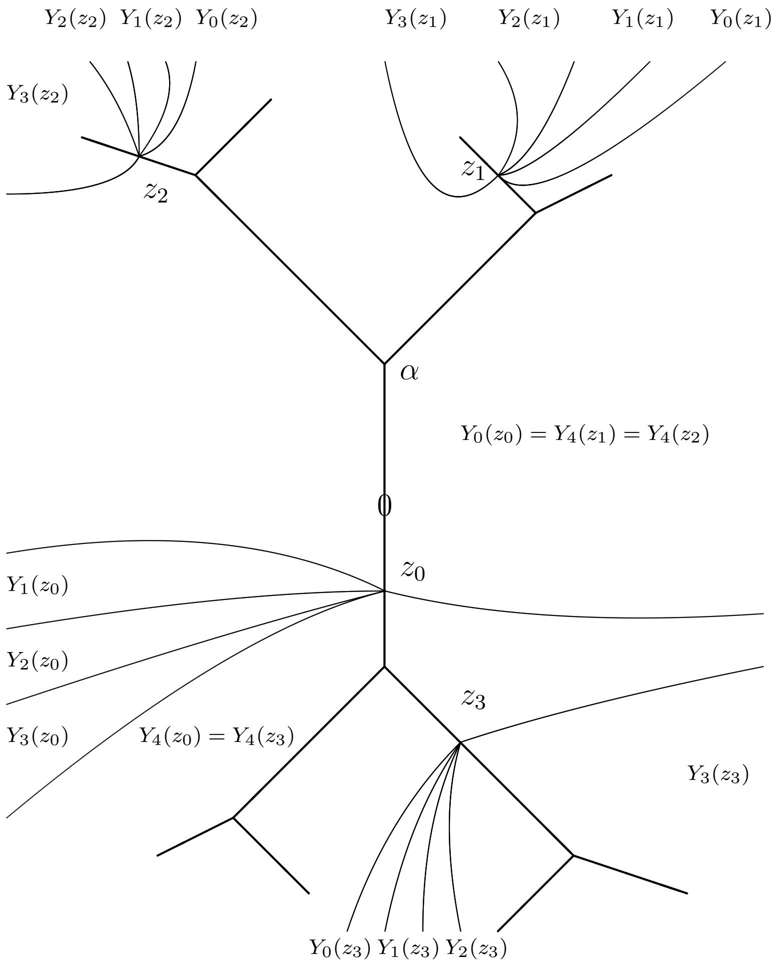

Figure 24: The definition of the sets $Y_i(z_j)$ for a parameter $c \in \mathcal{M}_2$ belonging to the $1/5$-sublimb of the hyperbolic component with internal address $1_{1/3} \to 3_{1/2} \to 4$ (here we have omitted the upper index η, since the line of potential η is not drawn in the picture). They are used to construct the homeomorphism from the partial $1/5$-tree $T_H^{(1/5)}$ to the partial $p/5$-trees $T_H^{(p/5)}$. Thick lines mark the Julia set, and thin lines mark the dynamic rays landing at the orbit $z_0, \ldots, z_3$.

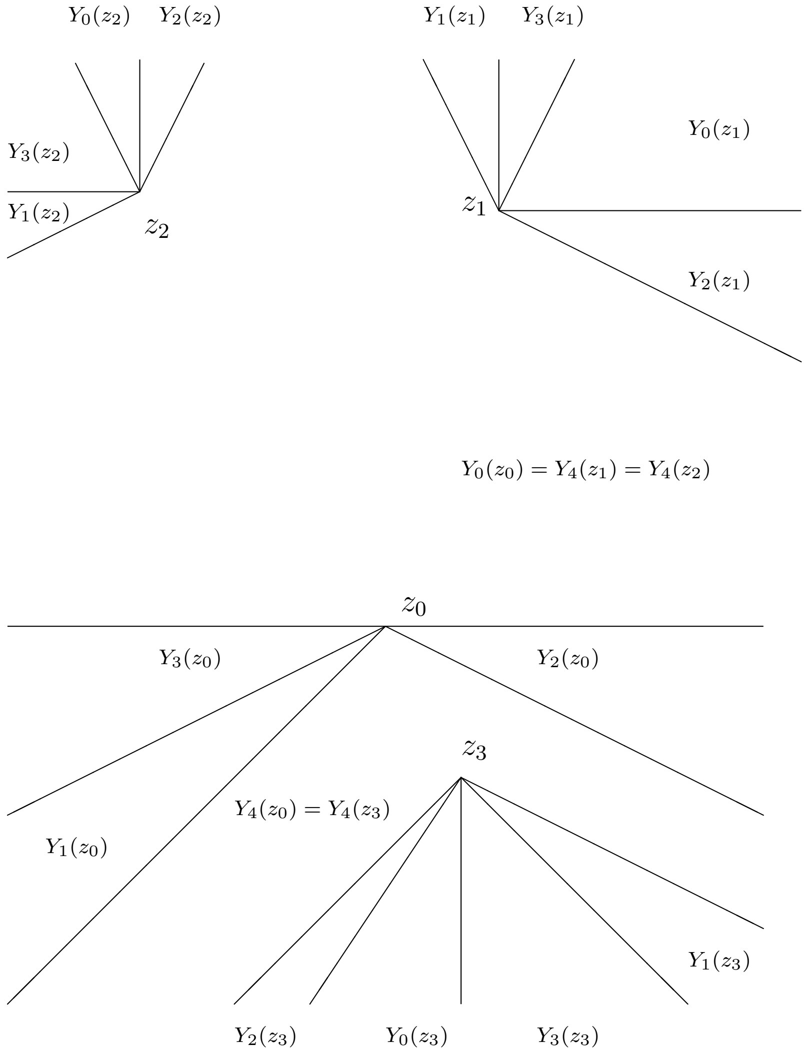

Figure 25: Reorganisation of the parts $Y_i(z_j)$ in the construction of the homeomorphism between part of the 1/5-sublimb and part of the 2/5-sublimb of the hyperbolic component with internal address $1_{1/3} \to 3_{1/2} \to 4$ (again we have omitted the equipotentials). The straight lines in the picture represent the dynamic rays landing at the points $z_0, \ldots z_3$.

(3) In a small neighborhood U of $z \in \{z_0, \ldots, z_{l-1}\}$ we again get complex coordinates by using the natural complex coordinates in each of the q sets $U \cap \operatorname{int}(Y_j^\eta(z))$ and putting them together continuously.

