

Combinatorics, geometry and attractors of quasi-quadratic maps.

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February 1992, revised October 1992

Abstract. The Milnor problem on one-dimensional attractors is solved for S -unimodal maps with a non-degenerate critical point c . It provides us with a complete understanding of the possible limit behavior for Lebesgue almost every point. This theorem follows from a geometric study of the critical set $\omega(c)$ of a “non-renormalizable” map. It is proven that the scaling factors characterizing the geometry of this set go down to 0 at least exponentially. This resolves the problem of the non-linearity control in small scales. The proofs strongly involve ideas from renormalization theory and holomorphic dynamics.

§1. Introduction.

Let $f : [0, 1] \rightarrow [0, 1]$ be an S -unimodal map (see the definitions later) of the interval with a non-degenerate critical point c . Let us call such a map *quasi-quadratic*. As usual, $\omega(x)$ denotes the limit set of the forward orb (x) . The following theorem solves the Milnor problem [M1].

Theorem on the Measure-Theoretic Attractor. *Let f be a quasi-quadratic map normalized by the condition $f : c \mapsto 1 \mapsto 0$. Then there is a unique set A (a measure-theoretic attractor in the sense of Milnor) such that $A = \omega(x)$ for Lebesgue almost all $x \in [0, 1]$, and only one of the following three possibilities can occur:*

- 1 . A is a limit cycle;
- 2 . A is a cycle of intervals;
- 3 . A is a Feigenbaum-like attractor.

This result gives a clear picture of measurable dynamics for the maps under consideration. Let us explain the words used in the statement. A limit cycle is the periodic orbit whose basin of attraction has non-empty interior. A cycle of intervals is the union of finitely many intervals I_n with disjoint interiors cyclically interchanged by the dynamics. A Feigenbaum-like attractor is an invariant Cantor set of the following structure:

$$A = \bigcap \mathcal{O}_n,$$

where $\mathcal{O}_1 \supset \mathcal{O}_2 \supset \dots$ is a nested sequence of cycles of intervals of increasing periods.

It makes sense to compare the above Theorem with its topological counterpart known since late 70s:

Theorem on the Topological Attractor ([MT], [G], [JR], [vS]). *Let f be an S -unimodal map normalized by the condition $f : c \mapsto 1 \mapsto 0$. Then there is a unique set Λ (a topological attractor) such that $\Lambda = \omega(x)$ for a generic $x \in [0, 1]$, and only one of the following three possibilities can occur:*

* Supported in part by NSF grant DMS-8920768 and a Sloan Research Fellowship.

- (i). Λ is a limit cycle;
- (ii). Λ is a cycle of intervals;
- (iii). Λ is a Feigenbaum-like attractor.

From this point of view the Theorem on the Measure-Theoretic Attractor says that the map f has a unique measure-theoretic attractor A coinciding with the topological attractor Λ . In cases (i) and (iii) this was proven by Guckenheimer [G]. In case (ii) it is known that

$$\Lambda = \bigcup_{k=0}^{p-1} I_k$$

where I_i form a cycle of intervals of period p , and $f^{\circ p}| I_k$ is topologically exact. The last property means that for any interval $J \subset I_k$ there is an n such that $f^{\circ pn} J = I_k$. So, we can reduce the Theorem on the Measure-Theoretic Attractor to the following statement:

Theorem A. *Let f be a quasi-quadratic map. Assume that f is topologically exact on $[0, 1]$, hence $\Lambda = [0, 1]$. Then $\omega(x) = [0, 1]$ for Lebesgue almost all $x \in [0, 1]$, that is $A = [0, 1]$.*

This measure-theoretic result follows from geometric properties of the critical set $\omega(c)$. In order to study microstructure of this set we introduce the concept of a generalized renormalization as the rescaled first return map restricted to a neighborhood of the critical set. This moves us out of the class of unimodal maps to a class \mathcal{T} of maps with a single critical point but defined on the union of disjoint intervals mapped onto a bigger interval.[†] Any map with recurrent critical point becomes infinitely renormalizable in this sense.

Let us consider the sequence of renormalized maps

$$R^{\circ n} f : \bigcup_i I_i^n \rightarrow I_0^{n-1}.$$

As basic geometric characteristics of the critical set we consider the scaling factors $\mu_n = |I_0^n|/|I_0^{n-1}|$ and the Poincaré lengths of the gaps between the intervals I_i^n of level n . Our main geometric result is

Theorem B. *Let f be a topologically exact quasi-quadratic map with the recurrent critical point. Then the Poincaré lengths of the gaps of the renormalized maps go up to ∞ .*

Moreover, to a first approximation they increase at least linearly. Unfortunately, there is one unpleasant circumstance which can slow the rate down, namely the long cascades of “central returns” (when the map is combinatorially close to being renormalizable). However, the rate is still under explicit control. In particular, if we number these cascades by $\kappa \in \mathbf{N}$ then the rate will be at least linear in κ .

As it is clear from the very name of our field, the basic problem of non-linear dynamics is to gain control of non-linearity. In our situation the non-linearity in small scales is

[†] Up to some point we allow the domain to consist of infinitely many intervals. However, the renormalization philosophy becomes really valuable only in the finite case.

controlled by the above mentioned scaling factors. Theorem B implies that *the scaling factors go down at least exponentially in κ* . This provides us with a perfect control of non-linearity: the high order renormalized maps are becoming purely quadratic exponentially fast. Observe that this contrasts drastically with the Feigenbaum-like case when the geometry of the critical set is bounded from below (provided the combinatorics is bounded) which creates a definite amount of non-linearity in all scales (see Sullivan [S]).

Let us now dwell in more detail on the ideas of this work. §2 contains the combinatorial treatment of maps of class \mathcal{T} . According to the recurrent properties of the $\text{orb}(c)$ we split the analysis into two subcases: the *reluctantly* and the *persistently* recurrent. The latter case presents a stronger recurrence of the critical point. For example, in this case the return time of points of the $\text{orb}(c)$ back to a neighborhood $U \ni c$ is bounded. Surprisingly this helps to provide the further analysis since the renormalized maps turn out to be of finite type (that is, defined on the finitely many intervals). We show that such maps are classified by the cascades of renormalization types, and that these types can be combined independently. The last statement gives us a big freedom in producing examples. In particular, the so called Fibonacci map naturally arises as a map of the simplest stationary type. This map studied in [LM] was a basic model which clarified the situation.

§§3-6 are occupied with the proof of Theorem B. In §3 we prove it under the assumption that we start with a small enough scaling factor. To this end we introduce a modification of the Poincaré lengths, the “asymmetric Poincaré lengths”, which behave more regularly under renormalization. We show that the asymmetric Poincaré lengths of the gaps increase almost monotonically. Though this is the longest technical piece of the paper, we are actually in a quite comfortable position since the starting condition provides us at once with a perfect control of non-linearity.

In order to get rid of the starting assumption we pass to the complex plane in the class of (*generalized*) *polynomial-like maps*, a complex counterpart of maps of class \mathcal{T} (§6). These maps are defined on the finite union of disjoint topological disks mapped onto a bigger disk. The crucial fact is that all such maps with the same combinatorics are quasi-conformally equivalent which yields quasi-symmetric equivalence on the real line. This part relies on recent developments in holomorphic dynamics, particularly on the work of Branner and Hubbard [BH]. Since the conclusion of Theorem B is quasi-symmetrically invariant, it is enough to produce just one example of a polynomial-like map f with a given combinatorics and arbitrarily small starting scaling factor. Exactly at this point it is important to have a freedom gained by passing to the class of *generalized* polynomial-like maps and its real counterpart, class \mathcal{T} .

A bridge between quadratic maps and generalized polynomial-like maps is given by the renormalization on a Yoccoz puzzle-piece introduced in [L2], a complex analogue of the renormalization mentioned above. This passage completes the proof of Theorem B for quadratic polynomials (see the end of §6). The argument for general quasi-quadratic maps is based upon two *Sullivan’s Principles* [S]:

- 1) *Renormalizations of sufficiently smooth maps are becoming real analytic;*
- 2) *High order renormalization of a real analytic map makes it polynomial-like.*

These principles turn out to be efficient in our setting of generalized renormalization. The

First Principle works when we have combinatorics of bounded type, that is the number of intervals I_i^n is bounded on all levels. To push it forward we need a priori distortion bounds obtained by Martens [Ma]. As to the case of unbounded combinatorics (as well as the non-minimal case), it is actually easier, and can be treated by a purely real argument. We discuss these issues in §4.

In §5 we push the Second Principle forward. In order to construct a polynomial-like map we consider the complex pull-backs of Euclidian disks based upon the real intervals, and estimate their sizes (“complex bounds”). This is another technical piece of the work.

Finally, in the last §7 we derive Theorem A from Theorem B by showing that the set $X = \{x : \omega(x) \ni c\}$ has zero lower density at c . This is our first application of geometric Theorem B but we expect a lot of others.

Remarks. 1. Since Milnor’s paper of 1985 there has been a good deal of effort to attack the problem of attractors, see [BL1-3], [GJ], [K], [Ma], [JS]. In particular, it was already known that there is only one measure-theoretic attractor (see [BL1-3] or [GJ], [K]). The reluctantly recurrent case was resolved in [BL3] and [GJ].

The geometric study of “non-renormalizable maps” was started in [GJ], [Ma] and [JS]. In [JS] the absence of Cantor attractors was proved for topologically exact maps sufficiently close to the Chebyshev polynomial $x \mapsto 4x(1 - x)$.

In [HK] the Fibonacci map was suggested as a candidate for a “wild” situation when the measure-theoretic and topological attractors are different. The paper [LM] showed that it is not the case for quasi-quadratic maps (an alternative purely real argument has been recently found in [KN]). However, a computer experiment carried out jointly with F.Tangerman indicates that this may well be true in higher degrees.

For a survey on the problem of attractors see [L3].

2. The complex counterpart of Theorem A says that the Lebesgue measure of the Julia set of a “non-tunable” quadratic polynomial is equal to 0 (Lyubich [L2] and Shishikura (unpublished)). The case of cubic polynomials with Cantor Julia set had been earlier treated by McMullen (see [BH]). The real result turns out to be harder than the complex one because conformal invariants (like annulus moduli or Poincaré metric) have only quasi-invariant analogues in the real setting. However, Theorem B yields that on deep levels they are becoming invariant exponentially fast with respect to pull-backs. This makes the real-complex dictionary much more precise.

3. There are two pieces of the paper which strongly depend on the quadratic-like nature of the critical point. First, the growth of the Poincaré lengths of the gaps relies on the fact that the square root map divides the Poincaré length at most by 2 (Lemma 3.8). (By the way, this is the place where the asymmetric Poincaré length comes to the scene). The second place is the Branner-Hubbard divergence property (§6).

Some definitions, notations and conventions. A continuous map $f : [0, 1] \rightarrow [0, 1]$ is called *unimodal* if it has only one extremum c . It is called *S-unimodal* if additionally it is three times differentiable with only critical point c and with negative Schwarzian

derivative outside c :

$$Sf = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 < 0.$$

If additionally the critical point c is non-degenerate, that is $f''(c) \neq 0$, then the map is called *quasi-quadratic*.

Let $\phi(x) = (x - c)^2$. We use the abbreviations qs and qc for “quasi-symmetric” and “quasi-conformal” respectively.

Saying “an interval” we mean a “closed interval”. The notation $I = [a, b]$ means that a and b are the endpoints of I but does not necessarily mean that $a < b$. Points a and a' are called “ c -symmetric” if $fa = fa'$. So, we can also talk about c -symmetric sets.

The n -fold iterate of a map g is denoted by g^{o_n} . The $\text{orb}(x) \equiv g$ - $\text{orb}(x)$ denotes $\{g^{o_m}x\}_{m=0}^\infty$. Also $\text{orb}_n(x) = \{g^{o_m}x\}_{m=0}^n$ denotes the initial piece of the orbit. Let $\mathbf{N} = \{0, 1, 2, \dots\}$.

We recommend the forthcoming book of de Melo and van Strien [MS] for the background in one-dimensional dynamics.

Acknowledgement. I’d like to thank Marco Martens for useful discussions, and for reading parts of the manuscript. I owe to John Milnor several nice suggestions which improved the exposition. I also take this opportunity to thank IHES and IMPA for their hospitality while I was doing parts of this work.

§2. Renormalization.

Renormalization in dynamical systems means the first return map to an appropriate piece of the phase space and then rescaling of this piece to the “original size”. For example, if we have a periodic c -symmetric interval $I \subset [0, 1]$ of period $p > 1$, we can consider the unimodal map $f^p|_I$ and then rescale I back to $[0, 1]$. This is the usual notion of renormalization in one-dimensional dynamics. So, we can talk about renormalizable and non-renormalizable maps. Repeating this procedure we can further talk about “twice renormalizable”, “thrice renormalizable”, ..., or at the end “infinitely renormalizable” maps. For example, looking through the Theorem on the Topological Attractor we see that case (ii) corresponds to at most finitely renormalizable maps, while the last case (iii) corresponds to infinitely renormalizable maps. The case we concentrated on, when the map f is topologically exact on $[fc, f^2c]$, is equivalent to being non-renormalizable.

However, this specific terminology highly accepted in one-dimensional dynamics is somewhat misleading. Indeed, we will see that the most interesting “non-renormalizable” maps can actually be treated as *infinitely renormalizable* in an appropriate sense, and this gives an efficient tool for studying the geometry of the critical set. This renormalization does not respect the class of unimodal maps. So, let us describe an appropriate class of maps.

Class \mathcal{T} . Let $q, p \in [0, \infty]$, and let I_k , $k = -q, \dots, p$, be a finite family of disjoint intervals compactly contained in a *base interval* J . The interval I_0 is marked, and will be called

central. Let us consider a map

$$g : \text{Dom}(g) = \bigcup_{k=-q}^p I_k \rightarrow J \quad (2-1)$$

with a single turning point $c \in I_0$ (we will still call it “critical”), which also satisfies the following property: $g(\partial I_k) \subset \partial J$. In particular, g maps each non-central interval I_k , $k \neq 0$, homeomorphically onto the whole interval J , while the central interval I_0 is c -symmetric.

Let us also assume that g does not have non-trivial wandering intervals. This condition holds automatically under some regularity assumptions, e.g., g has negative Schwarzian derivative, and the critical point c is non-flat (see [G] or [L1]).

Denote this class of maps by \mathcal{T}^∞ . Let us say that $g \in \mathcal{T}^\infty$ is of *finite type* if its domain consists of only finitely many intervals, that is $p, q \in \mathbf{N}$. Let \mathcal{T} denote the subclass of maps of finite type. Unimodal maps can also be viewed as maps of class \mathcal{T} whose domain contain a single interval.

A point x is said to be *non-escaping* if $g^{\circ n}x \in \text{Dom}(g)$ for all $n = 1, 2, \dots$. Denote by $K(g)$ the set of non-escaping points (the “filled-in Julia set” of g).