Repeating this procedure but now starting with potential η/d instead of η we construct the Riemann surface $\mathcal{X}'_c$; by construction,

$$\mathcal{X}'_c \subset\subset \mathcal{X}_c.$$

Let $\mathcal{A} := \{z_0, \ldots, z_{l-1}\}$ and $\mathcal{A}' := P_c^{-1}(\mathcal{A}) \setminus \mathcal{A}$, i.e.

$$\mathcal{A}' := \{\zeta z_0, \ldots, \zeta^{d-1} z_0,\ \zeta z_1, \ldots, \zeta^{d-1} z_1,\ \ \ldots,\ \ \zeta z_{l-1}, \ldots, \zeta^{d-1} z_{l-1}\}.$$

In particular, $\mathcal{A} = \{\alpha\}$ and $\mathcal{A}' = \{-\alpha\}$ in the case $d = 2$, $l = 1$. We define a degree d-mapping

$$f_c : \mathcal{X}'_c \to \mathcal{X}_c$$

in the following three steps:

Step 1: For every component $Y := Y_i^{\eta/d}(z_j)$ with $Y \cap \mathcal{A}' = \emptyset$, we have

$$P_c(Y_i^{\eta/d}(z_j)) = \begin{cases} Y_i^{\eta}(z_{j+1}) & : \quad j \neq l-1 \\ Y_{i+1}^{\eta}(z_0) & : \quad j = l-1 \end{cases}.$$

For these components, let

$$f_c|_Y := P_c|_Y.$$

(The precise definition of f_c on $Y \subset \mathcal{X}'_c$ is given by

$$f_c = \tilde{\iota}^{\,-1} \circ P_c \circ \iota,$$

where $\iota : \mathcal{X}'_c \supset Y \longrightarrow Y \subset \mathbb{C}$ and $\tilde{\iota} : \mathcal{X}'_c \supset P_c(Y) \longrightarrow P_c(Y) \subset \mathbb{C}$ are the natural embeddings. As we have mentioned before we will ignore the embeddings for simplicity of notation.)

Step2: Thus it remains to define f_c on every component $Y = Y_i^{\eta/d}(z_j)$ with $Y \cap \mathcal{A}' \neq \emptyset$. Exactly q dynamic rays land at every point in $\mathcal{A}' \cap Y$; these dynamic rays divide Y into $1 + m(q-1)$ connected components, where

$$m := |\mathcal{A}' \cap Y|.$$

The closures of these components are denoted by $Z_1, \ldots, Z_{1+m(q-1)}$.

For all components $Z \in \{Z_1, \ldots, Z_{1+m(q-1)}\}$ with $|\partial Z \cap (\mathcal{A} \cup \mathcal{A}')| \geq 2$, we define

$$f_c|_{\tilde{Z}} := P_c|_{\tilde{Z}};$$

here, the set $\tilde{Z}$ is the component Z with certain sectors of a certain width w (which is specified in Case 2 below) around some dynamic rays removed: for every point $a' \in \partial Z \cap \mathcal{A}'$ which disconnects the points in $(\mathcal{A} \cup \mathcal{A}') \setminus \{a'\}$, we remove the sectors of width w around all dynamic rays landing at a'. On these removed sectors we define the map f_c as in Case 2 below.

For all components $Z \in \{Z_1, \ldots, Z_{1+m(q-1)}\}$ with $|\partial Z \cap (\mathcal{A} \cup \mathcal{A}')| = 1$, there exists a unique integer $j \in \{1, \ldots, l-1\}$ and a unique point $z'_j \in \partial Z$ with

$$P_c(z'_j) = z_j.$$

Denote the closure of the components of $Z \setminus \text{rays}(z'_j)$ around z'_j in a counterclockwise order by $Z'_0, Z'_1, \ldots, Z'_{q-1}$ such that $P_c(Z'_0) = Y_0^{\eta}(z_j)$. Then

- $P_c : Z'_m \longrightarrow Y_m^{\eta}(z_j)$ is biholomorphic if $j \neq 1$ or $m \neq 0$;
- the map f_c is already defined at least on Z'_{q-1}.

To continue the construction of f_c we have to distinguish between two cases:

Case 1: *The map f_c is defined yet on only one component around z'_j.*
Then

$$(\mathcal{A} \cup \mathcal{A}') \cap Z'_m = \{z'_j\} \qquad \text{for } m \in \{1, \ldots, q-2\}.$$

Now we define f_c on $Z'_1 \cup \ldots \cup Z'_{q-2}$. For a sufficient small width $w > 0$, let

$$\triangle S^\eta(z'_j) := \bigcup_{\vartheta \in \Theta(z'_j)} S^\eta_w(\vartheta) \quad \text{and} \quad \triangle S^\eta(z_j) := \bigcup_{\vartheta \in \Theta(z_j)} S^\eta_w(\vartheta);$$

$\Theta(z'_j) \subset \mathbb{S}^1$ (respectively $\Theta(z_j) \subset \mathbb{S}^1$) is the set of angles describing the dynamic rays landing at z'_j (respectively z_j). Since we have changed the order of the components $Y^{\eta/d}_m(z_j)$ around z_j to define $\mathcal{X}'_c$, we have to assure that

$$f_c\Big(Z'_{q-1+m} \setminus \triangle S^{\eta/d}(z'_j)\Big) = Y^\eta_{q-1+mp'}(z_j) \setminus \triangle S^\eta(z_j) \quad (0 \le m \le q-1)$$

to get in the end a continuous mapping f_c (the m-th sector around $z'_j \in \mathcal{X}'_c$ starting from Z'_{q-1} has to be mapped to the m-th sector around $z_j \in \mathcal{X}_c$ starting from $Y^\eta_{q-1}(z_j)$).
This can be done as follows: let $m \in \{1, \ldots, q-1\}$. By Lemma 4.40, there exists $\eta' > 0$ and a biholomorphic mapping

$$P : Y^\eta_{q-1+m}(z_j) \longrightarrow Y^{\eta'}_{q-1+mp'}(z_j)$$

which is defined by a certain composition of backward and forward iterates of P_c. If we have chosen w small enough, then

$$P \circ P_c : Z'_{q-1+m} \setminus \triangle S^{\eta/d}(z'_j) \longrightarrow Y^{\eta'}_{q-1+mp'}(z_j) \setminus \triangle S^{\eta'}(z_j)$$

is biholomorphic. As in the proof of Lemma 5.3, we find a quasiconformal homeomorphism

$$\psi : Y^{\eta'}_{q-1+mp'}(z_j) \setminus \triangle S^{\eta'}(z_j) \longrightarrow Y^\eta_{q-1+mp'}(z_j) \setminus \triangle S^\eta(z_j)$$

such that

$$\psi(z) = z \quad \text{for } z \in Y^{\eta'/1.5}_{q-1+mp'}(z_j) \setminus \triangle S^{\eta'}(z_j).$$

Then we define

$$f_c(z) := (\psi \circ P \circ P_c)(z) \quad \text{for } z \in Z'_{q-1+m} \setminus \triangle S^{\eta/d}(z'_j).$$

For every integer $m \in \{1, \ldots, q-1\}$, we find P and ψ as described above to define f_c on $Z'_{q-1+m} \setminus \triangle S^{\eta/d}(z'_j)$. To finish the definition of f_c in $Z'_1 \cup \ldots \cup Z'_{q-1}$ we have to care about $\triangle S^{\eta/d}(z'_j) \setminus Z'_{q-1}$ which has to be mapped to $\triangle S^\eta(z_j) \setminus Y^\eta_{q-1}(z_j)$: this is done by choosing a quasiconformal homeomorphism between these sets such that f_c is continuous (in logarithmic Böttcher coordinates the considered sectors are triangles in the right half plane and Lemma 3.3 can be applied).

Case 2: *The map f_c is already defined in two components around z'_j.* Then

$$\begin{aligned}(\mathcal{A}\cup\mathcal{A}') &\cap \left(Z'_0\cup\ldots Z'_{q-3}\right)=\emptyset,\\ (\mathcal{A}\cup\mathcal{A}') &\cap Z'_{q-2}\neq\emptyset,\\ (\mathcal{A}\cup\mathcal{A}') &\cap Z'_{q-1}\neq\emptyset\end{aligned}$$

and since $\mathcal{R}_b(z_1)\cap(\mathcal{A}\cup\mathcal{A}')=\emptyset$, $j\neq 1$ and $z'_j\notin\{\zeta z_0,\ \ldots,\ \zeta^{d-1}z_0\}$. Let $\tilde{\vartheta}$ denote the angle of the dynamic ray landing at z'_j and containing $Z'_{q-1}\cap Z'_{q-2}$. For any width $w>0$, we define the P_c^{ql}-invariant sectors

$$Y_m^\eta(z_j,w):=Y_m^\eta(z_j)\setminus\bigcup_{\vartheta\in\Theta(z_j)}S_w^\eta(\vartheta)\qquad(1\leq m\leq q).$$

By Lemma 4.18, there is a unique $w>0$ such that

$$\operatorname{mod}\left(S_w^{\eta/d}(\tilde{\vartheta})\right)=(2p-1)\cdot\operatorname{mod}\left(Y_0^\eta(z_1,w)\right);$$

moreover, we shrink the potential η if necessary. Since the annuli $Y_n^\eta(z_j,w)/\sim$ are permuted biholomorphically by the dynamics (where the equivalence relation $\sim$ is defined as introduced in Section 4.2), all these annuli have the same opening modulus. We will redefine f_c on $S_w^{\eta/d}(\tilde{\vartheta})$ now. To do so, we subdivide this sector into $2p-1$ sectors with the same opening modulus $\operatorname{mod}\left(Y_0^\eta(z_1,w)\right)$; they are denoted counterclockwise by (compare Figure 26)

$$S_1,\ \ldots,\ S_{2p-1}.$$

Then we define the mapping f_c on the sectors

$$\begin{aligned}S_{2(p-1)} &\longrightarrow Y^\eta_{q-1-p'}(z_j,w)\\ S_{2(p-1)-2} &\longrightarrow Y^\eta_{q-1-2p'}(z_j,w)\\ &\ \ \vdots\\ S_2 &\longrightarrow Y^\eta_{q-1-(p-1)p'}(z_j,w)\end{aligned}$$

by the unique Riemann mappings such that vertices are mapped to vertices as usual. By Lemma 3.9, this map can be extended to a quasiconformal homeomorphism

$$S_w^{\eta/d}(\tilde{\vartheta})\longrightarrow\bigcup_{m=1}^{p-1}Y^\eta_{q-1-mp'}(z_j).$$

Analogously as in Case 1, we find quasiconformal homeomorphisms

$$\begin{aligned}Z'_{q-3}\setminus\triangle S^{\eta/d}(z'_j) &\longrightarrow Y^\eta_{q-2-p'}(z_j,w)\\ Z'_{q-4}\setminus\triangle S^{\eta/d}(z'_j) &\longrightarrow Y^\eta_{q-2-2p'}(z_j,w)\\ &\ \ \vdots\\ Z'_p\setminus\triangle S^{\eta/d}(z'_j) &\longrightarrow Y^\eta_{q-2-(q-(p+2))p'}(z_j,w)\end{aligned}$$

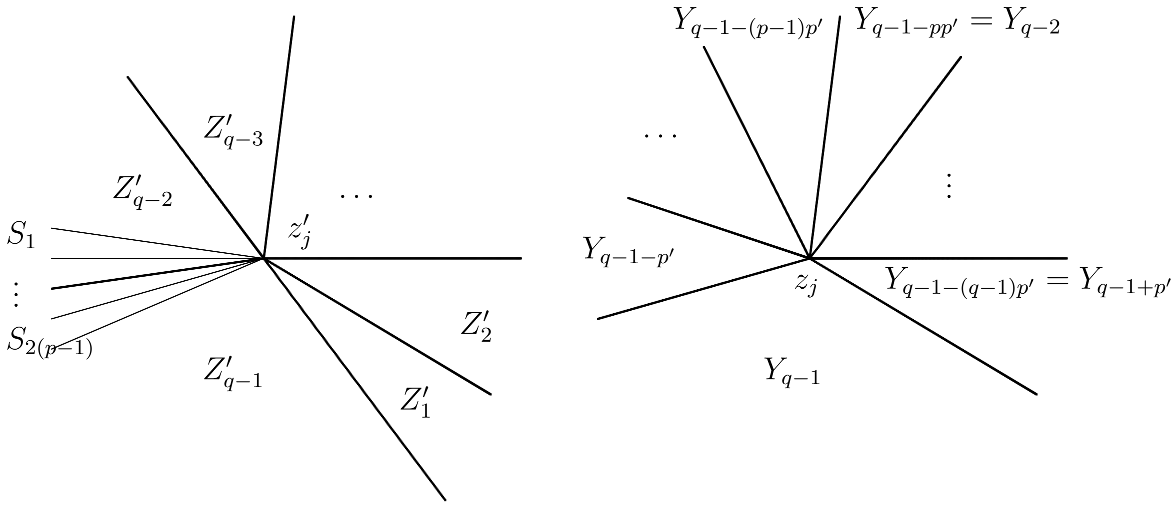

Figure 26: *Left:* the definition of the sectors Z'_m and S_n in $\mathcal{X}'_c$; *Right:* the components $Y_m(z_j)$ around z_j.

which are holomorphic except possibly for the points with potential greater than $\eta/(1.5d)$. To the remaining sectors around z'_j we extend f_c quasiconformally; in particular, the sectors $Z'_1, \ldots, Z'_{p-1}$ disappear in the complement of K_c and this is the point where the difference of the Julia sets of P_c and f_c comes from. The existence of such a quasiconformal extension is guaranteed by Lemma 3.9.

This finishes the definition of $f_c : \mathcal{X}'_c \to \mathcal{X}_c$ and we have to assure that it is quasi-polynomial-like of degree d.