Pull-Backs and nice intervals. The following *pull-back construction* plays an essential role in what follows. Given an interval T and a point x such that $g^{\circ n}x \in \text{int}T$, we can pull T back along the orbit $\{g^{\circ k}x\}_{k=0}^n$. This means that we inductively construct a sequence of intervals

$$T_n \equiv T \ni g^{\circ n}x, \quad T_{n-1} \ni g^{\circ(n-1)}x, \quad \dots, \quad T_0 \ni x$$

so that T_k is the maximal interval containing $g^k x$ whose image is contained in T_{k+1} . An interval T_k of the pull-back is called *critical* if $\text{int}T_k \ni c$. Since $g(\partial I_k) \subset \partial J$,

$$g(\partial T_k) \subset \partial T_{k+1}, \quad k = 0, 1, \dots, n-1. \quad (2-2)$$

Hence the non-critical intervals T_k diffeomorphically map onto T_{k+1} , and the critical intervals T_k are c -symmetric, $k = 0, 1, \dots, n-1$. A pull-back is called *monotone* if all intervals T_k except perhaps T are non-critical. It is called *unimodal* if T_0 is critical while T_1, \dots, T_{n-1} are non-critical.

As in [Ma] let us call a c -symmetric interval T *nice* if

$$g^{\circ n}(\partial T) \cap T = \emptyset, \quad n = 1, 2, \dots \quad (2-3)$$

For example, let α be a periodic point, and let β be its n -fold preimage, β' be a c -symmetric point. Then the interval $T = [\beta, \beta']$ is nice provided $\beta \notin \text{orb}(\alpha)$. It follows that there are nice intervals in any neighborhood of c (provided g has no limit cycles).

Let us denote by $\mathcal{M} \equiv \mathcal{M}_T$ the family of intervals obtained by pulling a nice interval T back along all possible orbits (\mathcal{M} comes from “Markov”). Following the analogy with the holomorphic setting (compare [H], [M2] or §6), the intervals of \mathcal{M} will also be called *puzzle-pieces*. We use the notation $T^{(n)}(x)$ for the puzzle-piece obtained by pulling T along $\text{orb}_n(x)$, and call n the *depth* of the puzzle-piece. (If we start with another interval,

say, J then, of course, we use the notations $J^{(n)}(x)$ for the corresponding puzzle-pieces). The basic properties of the family \mathcal{M} are:

- (i) any two intervals of \mathcal{M} are either disjoint or “strongly nested”; in the latter case the interval of higher depth is contained in the interior of the other one;
- (ii) any non-critical puzzle-piece $T^{(n)}(x)$ diffeomorphically maps onto $T^{(n-1)}(fx)$.
- (iii) For any critical puzzle-piece $T^{(n)}(x)$ of depth > 0

$$f(T^{(n)}(x), \partial T^{(n)}(x)) \subset (T^{(n-1)}(fx), T^{(n-1)}(fx)).$$

All critical puzzle-pieces are c -symmetric.

- (iv) If c is recurrent then for any puzzle-piece $T^{(n)}(x)$, the orb(c) does not cross $\partial T^{(n)}(x)$.

Lemma 2.1. *Let $K \in \mathcal{M}$ be a critical puzzle-piece, and x be a point whose orbit crosses $\text{int} K$. Take the first moment $n > 0$ for which $g^{on}x \in \text{int} K$. Then the pull-back of K along orb(x) is either monotone or unimodal.*

Proof. Let $K = K_m, K_{m-1}, \dots, K_0 \ni x$ be the above pull-back. Then the puzzle-pieces K_{m-1}, \dots, K_0 have higher depths than K . It follows that they are disjoint from K and hence non-critical. \square

In particular, if $x \notin K$ then the pull-back is monotone. If $x = c$ is the critical point then the pull-back is certainly unimodal.

Let $K, L \in \mathcal{M}$ be two critical intervals, $L \subset K$. The interval L is called a *kid* of K if it is obtained by a *unimodal* pull-back along a piece of the critical orbit $\{g^{ok}c\}_{k=0}^n, g^nc \in K$. The kid corresponding to the first return of the critical point back to K will be called the *first kid* K^1 . Lemma 2.1 can be improved in the following way.

Lemma 2.2. *Let $K \in \mathcal{M}$ be a critical puzzle-piece, L be its first kid, and x be a point whose orbit crosses $\text{int} L$. Take the first moment $n > 0$ for which $g^{on}x \in \text{int} L$. Then the pull-back of K along orb(x) is monotone if $x \notin L$, and unimodal otherwise.*

Proof. Consider the subsequent return moments $0 < n_1 < n_2 < \dots < n_l = n$ of orb(x) to K until the first meeting with L . According to the previous lemma, the first return produces a unimodal or monotone pull-back of K depending on whether $x \in L$ or not. All further returns produce monotone pull-backs of K . This is what was required. \square

Sequence of first grandkids. Cascades of central returns. Let $I^0 \equiv J$ be the base interval, $I^1 \equiv I_0$ be its first kid, I^2 be the first kid of I^1 etc. In such a way we construct a nested sequence of the “first grandkids”

$$I^0 \supset I^1 \supset I^2 \supset \dots$$

which will play fundamental role in what follows. Let $t(n)$ be the first return time of the critical point back to I^{n-1} . We say that the return to level $n-1$ is *central* if $g^{ot(n)}c \in I^n$. Otherwise the return is classified as *high* or *low* depending on whether $g^{ot(n)}I^n \supset I^n$ or $g^{ot(n)}I^n \cap I^n = \emptyset$ (compare [GJ]).

Denote by $\mathcal{L} \subset \mathbf{N}$ the sequence of all levels m such that $g^{ot(m)}c \in I^{m-1} \setminus I^m$ which means that the return to the level $m-1$ is non-central. Let $\kappa : \mathbf{N} \rightarrow \mathbf{N}$ be the monotone

surjective map such that $\kappa(m+1) = \kappa(m)+1$ for $m \in \mathcal{L}$ and $\kappa(m+1) = \kappa(m)$ otherwise. The series of levels with the same κ will be called *the cascades of central returns*. (The “cascade” can degenerate to a single level if the central return does not actually occur). So, κ numbers subsequently the cascades of central returns.

An important issue which we will discuss next is whether the sequence of grandkids shrink down to the critical point. (The main concern of this work will be the *rate of shrinking*).

Unimodal renormalization. Let us say that $g \in \mathcal{T}^\infty$ admits a unimodal renormalization if there exists a c -symmetric periodic interval. Then the return map to this interval is unimodal which justifies the terminology. It will be convenient to consider unimodal maps as “admitting unimodal renormalizations”. So, when we say that a map $g \in \mathcal{T}^\infty$ does not admit unimodal renormalizations, we assume automatically that it is not unimodal itself. (The standard meaning of a renormalizable unimodal map corresponds to admitting a unimodal renormalization with a period > 1 .)

Lemma 2.3. *The following properties are equivalent:*

- (i) g does not admit a unimodal renormalization.
- (ii) The sequence of levels with non-central returns is infinite.
- (iii) The grandkids I^n shrink to the critical point.
- (iv) The set $K(g)$ of non-escaping points has empty interior.

Proof. Let $t(n)$ be as above the return time of c back to I^{n-1} . Clearly, $t(n)$ is monotonically increasing.

(i) \Rightarrow (ii). Assume that the returns to all levels $m \geq n - 1$ are central. Then $\bigcap_{m \geq n} I^m$ is a c -symmetric g^{ol} -invariant interval. Contradiction.

(ii) \Rightarrow (iii). If $t(m) = t$, $m \geq n$, eventually stabilizes then the returns to all levels $\geq n - 1$ should be central contradicting (ii). If $t(n) \rightarrow \infty$ but I^n don't shrink to the critical point then the intersection $\bigcap I^n$ is a wandering interval.

(iii) \Rightarrow (iv). Since an appropriate iterate of I^n covers the whole base interval $I^0 = J$, there are escaping points in I^n . Since I^n shrink down to the critical point, there are escaping points in any neighborhood of c . But an appropriate iterate of any other interval must cover the critical point (no homtervals). Hence, escaping points are dense.

(iv) \Rightarrow (i). Any periodic interval is contained in the filled-in Julia set. \square

The condition (iv) can also be stated in the following way:

$$\text{diam}(J^{(n)}(x)) \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{2 - 4}$$

uniformly in x . If g is of finite type then it follows that $K(g)$ is a Cantor set.

Remarks. 1. One can also easily see that a map $g \in \mathcal{T}$ admits a unimodal renormalization if and only if there is a an interval I^{n-1} such that the critical point does not escape its kid I^n under iterates of $f^{ot(n)}$. In the complex setting it will be accepted as the main definition.

2. One can also state and prove *the Two kids Lemma* by the same argument as in the complex setting (see §6). However, we don't need it for the real discussion.

Persistent and reluctant recurrence.

Let $B(x, \epsilon)$ denote the interval centered at x of radius ϵ .

Assume c is recurrent. It is called *reluctantly recurrent* if there exist an $\epsilon > 0$ and an arbitrary long backward orbit $\bar{x} = \{x, x_{-1}, \dots, x_{-l}\}$ in $\omega(c)$ such that the $B(x, \epsilon)$ allows a monotone pull-back along \bar{x} . Otherwise c is called *persistently recurrent*.

Lemma 2.4. *Assume g does not admit unimodal renormalizations. Then the following two properties are equivalent:*

- (R1) c is reluctantly recurrent;
- (R2) There is a critical puzzle-piece $I^{(m)}(c)$ with infinitely many kids.

Proof. (R1) \Rightarrow (R2). By (2-4) we can find a puzzle-piece $J^{(n)}(x) \subset B(x, \epsilon)$. It is certainly a monotone pull-back of some critical puzzle-piece $J^{(m)}(c)$.

Let us pull $J^{(n)}(x)$ to depth $n+l$ along the backward orbit \bar{x} . Then let us consider the first moment when $\text{orb}(c)$ crosses $J^{(n+l)}(x_{-l})$ and pull this puzzle-piece further to the critical point. We get a kid of $J^{(m)}(c)$. Letting $l \rightarrow \infty$ we obtain infinitely many kids.

(R2) \Rightarrow (R1). Consider the pull-backs of $J^{(m)}(c)$ along the backward orbits of c creating the kids. \square

Remark. Persistently recurrent situation appeared in [BL3, Lemma 11.1] and [GJ] under different names. In the complex setting it was introduced in [Y] under the name we use here. The term “reluctantly recurrent” was suggested by McMullen.

An invariant set K is called *minimal* if the orbits of all points $x \in K$ are dense in K .

Lemma 2.5 (see [BL3], [Ma]). *In the persistently recurrent case the critical set $\omega(c)$ is a minimal Cantor set.*

Proof. Assume there exists an $x \in \omega(c)$ whose orbit does not accumulate on c . Then for sufficiently small $\epsilon > 0$ the pull-backs of $B(g^{\circ n}x, \epsilon)$ along the $\text{orb}_n(x)$ are monotone. The contradiction proves that $\omega(c)$ is minimal. In particular, $\omega(c)$ does not contain periodic points. Hence it is a Cantor set. \square

The minimality property yields that for any relative neighborhood $U \subset \omega(c)$ all points $x \in U$ eventually return back to U , and, moreover, the return time is bounded (but certainly depends on U). So, on $\omega(c)$ the return map is *of finite type*: it is defined on the finitely many intervals. This motivates the following consideration.

Return maps of finite type. Take a nice interval $L \ni c$. Let $K_0 \ni c$ be its first kid. Select also finitely many other pairwise disjoint pull-backs $K_i \subset L$, $-n \leq i \leq m$ corresponding to the first returns of some points back to L . Then we can define the first return map of class \mathcal{T} :

$$h : \bigcup_{j=-n}^m K_j \rightarrow L. \tag{2-5}$$

In the most interesting case when $L = I_0$ the combinatorial type of h can be described in terms of the g -itineraries of K_i through the intervals I_k of the previous level. To this end let us prepare a bit of algebraic language.

Spin semigroups. Let Γ be a semi-group. We call Γ a *spin semi-group* if it is supplied with a character $\epsilon : \Gamma \rightarrow \mathbf{Z}^2$. If Γ is free then the spin structure can be certainly prescribed arbitrary on the generators, and uniquely determined by this data.

Let $p, q \in \mathbf{N}$, and let us consider a free semi-group

$$\Gamma = \langle I_{-q}, \dots, I_0, \dots, I_p \rangle$$

with $p + q + 1$ generators which are linearly ordered with a marked generator I_0 . If $\epsilon(I_0) > 0$ we say that Γ is *positively oriented*. Otherwise Γ is *negatively oriented*.

Let $W(\Gamma) \subset \Gamma$ be the set of words $\gamma = I_{k(0)} \dots I_{k(s)}$ such that the last symbol is I_0 , while all others are different from I_0 . Let us order $W(\Gamma)$ as follows (compare [MT]). Let $\gamma_t = I_{k(0)} \dots I_{k(t)}$ be the initial part of γ , $t = 0, \dots, s$. Assign to γ an integer vector with components $\epsilon(\gamma_{t-1})k(t)$, $t = 1, \dots, s$. The ordering of Γ is induced by the lexicographic ordering of the corresponding vectors. Note that any two words of $W(\Gamma)$ are comparable since they end with the I_0 -symbol.

So, the object we have described should be called a *free ordered spin semigroup*. However, we will usually call it just “spin semigroup” keeping in mind the other structures.

Let (Γ', ϵ') be another spin semigroup of the same type with generators $I'_{-q'}, \dots, I'_0, \dots, I'_{p'}$, and let $\chi : (\Gamma', \epsilon') \rightarrow (\Gamma, \epsilon)$ be a semigroup homomorphism. Let us call it *unimodal* if $\chi(I_k) \in W(\Gamma)$ and χ is unimodal on the set of generators, that is, it is strictly monotone on the sets $I'_{-q'}, \dots, I'_0$ and $I'_0, \dots, I'_{p'}$, and has an extremum at I_0 . Moreover, we require I_0 to be the minimum or maximum depending on whether Γ is positively or negatively oriented.

A unimodal homomorphism χ is called *admissible* if additionally

$$\epsilon'(K_j) = \text{sgn} j \epsilon(\chi(K_j)). \quad (2-6)$$

Return type. Take a map $g \in \mathcal{T}$ as in (2-1) of finite type. Let the numbering of the intervals I_k be consistent with the line order. Consider a free group Γ_g generated by I_k . Let us supply Γ_g with a spin structure ϵ_g . Spin of a non-central interval I_k is equal to +1 or -1 depending on whether $g|I_k$ preserves or reverses orientation. The spin of the central interval I_0 is equal to +1 or -1 depending on whether c is the minimum or maximum point.

Assume now that the critical point returns to I_0 . Then we can consider a return map (2-5):

$$h : \bigcup_{j=-n}^m K_j \rightarrow I_0$$

of class \mathcal{T} . Let (G_h, ϵ_h) be an associated spin semigroup, and let us define a homomorphism

$$\chi : (\Gamma_h, \epsilon_h) \rightarrow (\Gamma_g, \epsilon_g)$$

assigning to each interval $K = K_j$ its itinerary through the intervals of the previous level

$$\chi(K) = I_{k(1)} \dots I_{k(s-1)} I_0 \in W(\Gamma_g) \quad (2-7)$$

until the first return back to I_0 , that is, $g^j K \subset I_{k(j)}$ with $k(j) \neq 0$ for $j = 1, \dots, s-1$.

Let us call the itinerary $\chi(K_0) \in W(\Gamma_g)$ of the central interval the *kneading sequence* of the return map.

We call the homomorphism χ the *type of the return map*. Observe that χ is admissible. Property (2-6) expresses the chain rule for the orientation type of the composition. The unimodal property of χ reflects the fact that the central branch of g is unimodal.

Denote by $\mathcal{T}(q, p, \epsilon)$ the subclass of \mathcal{T} with specified $q, p \in \mathbf{N}$ and spin character ϵ . Let $g^t \in \mathcal{T}(q, p, \epsilon)$, $t_0 \leq t \leq t_1$, be a continuous (in C^0 topology) one-parameter family of maps. This means that the intervals $I_k \equiv I_k(t)$ move continuously with t , and each rescaled branch $g_t|_{I_k(t)} \circ \psi_k^t$ depends continuously on t (here $\psi_k^t : [0, 1] \rightarrow I_k(t)$ is the orientation preserving rescaling). Let us call the family g^t *full* if $g^{t_0}I_0(t_0)$ covers all intervals $I_k(t_0)$ while $g^{t_1}I_0(t_1)$ does not intersect the interiors of $I_k(t_1)$ (or vice versa).

Lemma 2.6. *Let $g^t \in \mathcal{T}(q, p, \epsilon)$, $0 \leq t \leq 1$, be a full one-parameter family. Let $\chi : \Gamma' \rightarrow \Gamma_g$ be an admissible homomorphism. Then for some parameter value $t \in (0, 1)$ there is a return map $h^t \in \mathcal{T}$ whose return type is equal to χ . Moreover, there is a full family of return maps h^t , $t_0 \leq t \leq t_1$, with the same property.*

Proof. Without loss of generality we can assume that the critical point is minimum for all g^t , that is, Γ_g is positively oriented. In what follows we omit t in notations keeping in mind that everything actually depends on it. For every t let us consider the intervals L_j with itineraries $\chi(K_j)$. Such intervals exist (because outside the central interval g acts as the Bernoulli scheme), and continuously depend on t . Moreover, $g^{\circ(l_j-1)}L_j = I_0$ where l_j is the length of the itinerary $\chi(K_j)$.

Since $g = g^t$ is the full family, there is a parameter interval $T = [t_0, t_1]$ such that the critical value gc runs through the interval L_0 from one boundary point to another as t runs through T . Since χ is unimodal (and Γ_g is positively oriented), L_0 lies on the left of all other intervals L_j . Hence for $t \in T$ we have $gI_0 \supset L_i$, $i \neq 0$.

Let us now consider the pull-back $K_j \subset I_0$ of the intervals L_j by the central branch in such a way that K_j lies on the right of c if and only if $j > 0$. Then the first return map h on $\cup K_j$ has type χ . Moreover, as t runs through T , the critical value hc runs all way through I^0 , so that we have a full family. \square

Let $g = g_1 \in \mathcal{T}$. Assume that we can subsequently construct a sequence of return maps of class \mathcal{T}

$$g_n : \cup I_k^n \rightarrow I_0^{n-1}$$

where $I_0^0 \supset I_0^1 \supset \dots$ is a sequence of the first kids. Let χ_n , $n = 1, 2, \dots$, be the corresponding sequence of return types. The following statement says that these types can be combined independently. (“Do whatever you want”.)

Lemma 2.7. *Let $g^t \in \mathcal{T}(q, p, \epsilon)$, $0 \leq t \leq 1$, be a full one-parameter family, $\Gamma_0 \equiv \Gamma_g$. Let $\chi_n : \Gamma_n \rightarrow \Gamma_{n-1}$, $n = 1, 2, \dots$ be any sequence of admissible homomorphisms. Then there is a map $g = g^t$ which admits a sequence of return maps g_n of type χ_n .*

Proof. By the previous lemma we can subsequently construct a nested sequence of parameter intervals $T_1 \supset T_2 \supset \dots$ in such a way that the return types to T_k are required up to the level n , while the return maps g_{n+1}^t , $t \in T_n$, to the n th level form a full family. Intersecting these intervals we get a required parameter value. \square

In particular, we can fix the intervals I_k and consider the “standard” family of maps which are linear on the non-central intervals (and hence don’t depend on parameter) and quadratic on the central interval. Then by changing this central branch we can obtain a map of any type $\{\chi_n\}$.

The return graph. Let us consider a sequence

$$\begin{array}{ccccccc} \dots & \rightarrow & \Gamma_2 & \rightarrow & \Gamma_1 & \rightarrow & \Gamma_0 \\ & & \chi_3 & & \chi_2 & & \chi_1 \end{array}$$

of admissible homomorphisms of spin semigroups. We can associate to it the following graded graph Δ . Put on the n th level of Δ the generators $\{I_k^n\}$ of Γ_n . Connect a vertex I_k^n with a vertex I_j^{n-1} of the previous level by l edges if I_j^{n-1} has multiplicity l in the word $\chi_n(I_k^n)$. This graph contains the full “abelian” information about the sequence of homomorphisms.

Observe that each vertex of level n is connected by a simple edge with the central vertex I_0^{n-1} of the previous level. Let us call the graph (and the sequence $\{\chi_n\}$) *irreducible* if for any vertex I_k^n there is a path down to a central vertex I_0^{n+s} (such that we go strictly downstairs along this path).

Let us consider now a sequence of return maps g_n . Its type is given by a sequence of homomorphisms χ_n which provides us with a graph Δ . A vertex I_k^n is connected with I_j^{n-1} by l edges if the g_{n-1} -orbit of I_k^n passes through I_j^{n-1} l times before the first return back to the central interval I_0^{n-1} . (Such a graph was introduced by Marco Martens.) The following statement is immediate:

Lemma 2.8. *The graph Δ is irreducible if and only if the critical orb(c) crosses all intervals I_k^n . \square*

Realization of all types. Let $g \in \mathcal{T}$ have a recurrent critical point such that $f|_{\omega(c)}$ is minimal. Assume also that g does not admit a unimodal renormalization. Denote this class of maps by \mathcal{T}_{\min} . Let us consider the nested sequence of the first grandkids $I^0 \supset I^1 \supset \dots$. Let us define a sequence of return maps

$$g_n : \cup I_k^n \rightarrow I^{n-1}, \quad I^n \equiv I_0^n$$

in such a way that we select only those intervals I_k^n which cover the critical set. The number of these intervals is finite since $\omega(c)$ is minimal. We call g_n a *pre-renormalization* of g_{n-1} . So, we obtain a sequence $\bar{\chi}_g = \{\chi_n\}$ of admissible homomorphisms with an irreducible graph Δ_g . Let us call this sequence a *type* of g .

Lemma 2.9. *Let $g^t \in \mathcal{T}(q, p, \epsilon)$, $0 \leq t \leq 1$ be a full one-parameter family, $\Gamma_0 \equiv \Gamma_g$. Let $\bar{\chi} = \{\chi_n : \Gamma_n \rightarrow \Gamma_{n-1}\}_{n=1}^\infty$ be any irreducible sequence of admissible homomorphisms such that $\chi_n(I^n) \neq (I^{n-1})$ for infinitely many n . Then there is a map $g^t \in \mathcal{T}_{\min}$ of type $\bar{\chi}$.*

Proof. By Lemma 2.7 we know that there is a map g^t which admits a sequence of return maps of type χ_n . Since $\bar{\chi}$ is irreducible, this is a sequence of pre-renormalized maps (by Lemma 2.8). Hence $\bar{\chi}_g = \bar{\chi}$.

This map does not admit a unimodal renormalization since $\chi_n(I^n) \neq (I^{n-1})$ for infinitely many n . By Lemma 2.3 $\text{diam}(I^n) \rightarrow 0$. Since the critical point returns to all central intervals I^n , it is recurrent. In order to see the minimality property, let us consider the union of $\text{orb}(I_k^n)$ until their first return back to I^{n-1} . This provides us with a finite covering of $\omega(c)$ by the pull-backs of I^{n-1} . It follows that the orbit of any point $x \in \omega(c)$ crosses I^{n-1} and hence $\omega(x) = \omega(c)$. \square

Combinatorial model of the critical set. Let Γ_*^m be a sub-semigroup of Γ^m generated by non-central intervals I_k , $k \neq 0$. Consider a space Ω of all finite and infinite sequences $\bar{\delta} = (\delta_1, \delta_2, \dots)$ such that $\delta_n \in \Gamma_*^{l(n)}$ with a *strictly increasing* sequence of levels $l(n)$. Let us define a map $\sigma : \Omega \rightarrow \Omega$ in the following fashion. Set $n = l(1)$. If the word δ_1 has length greater than 1 then just forget the first symbol of this word. If $\delta_1 = (I_k^n)$ and $\chi(\delta_1) = (I_0^{n-1})$ then forget δ_1 . Finally, if δ_1 has only one symbol and $\chi(\delta_1) \neq (I_0^{n-1})$ then replace δ_1 with $\chi_*(\delta_1)$ where $\chi_*(\delta_1)$ coincides with $\chi(\delta_1)$ except for the last symbol I_0^{n-1} which is dropped.

Given a map $g \in \mathcal{T}_{\min}$, we can associate to a point $x \in \omega(c)$ an element of the space Ω in the following way. Let $x \in I^{n-1} \setminus I^n$. Then consider the itinerary δ_1 of x through the intervals of level $n = l(1)$ until the first moment t when it lands in the central interval I^n . Then find a level $l(2) = m > n = l(1)$ such that $y = g_n^{\circ t} x \in I^m \setminus I^{m-1}$. Coding now y in a similar way we will find δ_2 , etc. Since $\text{diam}(I_k^n) \rightarrow 0$ as $n \rightarrow \infty$, this provides us with a homeomorphism between $\omega(c)$ and Ω conjugating f and σ . So, we have the following lemma.

Lemma 2.10. *Two maps $f, g \in \mathcal{T}_{\min}$ have the same type, $\bar{\chi}_f = \bar{\chi}_g$, if and only if their restrictions on the critical sets are topologically conjugate by an orientation preserving homeomorphism respecting the critical points.* \square

Pull-back argument.

Lemma 2.11. *Two maps f and g of class \mathcal{T}_{\min} are topologically conjugate by an orientation preserving homeomorphism if and only if they have the same types, $\bar{\chi}_f = \bar{\chi}_g$.*

Proof. Let us set $g \equiv \tilde{f}$ and mark all objects related to g with the tilde. By Lemma 2.9, there is an orientation preserving conjugacy $h : \omega(c) \rightarrow \omega(\tilde{c})$ putting c to \tilde{c} . Let us continue it to an orientation preserving homeomorphism $h : (J, \cup I_k) \rightarrow (\tilde{J}, \cup \tilde{I}_k)$ of the base intervals.

Consider now a nested sequence of inverse images $J^{(k)} = f^{-k}J$, $k = 0, 1, \dots$ shrinking down to the Julia set $K(f)$, $J^{(1)} \equiv \cup I_k$ (and define $\tilde{J}^{(k)}$ similarly). Since $h(c) = \tilde{c}$, we can lift h via the folding maps f and \tilde{f} to an orientation preserving homeomorphism:

$$\begin{array}{ccc} & h_1 & \\ J^{(1)} & \rightarrow & \tilde{J}^{(1)} \\ f \downarrow & & \downarrow \tilde{f} \\ J & \rightarrow & \tilde{J} \\ & h & \end{array}$$

Then $h_1|_{\partial J^{(1)}} = h$, and hence h provides a continuation of h_1 to the whole base interval J .

Let us show that $h_1|_{\omega(c)} = h|_{\omega(c)}$. Indeed, since h is a conjugacy on $\omega(c)$, the points $h(x)$ and $h_1(x)$ either coincide or are \tilde{c} -symmetric. Since both of maps are orientation preserving, they must coincide.

Now let us repeat the construction and pull h_1 back, etc. In such a way we will construct a sequence $h_n : J \rightarrow \tilde{J}$ of orientation preserving homeomorphisms such that

- (i) $\tilde{f} \circ h_n = h_{n-1} \circ f$;
- (ii) h_n agrees with h on the critical set;
- (iii) $h_n|_{J \setminus J^{(n)}} = h_{n-1}$.

Hence there is a pointwise limit $H = \lim h_n$ which is an orientation preserving homeomorphism $J \setminus K(f) \rightarrow \tilde{J} \setminus K(\tilde{f})$ conjugating f and \tilde{f} . Since the both Julia sets $K(f)$ and $K(\tilde{f})$ are Cantor, H can be automatically continued across these sets. \square

Example: the Fibonacci recurrence. In this case we have two intervals I_0^n and I_j^n , $j \in \{-1, +1\}$, on all levels. The returns are high on all levels, and $\chi(I_0^n) = (I_j^{n-1}, I_0^{n-1})$, $\chi(I_j^n) = (I_0^{n-1})$. This data and the spin structure on the first level (four possibilities) uniquely determine the admissible spin structures on the further levels and the choice between $j = 1$ or -1 for the non-central intervals I_j^n . Namely, one can see that on two subsequent levels the I_j^n lie on one side of c , and then on the next two levels they lie on the opposite side, etc (the first level can be the only exception).

We see that there are four Fibonacci types. But actually two pairs of them are the same up to the choice of the orientation of the base interval. So, only two types are left which differ by the spin of the non-central branch. The renormalization interchanges these types.

Let us remark in conclusion that the name ‘‘Fibonacci’’ comes from the observation that the first return time t_n of the critical point to the n th level satisfies the Fibonacci recurrent equation $t_{n+1} = t_n + t_{n-1}$.

Renormalization. In the subsection ‘‘Realization of all types’’ we defined the pre-renormalization g_1 of a map $g \in \mathcal{T}_{\min}$. The *renormalization* Rg is obtained from g_1 just by rescaling of the base interval I_0^n (for g_1) to the original interval J . If we repeat this procedure, we will see that the n -fold renormalization $R^n g$ is obtained from g_n by rescaling I_0^n . The type of the renormalization is defined as the type of the corresponding pre-renormalization. Now we can summarize the above results as follows:

Theorem 2.12. *Let $g \in \mathcal{T}$. Then g is infinitely renormalizable with $R^n g \in \mathcal{T}_{\min}$. The topological type of g is uniquely determined by the sequence $\bar{\chi}_g$ of renormalization types. Any sequence of admissible types*

$$\begin{array}{ccccccc} \dots & \rightarrow & \Gamma_2 & \rightarrow & \Gamma_1 & \rightarrow & \Gamma_0 \\ & & \chi_3 & & \chi_2 & & \chi_1 \end{array}$$

can be realised in any full one-parameter family g^t with $\Gamma_g = \Gamma_0$.