Lemma 5.19 (The map f_c is Quasi-Polynomial-Like)
Every f_c-orbit passes the non-holomorphy region of f_c at most once. For $c \in T_H^{(1/q)}(i)$, the filled-in Julia set of f_c is connected and f_c is hybrid equivalent to a unique polynomial $P_{c'}$ such that c' and c are contained in the same sector of $\mathcal{M}_d$; more precisely, $c' \in T_H^{(p/q)}(i)$

Proof. By the definition of $T_H^{(1/q)}(i)$, the filled-in Julia set of f_c is connected. By construction, every f_c-orbit passes the regions where $\overline{\partial} f_c$ does not vanish at most once (compare Lemma 5.3). Putting the standard complex structure σ_0 into the annulus $\mathcal{X}_c \backslash \mathcal{X}'_c$ and pulling this structure back by the dynamics of f_c we get an f_c-invariant almost complex structure σ which can be integrated by a quasiconformal homeomorphism φ. Thus

$$\varphi^{-1} \circ f_c \circ \varphi$$

yields a polynomial-like mapping of degree d which is hybrid equivalent to a polynomial $P_{c'}$. By Theorem 4.24, the parameter c' is uniquely determined up to multiplication by a $(d-1)$-th root of unity and we can assume that c and c' are contained in the same sector of $\mathcal{M}_d$. The point z_0 is f_c-periodic with period l and the combinatorial rotation number at this point is equal to p/q. Since a quasiconformal conjugation does neither destroy the existence of a periodic point nor the corresponding combinatorial rotation number, $c' \in L_{p/q}$. In particular, by construction, $c' \in T_H^{(p/q)}(i)$ and we are done. □

5.2.2 Continuity and Bijectivity

Using the same arguments as in Lemma 5.5, Lemma 5.7, Lemma 5.10 and Lemma 5.11 we get

Lemma 5.20 (Behavior of χ)
The mapping $\chi\ :\ T_H^{(1/q)}(i) \longrightarrow T_H^{(p/q)}(i)\ ,\ c \mapsto c'$ *maps*

(1) $T_H^{(1/q)}(i) \cap \partial\mathcal{M}_d$ *to* $T_H^{(p/q)}(i) \cap \partial\mathcal{M}_d$*,*

(2) hyperbolic components to hyperbolic components,

(3) non-hyperbolic components to non-hyperbolic ones. □

Lemma 5.21 (Compatibility with Simple Renormalization)
Let $H_l \subset T_H^{(p/q)}(i)$ *be a hyperbolic component of period* l*,* $H_{\tilde{l}} := \chi(H_l)$*,* τ *the tuning map which sends the main cardioid of* $\mathcal{M}_d$ *to* H_l *and* $\tilde{\tau}$ *the tuning map which sends the main cardioid to* $H_{\tilde{l}}$*. Then* $\tau(\mathcal{M}_d) \subset T_H^{(1/q)}(i)$ *and there holds on* $\mathcal{M}_d$

$$\tilde{\tau}^{-1} \circ \chi \circ \tau = \mathrm{id}.$$

In particular,

$$\tilde{\tau} \circ \tau^{-1} = \chi\ :\ \tau(\mathcal{M}_d) \to \tilde{\tau}(\mathcal{M}_d)$$

is bijective. Moreover χ *maps* $\tau(\mathrm{int}(\mathcal{M}_d))$ *biholomorphically to* $\tilde{\tau}(\mathrm{int}(\mathcal{M}_d))$*.* □

Lemma 5.22 (Continuity and Bijectivity of χ)
The map χ *is continuous except possibly at parameters in non-hyperbolic components; the restriction of* χ *to the parameters* $c \in \mathcal{M}_d$ *where the Multibrot set is locally connected is injective.*
If $d = 2$*, then the mapping* $\chi\ :\ T_H^{(1/q)} \to T_H^{(p/q)}\ ,\ c \mapsto c'$ *is a homeomorphism.* □

Lemma 5.23 (Images of Dyadic Misiurewicz Points)
The dyadic Misiurewicz point(s) of lowest preperiod in $L_{1/q}$ *is (are) contained in* $T_H^{(1/q)}(i)$ *and it is (they are) mapped by* χ *to the dyadic Misiurewicz point(s) with lowest preperiod in* $T_H^{(p/q)}(i)$*.* □

5.3 Homeomorphic Embeddings of Sublimbs into Sublimbs

Theorem 5.24 (Homeomorphic Embeddings of $1/q$-Sublimbs)
Let $H \subset \mathcal{M}_d$ *be a hyperbolic component of period* l *and let* $q \geq 2$*. Then for every* $i \in \{1, \ldots, d-1\}$*, the* $1/q$*-sublimb in the* i*-th sector of* H *can be mapped into the* $1/(q+1)$*-sublimb of the* i*-th sector of* H *such that the dyadic Misiurewicz point(s) of lowest preperiod in the considered* $1/q$*-sublimb is (are) mapped to the dyadic Misiurewicz point(s) of lowest preperiod in the considered* $1/(q+1)$*-sublimb. This mapping is continuous on the boundary and analytic in the hyperbolic components.*
In the case $d = 2$ *this mapping is a homeomorphic embedding.*

Remark. This statement has been proved by B. Branner and A. Douady if $d = 2$, $q = 2$ and $l = 1$. Their proof generalizes to the case $q \geq 2$ and an arbitrary hyperbolic component $H \subset \mathcal{M}_2$; this has been remarked by A. Douady in [D]. Since the statement above is needed in this general form, we will sketch its proof in the following; we do not restrict to degree $d = 2$.
One could try to generalize this statement and construct a homeomorphic embedding of the p/q-sublimb into the $p'/(q+1)$-sublimb ($\gcd(p,q) = 1 = \gcd(p', q+1)$). But this is not possible in general, at least if the homeomorphism is orientation preserving. The reason is that for $p' \notin \{p-1, p, p+1\}$ this homeomorphism cannot be defined in the whole p/q-sublimb (compare Theorem 5.1 and Theorem 5.17).

5.3.1 Construction

For every parameter c in the $1/q$-sublimb $L_{1/q}$ of the i-th sector of H we will construct Riemann surfaces $\mathcal{X}'_c \subset\subset \mathcal{X}_c$ and a unicritical quasi-polynomial-like mapping $f_c : \mathcal{X}'_c \to \mathcal{X}_c$ of degree d. This mapping yields a polynomial within the $1/(q+1)$-sublimb $L_{1/(q+1)}$ of the i-th sector of H.