Maps of infinite type. In the case when the critical point is recurrent but $\omega(c)$ is not minimal, we still can renormalize the map on a nice interval by taking the first return map, keeping only the intervals intersecting $\omega(c)$ and rescaling. However, now the domain of the n -fold renormalization consists of infinitely many intervals starting from some level.

Still one can develop a similar combinatorial theory but we don't need it. Let us only agree that $I^n \equiv I_0^n$ is still the central interval, and the intervals I_k^n with $k > 0/k < 0$ lie on the right/left of c .

Getting started. Let $f : L \rightarrow K$, $L \subset K$, be a unimodal map, exact on $[fc, f^{\circ 2}c]$. Take a nice interval $J \equiv I^0 \subset [fc, f^{\circ 2}c]$, and renormalize f on J . We obtain a map $g \in \mathcal{T}^\infty$ which does not admit unimodal renormalizations. This is our object to study.

§3. Estimates of Poincaré lengths and scaling factors.

Hyperbolic line and asymmetric Poincaré length. Let us consider an interval $L = [a, b]$ as a *hyperbolic line* with the Poincaré metric $2dx/(x-a)(b-x)$. Let G be a subinterval of L , and U and V be the components of $L \setminus G$ (see Figure 1). We will use the following notations for the Poincaré length of G in L :

$$P(G) \equiv P(G|L) \equiv P(U, V) = \log\left(1 + \frac{|G|}{|U|}\right) + \log\left(1 + \frac{|G|}{|V|}\right).$$

Note that the bigger the space is around G in L , the smaller Poincaré length $P(G|L)$ is.

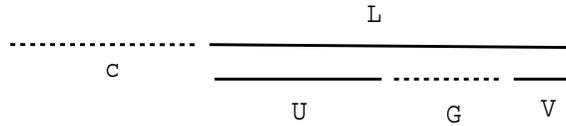


Figure 1

Let us now state the main analytic tools of real one-dimensional dynamics (see [MS]). They are called “the Schwarz Lemma” and “the Koebe Principle” by analogy with the classical facts in geometric function theory.

Schwarz Lemma. Any diffeomorphism $h : L \rightarrow L'$ with positive Schwarzian derivative contracts the Poincaré metric. \square

Hence, given an interval $G \subset L$ and its image $G' = hG$ we have: $P(G|L) \leq P(G'|L')$. So, if we have a definite space in L around G then we have also a definite space in L' around G' .

Given a diffeomorphism $h : I \rightarrow I'$, let us call

$$\max_{x, y \in I} \log \frac{|f'(x)|}{|f'(y)|}$$

its *distortion* or *non-linearity*. The case of zero non-linearity corresponds to linear maps.

Koebe Principle. Let $h : (L, I) \rightarrow (L', I')$ be a diffeomorphism with positive Schwarzian derivative, $r = P(I|L)$. Then the non-linearity of h on I is bounded by a constant $C(r)$ independent of h . Moreover, $C(r) = O(r)$ as $r \rightarrow 0$. \square

Lemma 3.1. If $L \supset T \supset I$ then $P(I|L) \leq \frac{1}{2}P(I|T)P(T|L)$.

Proof. Since the Poincaré metric is invariant under Möbius transformations, we can normalize the intervals in the following way: $L = [-1, 1]$, $T = [-\lambda, \lambda]$, $0 < \lambda < 1$. Let

us consider the map $h : x \mapsto \lambda x$. The calculation shows that $\|Dh(x)\|_L \leq \lambda$ in the L -Poincaré metric (with equality at 0). On the other hand, $h : L \mapsto T$ is an isometry from the L -Poincaré metric to the T -one. Hence $P(I|L) \leq \lambda P(h^{-1}I|L) = \lambda P(I|T)$. Observe finally that $2\lambda \leq P(T|L)$. \square

Remark. We will actually use Lemma 3.1 in a slightly different form:

$$P(I|L) \leq P(I|T) P^*(T|L)$$

where $P^*(T|L) = \min\{P(T|L), 1\}$.

Suppose we have a map g with a single non-degenerate critical point c . If the interval $\text{int } G$ does not contain c , then let us introduce the *asymmetric Poincaré length* defined as

$$Q(G) \equiv Q(G|L) \equiv Q(U, V) = \log\left(1 + \frac{|G|}{|U|}\right) + \frac{1}{2} \log\left(1 + \frac{|G|}{|V|}\right),$$

provided U is closer to c than V . Clearly, $Q(G) < P(G)$. The coefficient $1/2$ is related to the exponent 2 of the critical point c . It turns out that the asymmetric Poincaré length behaves more regularly under renormalizations of a quasi-quadratic map than the usual Poincaré length.

Parameters. From now on we will assume without change of notations that all maps of class \mathcal{T} have negative Schwarzian derivative and a non-degenerate critical point c . Let us consider a map $g \equiv q_1 \in \mathcal{T}$ which does not admit unimodal renormalizations, and has a recurrent critical point. Then we can construct the sequence of pre-renormalized maps

$$g_n : \bigcup I_k^n \rightarrow I^{n-1}$$

with the central intervals $I_0^n \equiv I^n$ shrinking down to c . In this section we don't assume that $\omega(c)$ is minimal, so that we allow infinitely many intervals I_k^n on all sufficiently high levels n .

By the gap between intervals U and V we mean the bounded connected component of $\mathbf{R} \setminus (U \cup V)$. Let us introduce the following parameters:

- K_n is the infimum of asymmetric Poincaré lengths $Q(I_s^n, I_t^n)$ of the gaps between intervals of level n with $st \geq 0$;
- $\mu_n = |I^n|/|I^{n-1}|$ is the scaling factor on level n ;
- $\lambda_n = \max_{i \neq 0} [I^{n-1} : I_i^n]$ is the maximal Poincaré length of the non-central intervals of level n . Set $\lambda_n^* = \min(\lambda_n, 1)$.

Further, let

$$\alpha_n = \sup_{k \neq 0} \frac{|I_k^n|}{\text{dist}(I_k^n, c)}$$

be a parameter which controls the non-linearity of the quadratic map $\phi(x) = (x - c)^2$ on non-central intervals.

By means of a C^3 -small change of variable (near the critical value) we can make f purely quadratic in a neighborhood of the critical point c . Then g_n can be decomposed

in the following way:

$$g_n|_{I_i^n} = h_{n,i} \circ \phi, \quad (3-0)$$

where $\phi(x) = (x - c)^2$ is the quadratic map and $h_{n,i}$ is a diffeomorphism with negative Schwarzian derivative of an appropriate interval onto J . Let us consider one more parameter:

- ρ_n is the maximal distortion of $h_{n,i}$, and call it the *distortion parameter*.

Let us remember that $\kappa : \mathbf{N} \rightarrow \mathbf{N}$ numbers the cascades of central returns (see §2). The goal of this section is to prove the following:

Conditional version of Theorem B. *There exist \bar{K} and $\bar{\mu} > 0$ (independent of a map) with the following property. If on some level N , $K_N > \bar{K}$ or $\mu_N < \bar{\mu}$ then $K_n \rightarrow \infty$. Moreover, there exist positive constants A , C and σ such that*

$$\begin{aligned} K_n &\geq A \kappa(n), \\ \mu_{n+1} &\leq C \exp(-\sigma \kappa(n)), \quad n \in \mathcal{L}, \\ \rho_n &\leq C \exp(-\sigma \kappa(n)). \end{aligned}$$

Remark. Observe that the scaling factors μ_n are exponentially small only on the special subsequence of levels (outside the long cascades of central returns). However, the estimates of the Poincaré lengths of the gaps, as well as the non-linearity control hold on all levels.

Let us fix constants $\bar{\mu}$ and $\bar{\rho}$ and $\bar{K} > 0$ such that $1 > \bar{\mu} > \bar{\rho} > 0$. We will assume (until the subsection “Cascades of central returns”) that the following estimates hold:

$$\rho_n < \bar{\rho}, \quad \mu_n < \bar{\mu}, \quad K_n > \bar{K}. \quad (3-1)$$

So, $\bar{\rho}$ controls the distortion, $\bar{\mu}$ controls the scaling factors, and \bar{K} controls the Poincaré lengths of the gaps. In what follows all constants depend on $\bar{\mu}$ (and actually on $\bar{\rho}$ which becomes non-important because we keep $\bar{\mu} > \bar{\rho}$) but not on the particular map. Sometimes we will abuse notations using the same letter for different constants.

Let us also fix small constants $\bar{\lambda}$ and $\bar{\alpha}$ which separate range of small values of parameters λ_n and α_n from big ones.

Strategy. Our strategy is the following. Let us consider two intervals $U' = I_k^{n+1}$ and $V' = I_j^{n+1}$ such that the gap G' between them does not contain the critical point. Let us push these intervals forward by g_n until the first moment p when $U = g_n^p U' \subset I_t^n$ and $V = g_n^p V' \subset I_s^n$ lie in different intervals of level n , that is $s \neq t$. Then loosely speaking, the Poincaré length of the gap between U and V can be estimated from below by $2K_n + \chi$ with an absolute constant $\chi > 0$. Pulling this back by an almost quadratic map we get an estimate of the asymmetric Poincaré length of the gap between U' and V' , namely $Q(U, V) \geq K_n + \chi/2$.

The argument depends on the positions of the intervals I_t^n and I_s^n . The Fibonacci-like situation when one of these intervals is central is the main one to look at (see Lemmas 3.3 and 3.9). In all other cases the estimates are actually getting better.

Estimates of $P(U, V)$. This will occupy lemmas 3.2 through 3.7.

Let us fix a level n and temporarily drop the index n in all notations so that $I^n \equiv I$, $g_n \equiv g$, $\mu_n \equiv \mu$ etc. However, let $I^{n-1} \equiv J$. Let us take a non-central interval I_t , $t \neq 0$, of level n , and consider an interval $U \subset I_t$ such that $g^l U = I$, $g^i U \cap I = \emptyset$, $i = 0, 1, \dots, l-1$. Sometimes we will write $l = l_U$. Let L and R be the components of $I_t \setminus U$ with L closer to c than R (see the following figure).

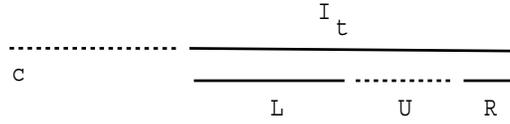


Figure 2

Let J^+ and J^- be the components of $J \setminus I$.

Lemma 3.2. *The following estimates hold:*

$$\frac{|U|}{|L|} + \frac{|U|}{|R|} \leq 4\mu(1 + O(\mu)). \quad (3-2)$$

If $l = 1$ then

$$\frac{|U|}{|L|} \leq 2\mu(1 + O(\rho + \mu)) \quad (3-3)$$

and

$$C(\rho, \mu)^{-1} \leq \frac{|R|}{|L|} \leq C(\rho, \mu), \quad (3-4)$$

with $C(\rho, \mu) = \sqrt{2}(1 + O(\rho + \mu))$. If $l > 1$ and then

$$\frac{|U|}{|L|} \leq 4\mu\lambda^*(1 + O(\mu)) \quad \text{and} \quad \frac{|U|}{|R|} \leq 4\mu\lambda^*(1 + O(\mu)). \quad (3-5)$$

Proof. There is an interval T such that $U \subset T \subset I_t$ which is diffeomorphically mapped by g^l onto J . By the Schwarz lemma,

$$P(U|I_t) \leq P(U|T) \leq P(I|J) = 4\mu(1 + O(\mu)). \quad (3-6)$$

But

$$P(U|I_t) = \left(\frac{|U|}{|L|} + \frac{|U|}{|R|} \right) (1 + O(P(U|I_t))), \quad (3-7)$$

provided there is an a priori bound on $[I_t : U]$. The last two estimates imply (3-2).

In order to get (3-3) let us make use of the decomposition (3-0):

$$\frac{|U|}{|L|} < \frac{|\phi U|}{|\phi L|} < \frac{|I_0|}{|J^+|}(1 + \rho) = 2\mu(1 + O(\rho + \mu)).$$

Estimate (3-4) follows from the fact that g is the composition of a quasi-symmetric map ϕ and a diffeomorphism with distortion ρ (the $\sqrt{2}$ comes as the qs norm of ϕ^{-1}).

Suppose now that $l > 1$. Then g^{l-1} maps the interval T introduced above onto a non-central interval I_s , $s \neq 0$. Hence there is another interval T' in between T and I_t which is mapped by g^{l-1} onto J . Hence $P(T|I_t) \leq P(T|T') \leq P(I_s|J) \leq \lambda$. By Lemma 3.1 and estimate (3-6),

$$P(U|I_t) \leq P(T|I_t)P(U|T) \leq 4\mu \min(\lambda, 1)(1 + O(\mu)), \quad (3-8)$$

and the estimates (3-5) follow from (3-7) and (3-8). \square

We will use the sign \prec or \succ if an estimate holds up to $O(\mu + \rho)$, and a sign \approx if an equality holds up to $O(\mu + \rho)$ (provided $\mu \leq \bar{\mu}, \rho \leq \bar{\rho}$).

Let $U \subset I_t$ be as above, H be the gap between I_0 and I_t , G be the gap between I_0 and U (see Figure 3). Let $P(H)$ denote the Poincaré length of H in $I_0 \cup H \cup I_t$, and $P(G)$ denote the Poincaré length of G in $I_0 \cup G \cup U$. Notations $Q(H)$ and $Q(G)$ mean the asymmetric Poincaré lengths of the same pairs of intervals. Let J^+ be the component of $J \setminus I$ containing I_t .

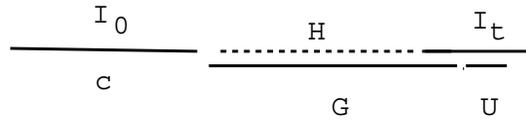


Figure 3

Lemma 3.3. *If $l = 1$ then there is an absolute constant $\chi > 0$ such that*

$$P(G) \succ 2Q(H) + \chi. \quad (3-10)$$

If $l > 1$ then

$$P(G) \succ 2Q(H) + \log^+ \frac{1}{\lambda} - \log 2. \quad (3-11)$$

Proof. We have

$$\begin{aligned} P(G) &\geq \log\left(1 + \frac{|H|}{|I|}\right) + \log\left(1 + \frac{|H| + |L|}{|U|}\right) = \\ &= \log\left(1 + \frac{|H|}{|I|}\right) + \log\left(1 + \frac{|H|}{|L| + |U|}\right) + \log\left(1 + \frac{|L|}{|U|}\right). \end{aligned} \quad (3-12)$$

The middle term is evidently bounded from below by $\log(1 + |H|/|I_t|)$. As to the last

term, then by (3-2) we have:

$$\begin{aligned} \log\left(1 + \frac{|L|}{|U|}\right) &\succ \log \frac{1}{4\mu} \approx \log\left(1 + \frac{|J^+|}{|I|}\right) - \log 2 \geq \\ &\geq \log\left(1 + \frac{|H|}{|I|}\right) + \log \frac{|J^+| + |I|}{|H| + |I|} - \log 2, \end{aligned} \quad (3-13)$$

and the estimate $P(G) \succ 2Q(H) - \log 2$ follows.

Let now $l = 1$. Then we can use (3-3) instead of (3-2), and $-\log 2$ disappears in the last estimate. We can also improve the estimate of the middle term of (3-12) as follows. Because of (3-3) and (3-4), there is a $\tau > 1$ such that $|I_t| \geq \tau(|L| + |U|)$. Hence

$$\log\left(1 + \frac{|H|}{|L| + |U|}\right) \succ \log\left(1 + \frac{|H|}{|I_t|}\right) + \chi, \quad (3-14)$$

provided H is not tiny as compared with I_t . On the other hand if H is tiny as compared with I_t then J^+ is big as compared with H and H is big as compared with I (namely, $\log(|H|/|I|) \geq \bar{K} - [a \text{ tiny term}]$). Hence the second term

$$\log \frac{|J^+| + |I|}{|H| + |I|}$$

in (3-13) is big, and suppresses $-\log 2$. These yield (3-10).

Finally, if $l > 1$ then we can improve (3-13) by using (3-5) instead of (3-2). \square

Now together with the above pair of intervals $U \subset I_t$, let us consider a similar pair $V \subset I_s$, $s \neq 0$, with V to be a monotone pull-back of I . Let us assume that both pairs lie on the same side of c , and the latter one is closer to c than the former (see Figure 4). Let G be the gap between V and U , H be the gap between I_s and I_t .

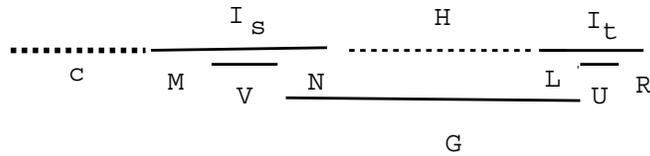


Figure 4

Lemma 3.4. *If I_s and I_t are non-central intervals lying on the same side of c then*

$$P(G) \geq P(H) + 2 \log \frac{1}{\mu} - O(1). \quad (3-16)$$

Proof. Let L and R be the components of $I_t \setminus U$ as defined above, while M and N

are the components of $I_s \setminus V$. Then we have:

$$\begin{aligned}
P(G) &= \log\left(1 + \frac{|H| + |N|}{|V|}\right) + \log\left(1 + \frac{|H| + |L|}{|U|}\right) = \\
&= \log\left(1 + \frac{|H|}{|N| + |V|}\right) + \log\left(1 + \frac{|N|}{|V|}\right) + \\
&\quad + \log\left(1 + \frac{|H|}{|L| + |U|}\right) + \log\left(1 + \frac{|L|}{|U|}\right).
\end{aligned} \tag{3-18}$$

The sum of the first and the third terms of (3-18) is certainly greater than $P(H)$. Estimating the second and the last terms by (3-2), we get (3-16). \square

The following lemma will allow us to handle the case when the non-linearity of ϕ is not small.

Lemma 3.5. *The following estimate holds: $\mu = O\left(1 + \frac{1}{\alpha}\right)e^{-K}$.*

Proof. Let us select an interval I_k for which $\alpha = |I_k|/\text{dist}(I_k, c)$. Let W be the gap between I and I_k . Then we have:

$$K \leq P(I, I_k) \prec \log \frac{1}{2\mu} + \log\left(1 + \frac{1}{\alpha}\right),$$

and the conclusion follows. \square

We will need the following lemma to analyze cascades of central returns.

Lemma 3.6. *Under the circumstances of Lemma 3.4, if $H \geq \text{dist}(H, c)$ then $P(G) \geq (5/2)K - O(1)$.*

Proof. Let W be the gap between I and I_s and X be the gap between I and I_t . Let us start with formula (3-18). Because of the assumption of the lemma, we can estimate from below half of its first term by $(1/2)\log(1 + |W|/|I_s|)$, and half of the third term by $(1/2)\log(1 + |X|/|I_t|) - \log 2$. The sum of the other halves we estimate as $(1/2)P(H)$.

Remember that J^+ denotes a component of $J \setminus I$. The second and the last terms we \succ estimate by (3-2) as $\log(1 + |I|/|J^+|)$ which is greater than both $\log(1 + |I|/|W|)$ and $\log(1 + |I|/|X|)$. Taking all these together, we get

$$P(G) \geq Q(I, I_s) + Q(I, I_t) + \frac{1}{2}P(H) - O(1) \geq (5/2)K - O(1).$$

\square

Finally, let us consider the case when I_s and I_t lie on the opposite sides of c .

Lemma 3.7. *If I_s and I_t lie on the opposite sides of c then $P(G) \succ (5/2)K$.*

Proof. The argument is the same as in the previous lemma. The point is that now we automatically have $|H| \geq |W|$ and $|H| \geq |X|$ where as above W denotes the gap between I and I_s , and X denotes the gap between I and I_t . \square

Quadratic pull-backs. Let us start with a lemma which says that the square root map divides the Poincaré length at most by 2.

Lemma 3.8. *Let us consider a quadratic map $\phi : x \mapsto (x - c)^2$. Let U and V be two disjoint intervals lying on the same side of c , V being closer to c than U . Then*

$$Q(V, U) > \frac{1}{2}P(\phi V, \phi U).$$

Proof. We can assume that $c = 0$ and V, U lie on the right of c . Let $V = [v, a]$, $U = [b, u]$. Then

$$\begin{aligned} P(V, U) - \frac{1}{2}P(\phi V, \phi U) &= \\ &= \frac{1}{2} \left(\log \frac{a+v}{a-v} - \log \frac{b+v}{b-v} \right) + \frac{1}{2} \left(\log \frac{u-a}{u-b} + \log \frac{u+b}{u+a} \right) > \frac{1}{2} \log \frac{u-a}{u-b}, \end{aligned}$$

which is exactly what is claimed. \square

In Lemmas 3.3 - 3.7 we have estimated the Poincaré length of the gap G between U and V . Now we are going to use Lemma 3.8 in order to estimate the asymmetric Poincaré length of the gap between U' and V' . Again let us start with the situation when one of the intervals, say I_s , is central (as in Lemma 3.3). As above, H denotes the gap between I_s and I_t . Set $T' = U' \cup G' \cup V'$, $T = g^p T' = U \cup G \cup V$.

Lemma 3.9. *Under the circumstances just described there is a constant $\chi > 0$ such that*

$$Q(G') \succ Q(H) + \chi, \tag{3-22}$$

or

$$Q(G') \succ K + \frac{1}{2} \log^+ \frac{1}{\lambda} - O(1). \tag{3-23}$$

Proof. *Case 1.* Let $p = 1$.

Let us use representation (3-0) of $g|I$ as the quadratic map ϕ postcomposed by a diffeomorphism h with distortion ρ . Pulling G back by h and then by ϕ (making use of Lemma 3.8), we see that $Q(G') \succ (1/2)P(G)$. Together with Lemma 3.3 this yields the claim.

Case 2. Let $p > 1$.

Then let us consider the intervals $\tilde{U} = g^{p-1}U'$, $\tilde{V} = g^{p-1}V'$ and $\tilde{G} = g^{p-1}G'$. All three of them belong to the same interval of level n , say I_j .

Because of (3-0) we can consider the following decomposition:

$$\psi \circ \phi|T', \tag{3-25}$$

where ψ is a diffeomorphism onto J . Remember that the non-linearity of the quadratic map $\phi|I_j$ is controlled by the quantifier α . Let us take a small $\bar{\alpha} > 0$ and consider several subcases.

Subcase (i). Assume $\alpha < \bar{\alpha}$.

This implies that $g|I_k$, $k \neq 0$, is an expanding map with the expansion > 2 and small non-linearity. Then the diffeomorphism ψ in (3-25) has small non-linearity as well. Together with Lemmas 3.3 and 3.8 this yields the desired estimates.

Let S be the gap between I and U .

Subcase (ii). Assume that $|S| \leq \bar{\alpha}|J|$.

Then $P(T|J) = O(\bar{\alpha} + \mu)$ (make use of Lemma 3.2). Hence $\psi|\phi T'$ has small non-linearity, and the result follows.

Subcase (iii). Finally, assume that

$$\alpha > \bar{\alpha} \tag{3-26}$$

and

$$\frac{|S|}{|J|} > \bar{\alpha}. \tag{3-27}$$

Then Lemma 3.5 and (3-26) imply

$$\frac{|I|}{|J|} = \mu = O(e^{-K}). \tag{3-28}$$

Together with (3-27) this implies

$$\frac{|I|}{|S|} = O(e^{-K}). \tag{3-29}$$

Let J^- be the component of $J \setminus I$ disjoint from S . Given an interval $X \subset I$, let \tilde{X} denote its pull-back by ψ . Pulling the interval $J^- \cup I \cup S$ back by ψ , we get by (3-28), (3-29) and the Schwarz lemma

$$\frac{|\tilde{I}|}{|\tilde{S}|} = O(e^{-K}). \tag{3-30}$$

Let Q be the component of $J \setminus U$ which does not contain c . Then

$$P(S, Q) = O(\mu\lambda_*) = O(\lambda_*e^{-K}),$$

and hence

$$\frac{|\tilde{U}|}{|\tilde{S}|} = O(\lambda_*e^{-K}) \tag{3-31}$$

as well. estimates (3-30) and (3-31) imply

$$P(\tilde{I}, \tilde{U}) \geq 2K + \log^+ \frac{1}{\lambda} - O(1). \tag{3-32}$$

Pulling this back by the quadratic map, we get (3-23). \square

Lemma 3.10. *Let both intervals I_s and I_t be non-central and lie on the same side of c . Then*

$$Q(G') \succ \frac{1}{2}K + \log \frac{1}{\mu} - O(1) \tag{3-33}$$

or

$$Q(G') \geq K + B(K, \lambda) - O(1) \tag{3-34}$$

where $B(K, \lambda) \geq 0$, and $B(K, \lambda) = K/4 - O(\lambda)$ for $\lambda \leq \bar{\lambda}$.

Proof. Let us again consider several cases depending on the non-linearity α of the quadratic map ϕ on the non-central intervals. Let $\bar{\alpha} > 0$ be small.

Case 1. Let $p = 1$ or $\alpha \leq \bar{\alpha}$.

For $p = 1$ let us use representation (3-0). For $p > 1$ the condition $\alpha \leq \bar{\alpha}$ holds. Hence for any $p \geq 1$

$$g^p|T' = \psi \circ \phi|T' \quad (3-36)$$

where ψ is a diffeomorphism with bounded non-linearity. Hence pulling T back by ψ , we don't spoil (3-16). Composing this with ϕ -pull-back, we get (3-33) by Lemma 3.8.

Case 2. Let $p > 1$ and $\alpha \geq \bar{\alpha}$. Then Lemma 3.5 yields $\mu = O(e^{-K})$.

Let $W \supset gT$ be a g^{p-1} -fold pull-back of J . Given an interval $X \subset J$, denote by $\tilde{X} \subset W$ its g^{p-1} -fold pull-back. Pulling the pairs of intervals $I_s \supset V$ and $I_t \supset U$ back to W we get

$$P(\tilde{V} : \tilde{I}_s) = O(\mu) = O(e^{-K}) \quad \text{and} \quad P(\tilde{U} : \tilde{I}_t) = O(\mu) = O(e^{-K}). \quad (3-37)$$

Let us pull the interval $I_s \cup H \cup I_t$ back subsequently by g and then by g^{p-2} . Apply Lemma 3.8 on the first step and the Koebe Principle on the second. This yields

$$P(\tilde{I}_s, \tilde{I}_t) \succ \frac{1}{2}P(H) \geq 2B(K, \lambda) \quad (3-38)$$

where $B(K, \lambda)$ is as was claimed. Estimates (3-37) and (3-38) yield

$$P(\tilde{U}, \tilde{V}) \geq 2K + 2B(K, \lambda) - O(1).$$

Pulling this back by g , we obtain (3-34). \square

Lemma 3.11. *Let I_s and I_t be non-central intervals lying on the opposite sides of c . Then*

$$Q(G') \geq K + B(K, \lambda) - O(1)$$

with $B(K, \lambda)$ as in Lemma 3.10.

Proof. Let us again consider two cases.

Case 1. Let $p = 1$ or $\alpha < \bar{\alpha}$.

Then argue as in Case 1 of the previous lemma but use Lemma 3.7 instead of Lemma 3.4. This yields $Q(G') \geq (5/4)K - O(1)$ which is better than what is claimed.

Case 2. Let $p > 1$ and $\alpha \geq \bar{\alpha}$.

Then argue as in Case 2 of the previous lemma. \square

More relations between the parameters. Let us mark the quantifiers of level $n + 1$ by ‘‘prime’’: $\mu' \equiv \mu_{n+1}$, $\lambda' \equiv \lambda_{n+1}$ etc. The following lemma provides us with rough estimates of the parameters of level $n + 1$ through $\mu = \mu_n$.

Lemma 3.12. *The following estimates hold:*

$$\lambda' = O(\sqrt{\mu}) \quad \text{and} \quad \mu' = O(\sqrt{\mu}), \quad (3-39)$$

$$\rho' = O(\mu), \quad (3-40)$$

$$K' \geq \frac{1}{2} \log \frac{1}{\mu} - O(1). \quad (3-41)$$

In the case of non-central return, (3-39) can be improved as follows:

$$\mu' = O(\sqrt{\mu\lambda^*}). \quad (3-42)$$

Proof. Let us take an interval U' of level $n+1$ and consider its image $U = gU' \subset I_t$. If $t = 0$ we have $P(U|J) \leq P(I_t|J) \approx 4\mu$. Otherwise by Lemma 3.1 and estimate (3-2) we have $P(U|J) = O(\lambda^*\mu)$. Now estimates (3-39) and (3-42) follow from decomposition (3-0).

It follows from (3-0) and Lemma 2.2 that $g_{n+1}|U' = \psi \circ \phi|U'$ where ψ is a diffeomorphism with the Koebe space spreading over J . Since $\psi(\phi U') \subset I$, (3-40) follows.