Fix a parameter c in the $1/q$-sublimb $L_{1/q}$ of the i-th sector of H. Exactly two parameter rays land at the root of this sublimb, and the period of their angles $\vartheta_1 < \vartheta_2$ is equal to ql. Let $\eta, v > 0$ (it may be necessary to modify η and v during the proof). In the dynamic plane of P_c the dynamic rays with angles ϑ_1 and ϑ_2 land at a point z_1 of exact period l; z_1 is the characteristic point of its orbit $\{z_1, P_c(z_1), \ldots, P_c^{l-1}(z_1)\}$, i.e. it disconnects the critical value from the rest of the orbit. Considering the notation of the sectors in Section 4.2, let

$$\Sigma_v^\eta := \Sigma_v^\eta(\vartheta_1, \vartheta_2)$$

and

$$\widetilde{\Sigma}_v^\eta := P_c^{-1}\left(\Sigma_v^\eta\right).$$

For l copies

$$W_0, \ldots, W_{l-1}$$

of Σ_v^η, let

$$\iota_j : \Sigma_v^\eta \to W_j, \quad j \in \{0, \ldots, l-1\}$$

be the canonical bijection.
A Riemann surface $\mathcal{X}_c$ is now defined by the disjoint union

$$\mathcal{X}_c := X_c^\eta \setminus \left(\bigcup_{j=0}^{l-1} S_v^\eta(d^j\vartheta_1)\right) \sqcup W_0 \sqcup \ldots \sqcup W_{l-1}$$

identifying the boundary points of $S_v^\eta(d^j\vartheta_1)$ with the boundary points of W_j as follows: the boundary of every W_j consists of three curves:

(1) the "upper" curve $\gamma_u^{(j)}$ describing all points in ∂W_j with potential η,

(2) the "left" curve $\gamma_l^{(j)}$ describing all points in ∂W_j with external argument bigger than $d^j\vartheta_1$ and

(3) the "right" curve $\gamma_r^{(j)}$ describing all points in ∂W_j with external argument smaller than $d^j\vartheta_1$.

For $0 \leq j \leq l-1$, the point on $\gamma_r^{(j)}$ with potential $p < \eta/d^j$ is identified with the point

$$\Phi_c^{-1}\left(\exp(d^j p) \cdot \exp\left(i(2\pi d^j \vartheta_1 - vp)\right)\right)$$

and the point on $\gamma_l^{(j)}$ with potential $p < \eta/d^{l-j}$ is identified with

$$\Phi_c^{-1}\left(\exp(d^{-(l-j)} p) \cdot \exp\left(i(2\pi d^j \vartheta_1 + vp)\right)\right).$$

Define a subset $\mathcal{X}'_c \subset \mathcal{X}_c$ by repeating this construction but now starting with potential η/d instead of η. We define the unicritical quasi-polynomial-like mapping f_c as follows:

$$f_c : \mathcal{X}'_c \setminus \bigcup_{j=0}^{l-1}\bigcup_{k=0}^{d-1} S_v^{\eta/d}\left(\frac{d^j\vartheta_1 + k}{d}\right) \longrightarrow \mathcal{X}_c,$$

$$z \mapsto \begin{cases} P_c(z) & : \quad z \in X_c^{\eta/d} \setminus \left(\bigcup_{j=0}^{l-1}\bigcup_{k=0}^{d-1} S_v^{\eta}(\frac{d^j\vartheta_1 + k}{d}) \cup \widetilde{\Sigma}_v^{\eta}\right) \\ \iota_0(P_c(z)) & : \quad z \in \widetilde{\Sigma}_v^{\eta} \\ \iota_{j+1}\iota_j^{-1}(z) & : \quad z \in W_j,\ 0 \leq j \leq l-2 \\ \iota_{l-1}^{-1}(z) & : \quad z \in W_{l-1} \end{cases}$$

By the identification properties above, this defines a continuous mapping which has to be extended continuously to the sectors $\bigcup_{j=0}^{l-1}\bigcup_{k=0}^{d-1} S_v^{\eta}(\frac{d^j\vartheta_1+k}{d})$. And here is where the opening modulus of these sectors plays an important role: to get a quasi-polynomial-like mapping

$$f_c : \mathcal{X}'_c \to \mathcal{X}_c,$$

the sector $S_v^{\eta/d}(\frac{d^j\vartheta_1+k}{d})$ have to be mapped quasiconformally to W_j such that the complex dilatation vanishes on the preimage of $K_c \cap W_j$: to do so we have to restrict to the parameters c which are not contained in the hyperbolic component H_{ql} immediately bifurcating from H_l. For all these parameters, there exists a sequence of pinching points in K_c converging to z_1 and Lemma 4.18 applies. Thus there are P_c^{ql}-invariant sectors

$$\widetilde{W}_j \subset W_j$$

centered at the orbit points $z_0, \ldots, z_{l-1}$ such that

$$\operatorname{mod}\left(\widetilde{W}_j\right) = \operatorname{mod}(S_{v/2}(\vartheta_1));$$

it may be necessary to enlarge v. After shrinking the potential η, we can assume that all the considered sectors do not intersect; we will have to modify η and v later again. Then $S_{v/2}^{\eta/d}(\frac{d^j\vartheta_1+k}{d})$ is a $\left(P_c^{-1} \circ P_c^{ql} \circ P_c\right)$-invariant sector; on this sector f_c is defined by the unique biholomorphic mapping to the P_c^{ql}-invariant sector $\widetilde{W}_j$ sending vertices to vertices. By Lemma 4.19, there is a quasiconformal extension

$$f_c : \bigcup_{j=0}^{l-1}\bigcup_{k=0}^{d-1}\left(S_v^{\eta/d}\left(\frac{d^j\vartheta_1 + k}{d}\right) \setminus S_{v/2}^{\eta/d}\left(\frac{d^j\vartheta_1 + k}{d}\right)\right) \longrightarrow \bigcup_{j=0}^{l-1}\left(W_j \setminus \widetilde{W}_j\right).$$

By construction, the mapping degree of f_c is d but it is not quasi-polynomial-like, since $\mathcal{X}_c \setminus \mathcal{X}'_c$ is not an annulus. So we have to modify these Riemann surfaces slightly (which means to shrink $\mathcal{X}_c$ and thus $\mathcal{X}'_c$ as well): to do so we first change the potential (with respect to the dynamics of P_c) bounding W_j for $0 \leq j \leq l-1$ from η to $\eta(1-j\epsilon)$ for $\epsilon > 0$ such that

$$d(1-l\epsilon) > 1.$$

But there has to be made one more modification: near all vertices where we have made our identifications we have to smoothen the Riemann surfaces. For simplicity of notation, we denote in the following the modified Riemann surfaces again by $\mathcal{X}'_c$ and $\mathcal{X}_c$.

Moreover, every f_c-orbit passes the non-holomorphy region of f_c at most once: this region is contained in

$$\bigcup_{j=0}^{l-1}\bigcup_{k=0}^{d-1} S_v^{\eta/d}\left(\frac{d^j\vartheta_1 + k}{d}\right)$$

and it is mapped to the union of certain sectors around the points $z_0, \ldots, z_{l-1}$. By construction, the map f_c is holomorphic on the orbit of all points within these sectors.