In order to get (3-41) let us take two intervals U' and V' of level $n+1$ and go through our basic construction (see the ‘‘Strategy’’). Represent $g^p|I'$ as a composition $\psi \circ \phi$ where ψ is a diffeomorphism with a Koebe space spreading over J . Now pull the pair of intervals $I_t \supset U$ back by g^p taking into account that $P(U|I_t) \prec 2\mu$. We see that $|U'|/|G'| = O(\sqrt{\mu})$, and the result follows. \square

In what follows we restore the index n . Let us now treat the problem of estimating μ_{n+1} through the parameters of lower levels.

Lemma 3.13. *Let $\epsilon > 0$. In the non-central return case one of the following estimates holds:*

$$\mu_{n+1} = O(\sqrt{\lambda_n^*} \exp(-(1-\epsilon)K_n/2)), \quad (3-43)$$

or

$$\mu_{n+1} = O(\exp(-(1+\epsilon/2)K_n/2)), \quad (3-44)$$

or

$$\mu_{n+1} = O\left(\sqrt{\mu_n \mu_{n-1} \lambda_{n-1}^*}\right). \quad (3-45)$$

Proof. Let us again consider several cases.

Case 1. Let $\alpha_n > \exp(-\epsilon K_n)$. Then by Lemma 3.5,

$$\mu_n = O(\exp(-(1-\epsilon)K_n)).$$

Using this and (3-42) we obtain (3-43).

Case 2. Let $\alpha_n \leq \exp(-\epsilon K_n)$. Let $U = g_n I_t^{n+1} \subset I_t^n$, $t \neq 0$. Let L be the component of $I_t^n \setminus U$ which is closer to c and R be the other component.

High return subcase. Then arguing as in Lemma 3.3 we see that

$$\begin{aligned} \log \frac{1}{\mu_{n+1}^2} &\succ \log \left(1 + \frac{\text{dist}(U, c)}{|U|}\right) \geq \log \left(1 + \frac{\text{dist}(I_t^n, c)}{|I_t^n|}\right) + \log \left(1 + \frac{|L|}{|U|}\right) \geq \\ &\geq \left[\log \frac{1}{2\mu_n} + \frac{1}{2} \log \left(1 + \frac{\text{dist}(I_t^n, c)}{|I_t^n|}\right) \right] + \frac{1}{2} \log \left(1 + \frac{\text{dist}(I_t^n, c)}{|I_t^n|}\right) - O(1) \geq \\ &\geq K_n + \frac{1}{2} \log \frac{1}{\alpha_n} - O(1) \geq (1 + \epsilon/2)K_n - O(1), \end{aligned}$$

and (3-44) follows.

Low return subcase. Let b be the boundary point of I^{n-1} lying on the same side of c as I_t^n . Set $X = [c, I_t^n]$, the convex hull of c and I_t^n , $Y = [I_t^n, b]$. We need to refine the situation again.

(i) Let $|Y|^2 \geq |X| \cdot |I_t^n| \exp(\epsilon K_n/2)$. Then

$$\begin{aligned} \log \frac{1}{\mu_{n+1}^2} &\approx \log \left(1 + \frac{\text{dist}(U, b)}{|U|} \right) \geq \log \left(\frac{|Y|}{|I_t^n|} \right) + \log \left(1 + \frac{|R|}{|U|} \right) \geq \\ &\geq \left[\frac{1}{2} \log \left(\frac{|X|}{|I_t^n|} \right) + \log \frac{1}{2\mu_n} \right] + \frac{1}{2} \epsilon K_n - O(1) \geq (1 + \epsilon/2) K_n - O(1), \end{aligned}$$

and we have (3-44) again.

(ii) Let $|Y|^2 \leq |X| \cdot |I_t^n| \exp(\epsilon K_n/2)$. Then “an exponentially low return” occurs:

$$|Y|/|I^{n-1}| \leq \exp(-\epsilon K_n/4).$$

It follows that

$$\text{dist}(g_{n-1}I_t^n, g_{n-1}b)/\text{dist}(g_{n-1}I_t^n, g_{n-1}c) = O(\exp(-\epsilon K_n/8)). \quad (3-46)$$

Let $g_{n-1}I_t^n \subset I_j^{n-1}$. Then (3-46) implies that I_j^{n-1} is a non-central interval, that is $j \neq 0$. Hence

$$|g_{n-1}I_t^n|/|g_{n-1}Y| = O(\mu_{n-1}\lambda_{n-1}^*).$$

Since $g_{n-1}|Y$ has distortion $O(\exp(-\epsilon K_n/4))$, we conclude that

$$|I_t^n|/|Y| = O(\mu_{n-1}\lambda_{n-1}^*)$$

as well. Together with $P(U|I_t^n) = O(\mu_n)$ this implies (3-45). \square

Now we are prepared to prove the Conditional Version of Theorem B. To make life easier, let us first treat the case when there are no central returns at all.

No central returns case. If μ_0 is small then K_1 is big by (3-41). So, we can make the following inductive assumption: There is a $\theta > 0$ such that

$$K_i \geq \theta i, \quad i = 1, 2, \dots, n, \quad (A_n)$$

and

$$\mu_i \leq \frac{1}{2} \exp(-\theta i/2), \quad i = 1, 2, \dots, n. \quad (B_n)$$

By (3-39) we have

$$\lambda_i \leq \exp(-\theta i/4) \ll \bar{\lambda}, \quad i = 1, 2, \dots, n. \quad (3-47)$$

Now Lemma 3.13 allows us to conclude that there is a $\delta > 0$ such that

$$\mu_i \leq \exp(-(\theta + \delta)i/2), \quad i = 1, 2, \dots, n+1. \quad (3-48)$$

which is certainly stronger than (B_{n+1}) .

In order to obtain (A_{n+1}) let us take a gap G' between two intervals U' and V' of level $n+1$ and push it forward as described in the above “Strategy”. Then we will find two intervals I_s^n and I_t^n . Let us consider three cases depending on the position of these intervals.

Case 1. Let I_s^n be the central interval. Then by Lemma 3.9 and estimate (3-47) we conclude that there is an absolute constant $\chi > 0$ such that $Q(G') > K_n + \chi$ which is greater than $\theta(n+1)$, provided θ was selected to be smaller than χ . Taking the infimum over all gaps G' we obtain A_{n+1} .

Case 2. Let I_s^n and I_t^n be two non-central intervals lying on the opposite sides of c . Then Lemma 3.10, assumption A_n and estimates (3-47), (3-48) give us a small $\delta > 0$ such that

$$Q(G') \geq (1 + \delta)(n + 1)\theta, \quad (3 - 49)$$

which implies A_{n+1} .

Case 3. Let I_s^n and I_t^n be two non-central intervals lying on the same side of c . Then Lemma 3.11, the assumption (A_n) and (3-47) yield (3-49) again.

Cascades of central returns. Let us have a non-central return on level $m - 1$ followed by the cascade of central returns on levels $m, m + 1, \dots, m + q - 1$, and completed by a non-central return on level $m + q$, $q \geq 1$. So, $m, m + q + 1 \in \mathcal{L}$. Set $g = g_{m+1}|I^{m+1}$ (see Figure 5). Then

$$g(c) \in I^{m+q} \setminus I^{m+q+1}, \text{ and } g_{m+i}|I^{m+i} = g|I^{m+i}, \quad i = 1, \dots, q + 1, \quad (3 - 50)$$

and I^{m+i+1} is the g -pull-back of I^{m+i} , $i = 0, \dots, q$. Let us call q the *length* of the cascade.

Figure 5

Let us fix a big natural number N . Let us define $\omega(m)$ in the following way. If a non-central return on level $m - 2$ occurs, that is $m - 1 \in \mathcal{L}$, then set $\omega(m) = 0$. Otherwise the level $m - 2$ completes a cascade of central returns of length p . Then set $\omega(m) = \min(p, N)$.

Let us assume by induction that there are $\theta > 0$ and $\delta > 0$ such that

$$K_{i+1} \geq ((\kappa(i) + \omega(i))\theta), \quad i \leq m, \quad i \in \mathcal{L} \quad (A_m),$$

and

$$\mu_{i+1} \leq \mu_1 \exp(-(\theta + \delta)\kappa(i)/2), \quad i \leq m, \quad i \in \mathcal{L}. \quad (B_m)$$

Our goal is to check (A_{m+q+1}) and (B_{m+q+1}) , provided θ and δ are small enough.

When we travel along the cascade of central returns the trouble is that the scaling factors μ_{m+i} is definitely increasing (and very fast: as $(\mu_m)^{1/2^i}$). However, they are still quite small ($< \bar{\mu}$) in the initial segment of the cascade, so that we can apply all above lemmas. If μ_1 is small enough then (B_m) guarantees that for $i \leq N$

$$\mu_{m+i} \leq \bar{\mu}. \quad (3-51)$$

Moreover, both λ_n and α_n are exponentially small, that is setting $\kappa = \kappa(m)$ we have

$$\lambda_{m+i} = O(\exp(-\kappa\theta/2)) \quad \text{and} \quad \alpha_{m+i} = O(\exp(-\kappa\theta/2)), \quad i = 2, \dots, q+1. \quad (3-52)$$

Indeed, take a non-central interval I_k^{m+i} and push it forward by g^{i-1} . Since $g^{i-1}(c) \in I^{m+1}$ while $g^{i-1}I_k^{m+i} \subset I^m \setminus I^{m+1}$, there is a non-central interval I_t^{m+1} containing $g^{i-1}I_k^{m+i}$. Let $X \supset I_k^{m+i}$ be the pull-back of I_t^{m+1} by g^{i-1} . Then X is contained in $I^{m+i-1} \setminus I^{m+i}$ and $P(I_k^{m+i}|X) = O(\mu_{m+1})$. This estimate together with (B_m) implies (3-52).

These considerations also show that g_{m+i} can be represented as a composition of the quadratic map ϕ and a diffeomorphism whose Koebe space is spread over I^m . Hence, the distortion parameters remain small:

$$\rho_{m+i} = O(\exp(-\kappa\theta/2)), \quad i = 2, \dots, q+1. \quad (3-53)$$

An estimate for K_{m+2} . A trouble with this estimate is that λ_{m+1} need not be small. However, by the induction assumption and (3-39) the only way this can happen is if $m-1 \notin \mathcal{L}$ and $m-2$ completes a long cascade of central returns, that is $\omega(m) = N$ is big which makes the assumption (A_m) stronger.

More specifically, let us follow the above ‘‘Strategy’’. Take a gap G' between two intervals of level $m+2$ and push it forward by iterates of g_{m+1} until its endpoints are separated by different intervals I_s^{m+1} and I_t^{m+1} of level $m+1$. As usual, let us consider several cases depending on the positions of these intervals.

Case 1. Let I_s^{m+1} be central. Then as we have explained either $\lambda_{m+1} < \bar{\lambda}$ or

$$K_{m+1} \geq \kappa\theta + A \quad (3-54)$$

with a big A . In both cases Lemma 3.9 yields

$$Q(G') > (\kappa + 1)\theta \quad (3-55),$$

provided θ is small enough.

Case 2. Let I_s^{m+1} and I_t^{m+1} be non-central lying on the same side of c . If $\lambda_{m+1} < \bar{\lambda}$ then Assumptions (A_m) , (B_m) and Lemma 3.10 imply that there is an $\epsilon > 0$ such that

$$Q(G') > (1 + \epsilon)\kappa\theta \quad (3-56),$$

which is certainly better than (3-55).

If $\lambda \geq \bar{\lambda}$ then (3-54) holds. Together with Lemma 3.10 this yields (3-55).

Case 3. Let I_s^{m+1} and I_t^{m+1} be non-central intervals lying on the opposite sides of c . Then argue as in the previous case using Lemma 3.11 instead of 3.10.

So, in all cases (3-55) holds, and hence $K_{m+2} \geq \theta(\kappa + 1)$.

Estimates for K_{m+i+1} , $i > 1$, in the initial segment of the cascade, (while (3-51) holds). Now λ_{m+1} is exponentially small by (3-52) but μ_{m+i} need not be exponentially small. Let us assume by induction that

$$K_{m+j+1} \geq (\kappa + j)\theta, \quad j = 1, \dots, i - 1. \quad (3-57)$$

To pass to the next level let us apply again our strategy and go through the same bunch of cases depending on the positions of I_s^{m+i} and I_t^{m+i} . Cases 1,2,3 mean the same as above.

Case 1. Then Lemma 3.9 and (3-52), (3-57) yield

$$Q(G') \geq (\kappa + i)\theta. \quad (3-58)$$

Case 2. Let H be the gap between I_s^{m+i} and I_t^{m+i} . If $|H| \geq \text{dist}(H, c)$ then Lemmas 3.6 and 3.8 yield the desired estimate. Otherwise g has a bounded distortion on $T' = U' \cup G' \cup V'$ (the notations are the same as in the Strategy description), and it follows from Lemma 3.4 that

$$P(G') > K_{m+i} + 2 \log \frac{1}{\bar{\mu}} - O(1) > K_{m+i} + \chi,$$

provided $\bar{\mu}$ is small enough.

Case 3 is treated in the standard way using Lemma 3.11 and (3-52).

Conclusion: (3-57) follows for $j = i$.

Distortion control in the tail of the cascade. Let M be the first moment for which $\mu_{m+M} \geq \bar{\mu}$. Since $\mu_{m+k+1} \approx \sqrt{\mu_{m+k}}$ for $k \leq q$,

$$|I_{m+q+1}| \geq c|I_M| \quad (3-59)$$

(with $c \approx \bar{\mu}^2$).

Lemma 3.14. *The map $g^{(j+1)}$ has bounded distortion on both components of $I^{M+j} \searrow J^{M+j+1}$, $0 \leq j \leq q$.*

Proof. As we know g is almost quadratic. Hence by (3-59) it has bounded non-linearity n_g on $I^M \searrow J^{m+q+1}$. Let L be a component of $I^{M+j} \searrow J^{M+j+1}$. Then $f^k L$ is a component of $I^{M+j-k} \searrow J^{M+j+1-k}$. Hence by the standard argument the non-linearity of g^{j+1} on L is bounded by

$$n_g \sum_{k=0}^j |g^k L| \leq n_g |I^M \searrow J^{m+q+1}| = O(1).$$

□

Pulling now the intervals from level M back to the tail of the cascade, we conclude that

$$K_{M+i} \geq K_M - O(1) \geq (\kappa + M)\theta, \quad i \leq q - M + 1 \quad (3-60)$$

(for, perhaps a bit smaller θ).

A Markov scheme. Let us build up a Markov map F . Let N be a big number as selected above. Set $K_j^{m+2N} = I_j^{m+2N}$, $j \neq 0$, and pull these intervals back by iterates of

g to levels $m+i$, $i = 2N, \dots, m+q+1$. Denote the corresponding intervals by K_j^{m+i} . Now set

$$F|K_j^{m+i} = g \quad \text{for } i > 2N \quad \text{and} \quad F|K_j^{m+2N} = g_{m+2N}.$$

This map F carries K_j^{m+i} onto K_j^{m+i-1} for $i > 2N$, and carries K_j^{m+2N} onto I^{m+2N-1} covering all intervals of our scheme.

Proof of (A_{m+q+1}) . Take two intervals $U' = I_k^{m+q+2}$ and $V' = I_l^{m+q+2}$ of level $m+q+2$, consider their images by g , and then push them forward by iterates of F until the first moment they don't belong to the same interval K_j^{m+i} of our scheme. This long-term composition is almost quadratic as one can see from (3-52). Denote the corresponding images of U' and V' by U and V . We again have to consider several cases.

Case 1. Assume that for some $N \leq j \leq q+1$ V belongs to a central interval I^{m+j} while $V \subset I_t^{m+j}$, $t \neq 0$. If $j > M$ push these intervals forward to level M . By Lemma 3.14 this results in a bounded change of the Poincaré length of the gap between the central interval I^{m+j} and V . Let us denote new intervals by \tilde{U} and \tilde{V} . Now they lie on the level $l = \min(M, j)$. Then Lemma 3.3, estimate (3-52) and the above estimates of K_{m+i} yield

$$P(I^l, V) \geq 2K_{m+2N} \geq 2(m+N)\theta.$$

Pulling this back by the quadratic map postcomposed with a bounded distortion map, we get the desired estimate.

Case 2. Let $V \subset I_s^{m+j}$, $U \subset I_t^{m+j}$ with $s \neq 0$ and $t \neq 0$. As in the previous case, pushing these intervals forward, we can assume that $j \geq M$. Let H be the gap between I_s^{m+j} and I_t^{m+j} .

Subcase (i). Let $H \ni c$. The the standard argument based on Lemma 3.7 gives the desired estimate.

Subcase (ii). Let $H \not\ni c$ and $|H| \geq \text{dist}(H, c)$. Use Lemma 3.6 instead of 3.7.

Subcase (iii). $H \not\ni c$ and $|H| < \text{dist}(H, c)$. Then let us push the intervals forward by iterated g . Set $U_n = g^n U$ etc. Assume there is a moment for which $|H_n| \geq \text{dist}(H, c)$. Then $g^n|U \cup H \cup V$ has a bounded distortion for the first such moment. Hence we can argue as in the previous subcase.

If there is no such a moment, then push the interval to the very beginning of the cascade (to level m) and apply Lemma 3.4.

Proof of (B_{m+q+1}) . We should estimate μ_{m+q+2} . Let us push I^{m+q+2} forward to higher levels:

$$T^l = G_l I^{m+q+2} \equiv g_l \circ g_{l+1} \circ \dots \circ g_{m+q+1} I^{m+q+2}.$$

Let us stop on the highest level $l < M$ for which one of the following properties hold:

- (i) the map G_l is not exponentially low in the sense of Lemma 3.13, Case 2-ii (that is $G_l I^{m+q+1}$ belongs to the $\exp(-\epsilon K_{l-1})|I^{l-1}|$ -neighborhood of ∂I^{l-1}), or
- (ii) $l-1 \in \mathcal{L}$ and $\kappa(l-1) = \kappa(m) - 1$. This means that we have arrived at the beginning of the previous cascade.

It follows from Lemma 3.14 that G is a quadratic map postcomposed with a bounded distortion map. This allows us to apply Lemma 3.13 for G_l instead of g , and to estimate

μ_{m+q+2} through the parameters of level l . If (i) occurs then (3-43) or (3-44)-like estimates hold. Together with the above estimates of K_l and λ_l they yield the desired estimate.

Otherwise there is a non-central interval $I_s^l \supset T_l$ which can be monotonically pulled back by G_l . Since

$$P(T_l|I_s^l) = O(\mu_l \mu_{l-1} \dots \mu_{m+q+1}) = O(\exp(-5\theta/4)),$$

we conclude that

$$\mu_{m+q+2} = O\left(\sqrt{P(T_l|I_s^l)}\right) = O(\exp(-5\theta/8)),$$

and (B_{m+q+1}) follows.

§4. Real bounds and limits of renormalized maps.

In this section we will prove Theorem B for maps with a non-minimal critical set $\omega(c)$ and for maps of unbounded type. For maps of bounded type will show that if the scaling factors stay away from zero then the family of renormalized maps is compact, and all limit maps are real analytic. As usual, let us first assume that *there are no central returns*. Set $x_m = f^m x$.

A priori bounds. As in §3 let us consider the decomposition (3-0) $g_n|I_i^n = h_{n,i} \circ \phi$, and denote by ρ_n the maximal distortion of the diffeomorphisms $h_{n,i}$.

Theorem 4.1 (Martens [Ma]). *The distortions ρ_n are uniformly bounded.*

Lemma 4.2. *The scaling factors $\mu_n = |I^n|/|I^{n-1}|$ are bounded away from 1.*

Proof. The case of low return on level $n-1$ was treated in ([Ma], Lemma 3.7).

In the case of high return let us assume that μ_n is close to 1. Then because of the bounds given by Theorem 4.1, the next scaling factor $\mu_{n+1} = O(\sqrt{1-\mu_n})$ will be very small. Then by the results of §3 $\mu_m \rightarrow 0$. \square

Lemma 4.3. *All Poincaré lengths $P(I_j^n|I^{n-1})$ are bounded away from ∞ .*

Proof. This follows from the previous lemma and Lemma 3.12. \square

Let $\bar{\lambda}$ denote an upper bound of Poincaré lengths $P(I_j^n|I^{n-1})$. Consider two intervals $T \supset G$ with L and R to be the components of $T \setminus G$. Denote by

$$\bar{\sigma} = \frac{e^{\bar{\lambda}}}{1 + e^{\bar{\lambda}}} < 1 \tag{4-1}$$

an upper bound of $|G|/(|G| + |L|)$ provided $P(G|T) \leq \bar{\lambda}$.

Orders and ranks. Let us define the *order* ord_0^n of the return to level n as the return time of g_n -orb(c) back to I^n . Let us also define l -orders ord_l^n as the return time of g_{n-l} -orb(c) back to I^n . In terms of the return graph the ord_0^n is just the number of edges beginning at I^{n+1} (and leading to the previous level n). The ord_l^n is the number of paths of length $l+1$ beginning at I^{n+1} (and leading to the level $n-l$).

Lemma 4.4. *If the scaling factors μ_n stay away from 0 then for each l the l -orders ord_l^n of returns to all levels are uniformly bounded.*

Proof. If on level n a return of high order p occurs then by Lemma 4.3 the next scaling factor $\mu_{n+1} = O(\sqrt{\lambda^p})$ is very small. Similarly, for a given l , if ord_l^n is big then traveling down the graph from level $n - l$ to $n + 1$ we see that μ_{n+1} is small as well. \square

Let us consider now the Markov family \mathcal{M} of intervals obtained by pull-backs of the initial interval I^0 (see §2).

Let us assign to the critical intervals $K \in \mathcal{M}$ rank 0. Let us say that an interval $K \in \mathcal{M}$, $K \subset I^{n-1} \searrow J^n$ has rank k , $k \geq 1$, if $\text{orb}(c)$ passes through it before the first return to I^{n+k-1} but after the first return to I^{n+k-2} . For example, $k = 1$ if $\text{orb}(c)$ passes through K before the first return to I^n . For $K = I_j^n$ this can be nicely expressed in terms of the return graph as the length k of the shortest path leading from I_j^n down to a central interval I^{n+k} .

Lemma 4.5. *Let $K \in \mathcal{M}$, $K \subset I^{n-1} \searrow J^n$, and $\text{rank}(K) \geq k > 0$. Let us take a point $x \in I^{n+k-2} \searrow J^{n+k-1}$, and consider the first moment l when $f^l x \in K$ (provided there is one). Then the interval K can be diffeomorphically pulled back along the $\text{orb}_l(x)$ to an interval $K' \subset I^{n+k-2} \searrow J^{n+k-1}$.*

Remark. We don't claim that $K' \ni x$ but just some iterate x_s , $s \leq l$, so that the pull-back has length $l - s$.

Proof. If $k = 1$ there is nothing to prove (set $K' = K$). If $k > 1$ let us consider the last moment $s < l$ when the $\text{orb}(x)$ visits I^n . Then there is a moment p such that $g_n^{\circ p}(x_s) \in K$ and all intermediate iterates $g_n^{\circ m}(x_s) \notin I^n$, $0 < m < p$. I claim that the pull-back of K along the g_n -orbit of x_s is monotone. Indeed, otherwise $g_n^{\circ p}(c) \in K$ while $g_n^{\circ m}(c) \notin I^n$, $0 < m < p$, so that $\text{rank}(K) = 1$.

Let $K_1 \ni x_s$ be the monotone pull-back of K along the g_n -orbit of x_s . Then $K_1 \subset I^n \searrow J^{n+1}$ and $\text{rank}(K_1) \geq k - 1$. So, we can proceed by induction. \square

Lemma 4.6. *If the scaling factors μ_n stay away from 0 then ranks of all intervals I_j^n are uniformly bounded.*

Proof. Let $\text{rank} I_j^n = k$. Let l be the first moment when $\text{orb}(c)$ visits I_j^n . Then by the definition of rank there is an $s < l$ such that $c_s \in I^{n+k-2} \searrow J^{n+k-1}$. By the previous lemma we can monotonically pull I_j^n to the level $n + k - 2$ along the $\text{orb} c_s$. We obtain an interval $K' \subset I^{n+k-2} \searrow J^{n+k-1}$.

Let us consider the interval I_i^{n+k-1} containing c_s . Then one can see by induction (involving the Schwarz lemma) that

$$P(I_i^{n+k-1} | I^{n+k-2}) \leq P(I_i^{n+k-1} | K') \leq \bar{\sigma}^{k-1}$$

with $\bar{\sigma}$ from (4-1). Since $\text{orb}(c)$ passes through I_i^{n+k-1} before its return to I^{n+k-1} , the scaling factor $\mu_{n+k} = O(\bar{\sigma}^{(k-1)/2})$ is small if k is big. Contradiction. \square

Non-minimal case. Now we are ready to proof Theorem B in the non-minimal case.

Lemma 4.7. *Assume that the critical set $\omega(c)$ is not minimal. Then the scaling factors μ_n go down to 0, $n \in \mathcal{L}$.*

Proof. Indeed, in the non-minimal case there is a level n and a point $x \in \omega(c) \cap I^{n-1}$ which never passes through the central interval I^n . It follows that the return time of points of $\text{orb}(c)$ back to I^n is unbounded and hence there are infinitely many intervals I_k^n of level n . The ranks of these intervals certainly must grow up to ∞ . Now lemma 4.6 provides us with the starting condition for Theorem 3.0. \square

Unbounded Combinatorics. Assume that $\omega(c)$ is minimal. Let us say that f is of *bounded type* if the number of intervals on all levels is uniformly bounded, and of *unbounded type* otherwise. The unbounded case can also be treated by a purely real argument (I owe this remark to Swiatek).

Lemma 4.8. *If f has unbounded combinatorics then the scaling factors μ_n go down to 0, $n \in \mathcal{L}$.*

Proof. If the combinatorics are unbounded then either the l -orders of returns or ranks of the intervals are unbounded (consider the return graph from §2). Now the required starting condition for Theorem 3.0 follows from lemmas 4.4. and 4.6. \square

Bounded Combinatorics. This is the main case when we need to involve complex analytic methods. In this subsection we will show that provided the scaling factors stay away from 0, there is a sequence of renormalized maps C^1 -converging to an analytic map.

Here by the gaps of level n we will mean the components of $I^{n-1} \setminus \cup I_j^n$.

Lemma 4.9. *If the scaling factors stay away from 0 then all intervals and all gaps of level n are commensurable with I^{n-1} .*

Proof. Let us show first that the intervals I_i^n may not be tiny as compared with I^{n-1} . Indeed, because of Lemma 4.3 such an interval should lie very close to ∂I^{n-1} . On the other hand, it follows from Theorem 4.1 that g_{n-1} is quasi-symmetric. Hence, $g_{n-1}I_j^n$ lies very close to ∂I^{n-2} and, moreover, $g_{n-1}I^{n-1}$ covers the interval $I_s^{n-1} \supset g_{n-1}I_j^n$ (again because of Lemma 4.3). Hence we can monotonically pull I_s^{n-1} back by g_{n-1} .

Now we can apply the same argument to the interval I_s^{n-1} and map g_{n-2} and so on. In such a way we will find a big l and an interval I_t^{n-l} which can be monotonically pulled back by $g_{n-l} \circ \dots \circ g_{n-1}$ along the orbit of I_i^n . Let $K \subset I^{n-1}$ be this pull-back. This interval provides us with a big space around I_i^n , namely $|I_i^n|/|K| \leq \bar{\sigma}^l$. Since rank I_i^n is bounded by Lemma 4.6, we can pull this space back and obtain a small scaling factor. Contradiction.

Let us now consider the gaps. They may not be too big as compared with the intervals since otherwise the intervals would be tiny as compared with I^{n-1} . Let us consider any gap G in between I_i^n and I_j^n . Arguing as in the proof of estimate (3-41) one can see that the Poincaré length $P(G)$ is bounded from below provided the scaling factors are bounded away from 1. This means that G is not tiny as compared with one of the intervals I_j^n , I_i^n . Since these intervals are commensurable with I^{n-1} , so the interval G is as well.

As to the two “boundary” gaps, they are not too small as compared with the attached intervals because of Lemma 4.3. \square

Consequently, we can select a sequence of renormalized maps $R^n f$ in such a way that the configurations of intervals and gaps converge to a non-degenerate configuration of intervals and gaps. Use now the rescaled representation (3-0) for these maps:

$$R^n f | \tilde{I}_i^n = G_{n,i} \circ \phi$$

where \tilde{I}_i^n are the rescaled intervals of level n and $G_{n,i}$ are diffeomorphisms of appropriate intervals onto the unit interval. By Theorem 4.1 and the previous lemma, the inverse maps $G_{n,i}^{-1}$ have uniformly bounded C^2 -norms, and hence form a C^1 -compact family. So, we can select a C^1 -convergent sequence of renormalized maps,

$$G_{n(s),i}^{-1} \rightarrow G_i^{-1} \quad \text{as } n(s) \rightarrow \infty.$$

Each G_i is a long composition of the square root maps and diffeomorphisms whose total distortion is controlled by

$$\omega_n = \sum_j \sum_{m=1}^{p(n,j)-1} |f^m(I_j^n)|$$

where $p(n, j)$ is the return time of I_j^n back to I^{n-1} . But $\omega_n \rightarrow |\omega(c)| = 0$ by [Ma]. Hence the total distortion of the diffeomorphisms involved is vanishing.