As usual one finds an f_c-invariant almost complex structure σ which is equal to the standard complex structure on $\mathcal{X}_c \setminus \mathcal{X}'_c$ and on the filled-in Julia set of f_c. By Theorem 3.4, this almost complex structure is mapped to the standard complex structure by a quasiconformal homeomorphism and conjugating f_c with this quasiconformal homeomorphism yields a unicritical polynomial-like mapping of degree d; by construction, the critical orbit is contained in $\mathcal{X}'_c$ and thus the small filled-in Julia set of f_c is connected. By Theorem 4.24, the polynomial-like mapping found before is hybrid equivalent to a uniquely determined polynomial $P_{c'}$ such that c and c' are contained in the same sector of $\mathcal{M}_d$. This defines a mapping

$$L_{1/q} \setminus H_{ql} \to L_{1/(q+1)} \setminus H_{(q+1)l}.$$

For the parameters $c \in \overline{H_{ql}}$, i.e. if c is contained in the closure of the hyperbolic component bifurcating immediately from H_l, the multiplier mapping defines a map

$$\overline{H_{ql}} \to \overline{H_{(q+1)l}},\ c \mapsto c'.$$

Altogether, this yields a mapping χ from the $1/q$-sublimb to part of the $1/(q+1)$-sublimb. It remains to prove that χ is actually continuous and injective. We denote by $L_{q,q+1}$ the subset of the parameters in the $1/(q+1)$-sublimb which is reached by the construction above.

5.3.2 Continuity and Bijectivity

Using arguments which are quite standard (compare Section 5.1 and [BD]) we get:

Lemma 5.25 (Continuity of χ on the Boundary)
The mapping $\chi : L_{1/q} \to L_{q,q+1}$ is continuous on the boundary and analytic on the hyperbolic components of $\mathcal{M}_d$.

□

Lemma 5.26 (Bijectivity of χ)
For $d = 2$, the mapping $\chi : L_{1/q} \to L_{q,q+1}$ is a homeomorphism.

Proof. To prove the bijectivity of χ we construct the inverse $\tilde{\chi}$ of this mapping as it is done in [BD]. We will see that it is much less difficult to remove a sector at a periodic point than to add one, since we do not have to consider any opening moduli.

Let $c \in L_{q,q+1}$. Starting from X_c^η we again construct new Riemann surfaces by cutting out certain sectors: let ϑ_1 and ϑ_2 be the angles of the parameter rays landing at the root of the $1/(q+1)$-sublimb in the i-th sector of H_l. For $j \in \{0, \dots, l-1\}$, we remove from $X_c^{\eta/d}$ and X_c^η the region between the dynamic rays with angles $d^j\vartheta_1$ and $d^j\vartheta_2$:

$$X_c^\eta \setminus \bigcup_{j=0}^{l-1} \Sigma_v^\eta(d^j\vartheta_1, d^j\vartheta_2) \quad \text{and}$$

$$X_c^{\eta/d} \setminus \bigcup_{j=0}^{l-1} \Sigma_v^\eta(d^j\vartheta_1, d^j\vartheta_2).$$

Identifying in both cases the boundaries equipotentially, we construct new Riemann surfaces

$$\mathfrak{X}_c' \subset \mathfrak{X}_c.$$

Let

$$f_c : \mathfrak{X}_c' \longrightarrow \mathfrak{X}_c$$

with

$$f_c(z) := P_c(z)$$

for

$$z \in \mathfrak{X}_c' \setminus \bigcup_{j=0}^{l-1} \bigcup_{k=1}^{d-1} \left(S_v^{\eta/d}\left(\frac{d^j\vartheta_1 + k}{d}\right) \cup \Sigma_v^{\eta/d}\left(\frac{d^j\vartheta_1 + k}{d}, \frac{d^j\vartheta_2 + k}{d}\right) \cup S_v^{\eta/d}\left(\frac{d^j\vartheta_2 + k}{d}\right)\right)$$

On the remaining sectors we have to find a quasiconformal continuation; since the considered sectors do not contain points of the Julia set of the new mapping, it is not necessary to have vanishing complex dilatation anywhere in these sectors. Moreover these sectors are only centered at strictly preperiodic points and thus Shishikura's principle holds, independently on the choice of the quasiconformal continuation. After changing the complex structure we get a polynomial-like mapping which itself is hybrid equivalent to a polynomial $P_{c'}$ such that c' is contained in the $1/q$-sublimb of the i-th sector of H_l. Then $\tilde{\chi}(c) := c'$. By construction, $P_{\chi\circ\tilde{\chi}(c)}$ and P_c are hybrid equivalent for $c \in L_{q,q+1}$; thus $\chi \circ \tilde{\chi} = \mathrm{id}|_{L_{q,q+1}}$. Similarly, $\tilde{\chi} \circ \chi = \mathrm{id}|_{L_q}$. □

Lemma 5.27 (Images of Dyadic Misiurewicz Points)
The dyadic Misiurewicz point(s) of lowest preperiod in $L_{1/q}$ is (are) mapped by χ to the dyadic Misiurewicz point(s) with lowest preperiod in $L_{1/(q+1)}$. □

6 Applications

In this section we present two applications of the homeomorphisms from Section 5. The first one is to construct veins of the Mandelbrot set; these are injective paths in $\mathcal{M}_d$ connecting the origin with certain antenna tips of $\mathcal{M}_d$. We will prove in Section 6.1 that every antenna tip of $\mathcal{M}_2$ can be connected by such an injective path in $\mathcal{M}_2$ with the origin.

One could try to generalize this result from the Mandelbrot set to the Multibrot sets. But there occur mainly two problems. The first one is that Theorem 5.2 is not known yet to be true for degree $d > 2$. The second problem, and this is much harder, is that we do not have a surgery for sectors: we need a homeomorphic embedding of every sector of a hyperbolic component (or part of it) into every other sector of this hyperbolic component (or part of it) such that the dyadic Misiurewicz point(s) of lowest preperiod is (are) mapped to the dyadic Misiurewicz point(s) of lowest preperiod.

In the introduction of this thesis we claimed that we will not use Yoccoz results on local connectivity to prove the existence of the veins in $\mathcal{M}_2$. But in the proof of continuity on non-hyperbolic components and in the proof of bijectivity of our mappings we have used his results. How does this fit together?
First we argue that we do not need to know anything about non-hyperbolic components: by Theorem 6.6 there is no parameter $c \in \mathbb{R}$ which belongs to a non-hyperbolic component of $\mathcal{M}_2$. Since all the mappings from Section 5 map the boundary of $\mathcal{M}_d$ to the boundary, hyperbolic components to hyperbolic ones and non-hyperbolic components to non-hyperbolic ones, we get independently on the result of Yoccoz (not necessary injective) paths in $\mathcal{M}_2$ between the origin and every dyadic Misiurewicz point; the continuity statement on queer components is thus not important for the existence of the veins.
Secondly, we do not really have to know that the constructed mappings are bijective: In [Mi, §17] one finds the following theorem:

Theorem 6.1 (Paths and Arcs)
Let X be a compact connected metric space and $x_1, x_2 \in X$. If there exists a path in X connecting x_1 and x_2, then there also exists an injective path (="arc") connecting these two points.

Thus every path in $\mathcal{M}_2$ connecting two points of the Mandelbrot set can be used to find an arc in $\mathcal{M}_2$ between these two points.

6.1 Veins in the Mandelbrot Set

Theorem 6.2 (Veins to Dyadic Misiurewicz Points)
For every dyadic Misiurewicz point $c \in \mathcal{M}_2$, there exists a path in $\mathcal{M}_2$ from 0 to c.

Proof. For the proof we use the following three statements:

(1) Let $H \subset \mathcal{M}_2$ be a hyperbolic component and let p, q, p' q' be positive integers such that $p < q$, $p' < q'$ and $\gcd(p, q) = 1 = \gcd(p', q')$. Moreover, denote by $c_{p/q}$ (respectively $c_{p'/q'}$) the unique dyadic Misiurewicz point in the p/q- (respectively p'/q'-) sublimb with smallest preperiod. Then there exists a connected set L in $\mathcal{M}_2$ containing the center of H and $c_{p/q}$ such that we have a bijective continuous mapping from L to a connected set in $\mathcal{M}_2$ containing the center of H and $c_{p'/q'}$. (Theorem 5.17 and Theorem 5.24).

(2) Consider a Misiurewicz point $c_0 \in \mathcal{M}_2$ which is the landing point of $q \geq 3$ parameter rays. Then the partial trees behind c_0 are pairwise dynamically homeomorphic; in particular, for $j \in \{1, \ldots, q\}$ the j-th partial tree behind c_0 contains the unique dyadic Misiurewicz point with smallest preperiod of the j-th branch behind c_0 (Theorem 5.2).

(3) Let c, $c' \in \mathcal{M}_2$ be two different dyadic Misiurewicz points. Then either there exists a hyperbolic component H or a Misiurewicz point $c'' \in \mathcal{M}_2$ such that 0, c and c' are contained in different components of $\mathcal{M}_2 \setminus \overline{H}$ or $\mathcal{M}_2 \setminus \{c''\}$ (this follows from [Schl2, Theorem 2.2]).

Using these statements we now construct an arc $\gamma \subset \mathcal{M}_2$ from the parameter 0 to any dyadic Misiurewicz point:

Step 1: We start with the path

$$\gamma_0 : [0, 1] \to \mathcal{M}_2,\ t \mapsto -2t.$$

Denote by c_1 the dyadic Misiurewicz point which is contained in the same limb as c such that the preperiod of c_1 is as small as possible. Using statement (1) from above, there is an arc γ_1 connecting 0 with c_1. We are done if $c_1 = c$, otherwise we continue with Step 2.

Step 2: Using Statement (3), there exists either a hyperbolic component H or a Misiurewicz point c' such that 0, c_1 and c are contained in different components of $\mathcal{M}_2 \setminus \overline{H}$ (respectively $\mathcal{M}_2 \setminus \{c'\}$). Denote by c_2 the dyadic Misiurewicz point of smallest preperiod behind H (respectively c').
If a hyperbolic component H separates 0, c_1 and c, we use Statement (1) to construct a new arc γ_2 from 0 to c_2.
If a Misiurewicz point c' separates 0, c_1 and c we use Statement (2) to construct a new arc γ_2 connecting 0 with c_2.
In the case $c = c_2$ we are done; otherwise we repeat Step 2 with c_2 instead of c_1.

By construction of the sequence c_1, c_2, ... of dyadic Misiurewicz points, the preperiod of c_{j+1} is not smaller than the preperiod of c_j. Thus the preperiods of the Misiurewicz points c_1, c_2, ... form a not necessarily strictly increasing sequence of integers which is bounded by the preperiod of c. Since all the points c_j are pairwise distinct by construction and since the number of dyadic Misiurewicz points of a fixed preperiod is finite, the sequence above is finite, i.e. there exists an integer n such that $c_n = c$. This proves the theorem. □

Corollary 6.3 (Veins through Hyperbolic Components)
For every hyperbolic component $H \subset \mathcal{M}_2$, there exists an arc in $\mathcal{M}_2$ from 0 to the center of H.

Proof. The root of H is the landing point of two periodic parameter rays with angles $\vartheta_1 < \vartheta_2$. Let Θ be the set of angles such that the corresponding parameter rays land at the dyadic Misiurewicz points in $\mathcal{M}_2$. Since Θ is dense in $\mathbb{S}^1$, there exists $\vartheta \in]\vartheta_1, \vartheta_2[\cap \Theta$. Thus there exist dyadic Misiurewicz points behind H. By Theorem 6.2, every dyadic Misiurewicz point behind H can be connected with 0 by an arc in $\mathcal{M}_2$; every such arc yields an arc in $\mathcal{M}_2$ connecting 0 with the center of H. □

By the last two statements, dyadic Misiurewicz points and hyperbolic components can be connected with the parameter 0 by an arc in $\mathcal{M}_2$. In [Schl1] and [Schl2] D. Schleicher has introduced the theory of fibers leading to another description of local connectivity of Julia sets and Multibrot sets. Using this concept we find more veins in the Mandelbrot set:

Corollary 6.4 (Veins to Parameters with Trivial Fiber)
For every parameter $c \in \mathcal{M}_2$ with trivial fiber, there exists an arc in $\mathcal{M}_2$ from 0 to c.

Proof. By definition of the fibers [Schl2, Definition 3.1, Definition 3.2], for every parameter $c \in \partial\mathcal{M}_2$ with trivial fiber, there exists a sequence $c_1, c_2, \dots$ of parameters in $\mathcal{M}_2$ such that

(1) every point c_j is the landing point of at least two periodic or preperiodic parameter rays and

(2) $\lim_{j\to\infty} c_j = c$.

Thus every point c_j is either a Misiurewicz point which disconnects the Mandelbrot set or it is the root of a hyperbolic component. Since $\{a/2^n : n \in \mathbb{N}, 1 \le a \le 2^n - 1\}$ is dense in $\mathbb{S}^1$, there exists a dyadic Misiurewicz point c'_j behind c_j. By Theorem 6.2, there exists an arc connecting 0 and c'_j. Since c_j separates 0 from c'_j and since the Mandelbrot set is full, the points c_{j-1} and c_j are contained in that arc; this yields an arc connecting c_{j-1} and c_j. Since $\lim_{j\to\infty} c_j = c$, connecting all the points $c_0 := 0, c_1, c_2, \dots$ gives an arc from 0 to c. □

6.2 Local Connectivity of Julia Sets

In the context of fibers one can see that all the homeomorphisms we have considered above preserve triviality of fibers and thus local connectivity of the Julia set (compare [Schl2, Corollary 5.1]):

Corollary 6.5 *The homeomorphisms χ from Theorem 5.1, Theorem 5.2, Theorem 5.17 and Theorem 5.24 preserve local connectivity of the Julia sets, i.e. for every parameter c in the domain of definition of χ the Julia set of the polynomial P_c is locally connected if and only if the Julia set of the polynomial $P_{\chi(c)}$ is locally connected.* □

This can be used to prove that the Julia sets of all polynomials on the arcs connecting 0 with any point of trivial fiber are locally connected. For the proof, we need the following statement from [LvS]:

Theorem 6.6 (Locally Connected Julia Sets)
If $d \geq 2$ is even and $c \in \mathbb{R}$ such that the Julia set J_c of $z \mapsto z^d + c$ is connected, then J_c is locally connected.

Since this theorem does not restrict to the case $d = 2$, we have found many veins within every Multibrot set $\mathcal{M}_d$ of even degree such that the Julia set of every polynomial on such a vein is locally connected: starting with the canonical vein on $\mathbb{R} \cap \mathcal{M}_d$ as in the quadratic case we find new veins by applying Theorem 5.2, Theorem 5.17 and Theorem 5.24. But for $d \geq 4$, we do not get veins to all the dyadic Misiurewicz points of $\mathcal{M}_d$ as discussed before. But for all the veins which can be constructed within the Multibrot sets by the method above, the parameters c on these arcs correspond to polynomials with locally connected Julia sets.

References

[Ahl] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand Mathematical Studies #10, 1966.

[BD] B. Branner and A. Douady, *Surgery on Complex Polynomials*, Proceedings of the Symposium on Dynamical Systems, Mexico, 1986, in: Lecture Notes in Math., **1345**, p. 11 - 72, Springer. 20, 21, 22, 25 4, 6, 23, 37, 45, 81, 110, 111

[BF] B. Branner, N. Fagella, *Homeomorphisms between Limbs of the Mandelbrot Set*, Journal of Geom. Anal., Vol 9, **1999**, p. 327-390. 81

[Bi] B. Bielefeld, *Changing the order of Critical Points of Polynomials using Quasiconformal Surgery*, Dissertation at Cornell University, 1989. 23

[Car] C. Carathéodory, *Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis*, Mathematische Annalen **78**. 7

[CG] L. Carleson, T.W. Gamelin, *Complex Dynamics*, Spinger, 1993. 28, 33

[D] A. Douady, *Descriptions of compact sets in* $\mathbb{C}$, in Topological Methods in Modern Mathematics, Proceedings of a Symposium in Honor of John Milnor's sixtieth birthday, 1993. 108

[DH1] A. Douady and J.H. Hubbard, *Etude dynamique des polynomes complexes, I, II*, Publ. Math. Orsay (1984, 1985). 33

[DH2] A. Douady and J.H. Hubbard, *On the Dynamics of Polynomial-like Mappings*, Ann. scient. Éc. Norm. Sup., 4^{e} série, **18** (1985), p. 287 - 343. 2, 45, 46, 47, 81

[Du] P. L. Duren, *Univalent Functions*, A Series of Comprehensive Studies in Mathematics 259, Springer, 1983. 7

[E] D. Eberlein, *Rational Parameter Rays of Multibrot Sets*, Diploma Thesis, TU München, 1999. 28, 33, 34

[EY] A. Epstein, M. Yampolsky, *Geography of the cubic connectedness locus I: Intertwining Surgery*, Ann. scient. Éc. Norm. Sup., 4^{e} série, **32** (1999), p. 151 - 185. 45

[GM] R. Goldberg, J. Milnor, *Fixed Points of Polynomial Maps, Part II, Fixed Point Portraits*, Ann. scient. Éc. Norm. Sup., 4^{e} série, **26** (1993), p. 51 - 98. 35

[J] W. Jung, *Homeomorphisms on Edges of the Mandelbrot Set*, in preparation. 6

[LS] E. Lau, D. Schleicher, *Internal Addresses in the Mandelbrot Set and Irreducibility of Polynomials*, IMS at Stony Brook Preprint 1994/19. 35, 58

[LvS] G. Levin, S.v. Strien, *Local Connectivity of the Julia Set of Real Polynomials*, Annales of Math., **147** (1998), p. 471 - 541. 114

[LV] Lehto, Virtanten, *Quasiconformal mappings in the plane*, Die Grundlehren der mathematischen Wissenschaften **126**, Springer Verlag 1973. 21

[Ly1] M. Lyubich, *Dynamics of Quadratic Polynomials, I-II*, Acta Math. **178** (1997), p.185-297.

[McM1] C.T. McMullen, *Complex Dynamics and Renormalization*, Annales of Mathematics Studies, 135, Princeton University Press, 1994. 28, 48, 49, 71, 72

[McM2] C.T. McMullen, *The Mandelbrot Set is Universal*, in: *The Mandelbrot Set, Theme and Variations*, Tan Lei ed., LMS Lecture Notes **274**, Cambridge University Press. 2

[Mi] J. Milnor, *Dynamics on One Complex Variable: Introductory Lectures*, 2^{nd} Edition, Vieweg Verlag 2000. 7, 8, 32, 49, 112

[MSS] R. Mañé, P. Sad, D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. Éc. Norm. Sup. Paris (4) **16** (1983), p. 193-217. 1

[PR] Ch. Pommerenke, B. Rodin, *Holomorphic Families of Riemann Mapping Functions*, J.Math.Kyoto Univ., 1986. 7, 9

[RS] J. Riedl, D. Schleicher, *On Crossed Renormalization of Quadratic Polynomials*, In: Proceedings of the 1997 conference on holomorphic dynamics, Kyoto (1998) 49

[Ro] B. Rodin, *Behaviour of the Riemann Mapping Function under Complex Analytic Deformations of the Domain*, Complex Variables, 1986, **5**. 7

[Schi] M. Schiffer, *Variation of the Green Function and Theory of the p-valued Functions*, Amer.J.Math. **65**, 1943, 341-360. 7

[Schl1] D. Schleicher, *On Fibers and Local Connectivity of Compact Sets in C*, IMS at Stony Brook Preprint 1998/12. 48, 114

[Schl2] D. Schleicher, *On Fibers and Local Connectivity of Mandelbrot and Multibrot Sets* and *On Fibers and Renormalization of Julia Sets and Multibrot Sets*, IMS at Stony Brook Preprint 1998/13. 48, 113, 114

[Schl3] D. Schleicher, *The Structure of the Mandelbrot Set*, manuscript.

[Schl4] D. Schleicher, *Rational Parameter Rays of the Mandelbrot Set*, in Astérisque **261**, 2000, p. 405-443. 33, 34