Now the ‘‘Shuffling Lemma’’ (see [S] or [MS], Ch. VI, Theorem 2.3) yields that the limit of renormalized maps is real analytic and, moreover, belongs to so called Epstein class which we are going to study in the next section.

Cascades of central returns. Let us have (as in the end of §3) a non-central return on level $m - 1$ followed by the cascade of central returns on levels $m, m = 1, \dots, m + q - 1$, and completed by a non-central return on level $m + q$. The following remarks allow to adjust the previous analysis to this case.

First of all, the first scaling factor μ_m of the cascade stays away from 1 by lemma 4.2.

Assume now that the starting conditions don't hold, that is, the scaling factors stay away from 0. Then μ_{m+q} stay away from 1. For, otherwise we have a high return on the level $m + q - 1$, and the next scaling factor μ_{m+q+1} is tiny (see the argument of Lemma 4.2).

Furthermore, the ratio

$$\frac{|I^{m+q}|}{|I^m|} \geq \delta > 0 \tag{4-2}$$

also stays away from 0. For, otherwise the non-central intervals I_k^{m+q} have a small Poincaré length in the appropriate component of $I^{m+q-1} \searrow I^{m+q}$. This would enforce μ_{m+q+1} to be small again.

Consequently the map

$$g_m^{\circ(q-1)} : I^{m+q-1} \searrow I^{m+q} \rightarrow I^m \searrow I^{m-1}$$

has a bounded distortion. Indeed, since g_m is quadratic up to a bounded distortion, by (4-2) it has bounded non-linearity on $I^m \searrow I^{m+q}$. Since the iterates of $I^{m+q-1} \searrow I^{m+q}$ are disjoint, the claim follows.

Now we can consider the return graph skipping all intermediate levels between $m+q-1$ and m , and to define the orders and ranks of the intervals through this graph. As we have shown, on all levels of the graph we have a priori bounds of the scaling factors, and passage from one level to another has a bounded distortion. Now we can repeat the above argument.

§5. Epstein class and complex bounds.

The goal of this section is to show that an appropriate renormalization of an analytic map of Epstein class is polynomial-like.

Polynomial-like maps. By a *polynomial like map* we mean an analytic branched covering

$$f : \cup_{i=0}^l U_i \rightarrow V$$

where U_i and V are topological disks, $\text{cl } U_i \subset V$. Let us consider a class \mathcal{A} of polynomial-like maps f having a single non-degenerate critical point $c \in U_0$.

Epstein class. Given an interval $I \subset \mathbf{R}$, let $P(I) = \text{int}(\mathbf{C} \setminus (\mathbf{R} \setminus I))$ denote the complex plane slitted along two rays, $D(I)$ denote the disk based upon I as a diameter. Let g be a real analytic map $\cup I_k \rightarrow J$ satisfying the following properties:

(i). For $k \neq 0$ there is the inverse map $(g|_{I_k})^{-1}$ which univalently maps $P(J)$ onto $P(I_k)$.

(ii). $g|_{I_0} = h \circ \phi$, $\phi(z) = (z - c)^2$ is the quadratic map, and h^{-1} has a univalent analytic continuation to $P(J)$.

Let us call this class of maps *Epstein class* \mathcal{E} (compare [S]). Let us start with a general lemma from hyperbolic geometry which was an ingredient of Sullivan's Sector Lemma. It is essential for our complex bounds as well.

Lemma 5.1. *Let $\phi : P(I) \rightarrow P(J)$ be an analytic map which maps I diffeomorphically onto J . Then $\phi D(I) \subset D(J)$.*

Proof. The interval I is a Poincaré geodesic in $P(I)$, and the disk $D(I)$ is its Poincaré neighborhood (of radius independent of I). Since ϕ contracts the Poincaré metric, we are done. \square

High returns. In order to make the following discussion more comprehensible let us dwell first on the case when high returns occur on all levels (compare [LM], §8).

Lemma 5.2. *Let $f \in \mathcal{E}$. Assume that we have high returns on all levels. Then g_n is polynomial-like for some n .*

Proof. We can assume that the scaling factors stay away from zero. Then given an arbitrary small $\sigma > 0$, we can select a moment n such that

$$\mu_n \geq \mu_{n-1}(1 - \sigma). \tag{5-1}$$

Set $q = g_n c \in I^{n-1} \setminus J^n$, $I^m = [\alpha_m, \beta_m]$ where β_m lies on the same side of c as q (see Figure 6).

Let us estimate the Poincaré length P_0 of $[q, \beta_{n-1}]$ in $[\alpha_{n-1}, \beta_{n-2}]$:

$$\begin{aligned} P_0 &\leq \log \left(1 + \frac{1 - \mu_n}{1 + \mu_n} \right) + \log \left(1 + \frac{\mu_{n-1}(1 - \mu_n)}{1 - \mu_{n-1}} \right) \leq \\ &\log \frac{2}{1 + \mu_n} + \log(1 + \mu_n) + o(1) = \log 2 + o(1) \quad \text{as } \sigma \rightarrow 0 \end{aligned} \tag{5-2}$$

(we have replaced $\mu_{n-1}/(1 - \mu_{n-1})$ by $\mu_n/(1 - \mu_n)$ using (5-1) and monotonicity of the linear-fractional function $x/(1 - x)$, $x \in (0, 1)$, and boundedness of μ_n away from 1 (Lemma 4.2)). Let us consider now the representation $g_n = h \circ \phi$ where ϕ is the quadratic map and h is a diffeomorphism of an interval $T \supset \phi I^n$ onto $[\alpha_{n-1}, \beta_{n-2}]$. Set $T = \phi I^n \cup M \cup R$ where M and R are h -pull-backs of $[q, \beta_{n-1}]$ and $[\beta_{n-1}, \beta_{n-2}]$ respectively. By the Schwarz lemma and (5-2)

$$\log \left(1 + \frac{|M|}{|\phi I^n|} \right) \leq P_0 \leq \log 2 + o(1),$$

hence

$$\frac{|M|}{|\phi I^n|} \leq 1 + o(1) \quad \text{as } \sigma \rightarrow 0. \tag{5-3}$$

Figure 6

Let us now take the set $V = D(I^{n-1})$ and pull it back by h . By Lemma 5.1 we will obtain a domain U' contained in $D(\phi I^n \cup M)$. Pulling this domain back by the quadratic map ϕ we obtain by (5-3) a convex domain U_0 which is almost contained in the disk $D(I^n)$.

On the other hand, pulling V back by the univalent branches of g_n , we will get by Lemma 5.1 domains U_k such that $U_k \subset D(I_k^n)$, $k \neq 0$. We conclude that the sets $\text{cl}(U_k)$ are pairwise disjoint and are contained in V . \square

Cut-off iterates. Let us define now a “cut-off” iterate $g_n^{ol}K$ of an interval $K \ni c$

inductively in the following way:

$$g_n^{\circ l} K = g_n(g_n^{\circ(l-1)} K \cap I_n^j)$$

where $I_j^n \ni g_n^{\circ(l-1)} c$. If $K = I^n$ then one boundary point of its cut-off iterates belongs to the ∂I^{n-1} . Let us select the first moment l for which $g_n^{\circ l}(I^n, c) \cap I^n \neq \emptyset$. In the low return case $l > 1$.

Low returns (a particular case).

Lemma 5.3. *Let us select a level n for which (5-1) holds, and find an l as described above. Assume that $g_n^{\circ l} c \notin I^n$. Then appropriate pull-backs of $D(I^{n-1})$ form a polynomial-like map.*

Proof. Denote $q = g_n^{\circ l} c$, $p = g_n^{\circ(l-1)} c$, $I^m = [\alpha_m, \beta_m]$ where β_m lies on the same side of c as q . Let I_i^n be the interval of level n containing p , and let p divides it into the intervals L and G with G closer to c than L . Note that $g_n L = [\alpha_{n-1}, q]$, $g_n G = [q, \beta_{n-1}]$.

Let us consider representation (3-0). Since $h_{n,i}$ has a Koebe space spread over I^{n-2} , one can get the following estimate in the same way as (5-3):

$$\frac{|\phi G|}{|\phi L|} \leq 1 + o(1) \quad \text{as } \sigma \rightarrow 0. \quad (5-4)$$

Set $\mu_n = \mu$. Since the non-linearity of ϕ on I_i^n is at most $\log(1/\mu)$, we obtain the following estimate:

$$\frac{|G|}{|L|} \leq \frac{1}{\mu} (1 + o(1)) \quad \text{as } \sigma \rightarrow 0. \quad (5-5)$$

Let R be the component of $I^{n-1} \setminus I_i^n$ containing the critical point c . Let us estimate now the Poincaré length $P(L, R)$. It follows from (5-5) that

$$|G| \leq \frac{|I_i^n|}{1 + \mu} (1 + o(1)) \leq \frac{1 - \mu}{1 + \mu} |I^{n-1}| (1 + o(1)). \quad (5-6)$$

Hence

$$\begin{aligned} P(L, R) &\leq \log \left(1 + \frac{|G|}{|L|} \right) + \log \left(1 + \frac{|G|}{|R|} \right) + o(1) \leq \\ &\leq \log \left(1 + \frac{1}{\mu} \right) + \log \left(1 + \frac{1 - \mu}{(1 + \mu)^2} \right) + o(1) = \\ &= \log \left(1 + \frac{2}{\mu^2 + \mu} \right) + o(1) < \log \left(1 + \frac{1}{\mu^2} \right) + o(1). \end{aligned} \quad (5-7)$$

Let $T \ni c$ be the pull-back of I^{n-1} by g_n^l . Then $g_n^{l-1} T = L$. Decompose $g_n^{l-1} T = h \circ \phi$ where

$$h : \phi T \cup G' \cup R' \rightarrow L \cup G \cup R = I^{n-1}$$

is a diffeomorphism. By the Schwarz lemma and (5-7),

$$\frac{|G'|}{|\phi T|} \leq \frac{1}{\mu^2} (1 + o(1)).$$

Set $\nu = |T|/|I^{n-1}|$. It follows from the a priori bounds of §4 that $\nu/\mu = |T|/|I^n|$ stays away from 1. Hence the last estimate can be rewritten as

$$\frac{|G'|}{|\phi T|} \leq \frac{\tau^2}{\nu^2} \quad (5-8)$$

with an absolute constant $\tau < 1$.

Let us consider now the disk $D(I^{n-1})$. By Lemma 5.1 its pull-back V'_0 by h is contained in the disk $D(\phi T)$. Pulling V' by the quadratic map ϕ we obtain a domain V_0 based upon the interval T . By (5-8) the outer radius of V_0 around c is less than $(\tau/\nu)|T| = \tau|I^{n-1}|$. Hence $\text{cl}(V_0) \subset D(I^{n-1})$.

We have completed the construction of the central domain V_0 . Let us now construct non-central domains of definition of our polynomial-like map. First of all, take a non-central interval I_i^n of level n and pull the disk $D(I^{n-1})$ back by the corresponding branch of g_n^{-1} . By Lemma 5.1 we obtain a domain $W_i^n \subset I_i^n$.

Let now $x \in I^n \setminus T$. Then there is a moment $s < l$ when g_n -orb(x) is separated from g_n -orbit of T by intervals of level n . In other words $g_n^{\circ s}x \in I_j^n$ while $g_n^{\circ s}T \cap I_j^n = \emptyset$. Hence we can monotonically pull I_j^n back to x by $g_n^{\circ s}$ and obtain an interval $T(x)$. Moreover, we can univalently pull the domain W_j^n back to x and obtain a domain $W(x) \subset D(T(x))$ based upon $T(x)$. A map $g_n^{\circ(s+1)}$ univalently carries this domain onto D^{n-1} .

Let us consider the whole (finite) bunch of intervals I_i^n and $T(x)$, and the corresponding bunch of domains W_i^n and $W(x)$. Let us redenote them as T_i and V_i , $i = 1, \dots$ respectively. Since $V_i \subset D(T_i)$, these domains are pairwise disjoint. They are also disjoint from the central domain V_0 . Indeed, the domains ϕV_i , $i = 0, 1, \dots$ are pairwise disjoint since they are contained in the disks based upon disjoint intervals.

So, we have a polynomial-like map $H : \cup V_i \rightarrow D(I^{n-1})$. \square

A remark on scaling factors. The map H constructed above satisfies the following property: the g_n -first return map to T_0 coincides with the H -first return map to T_0 . Indeed, if $H|T_i = g_n^{\circ m}$, $i \neq 0$, then by looking through the construction we see that $g_n^{\circ k}T_i$ does not intersect T_0 .

Let $T_0 \equiv T^0 \supset T^1 \supset \dots$ be the sequence of the central intervals of the renormalized maps $R^{\circ N}H$, and ν_n be the corresponding sequence of the scaling factors. Then the above property of the first return maps yield that there is an N such that

$$I^{n+N} \subset T^n \subset I^{n+N-1}.$$

Hence the scaling factors ν_n can be estimated through the μ_n and vice versa. It follows that $\nu_n \rightarrow 0$ if and only if $\mu_n \rightarrow 0$.

Low returns: a general construction. The particular construction described above will be one step of the general construction. Let us start with a map

$$h_1 : \cup T_i^{1,0} \rightarrow T^0$$

of class \mathcal{A} . Let us order pairs (n, j) of integer numbers lexicographically. We will construct a finite hierarchical family of intervals $T_i^{n,j}$, $(n, j) \in E \subset \mathbf{Z}^2$, satisfying the following properties:

0) $E = \{(n, j) : 0 \leq n \leq N, 0 \leq j \leq j(n)\}$, $T^0 \equiv T_0^{0,0}$, $T_0^{n,j} \equiv T^{n,j}$ are symmetric intervals containing c ;

Let $(n', j') \in E$ be the lexicographic successor of $(n, j) \in E$. Then

- 1) $T^{n,j} \supset \text{cl}(T_i^{n',j'})$;
- 2) There is a map $h_n : \cup T_i^{n,0} \rightarrow T^0$ of class \mathcal{A} induced by h_1 ;
- 3) For $j > 0$ the intervals $T_i^{n,j}$ are obtained from $T_i^{n,0}$ by pulling back by the central branch of h_n . Moreover, $h_n c \in T^{n,j}$ for all j except for the last one $j(n)$ (the reader can recognize here the cascades of central returns).
- 4) For $n < N$, $h_n T^{n,0} \not\supset c$, while $h_N T^{N,0} \ni c$.

Under such circumstances we will also consider the map

$$H_n : \left(\bigcup_{m < n, i \neq 0} T_i^{m,j} \right) \cup T_0^n \rightarrow T^0$$

such that

$$H_n | T_i^{m,j} = h_m.$$

Note that H_n is a map of class \mathcal{A} with the non-escaping critical point.

Assuming we have already constructed all intervals and maps up to the level $T^{n,0}$, let us do the next step.

High case. If $h_n T^{n,0} \supset T^{n,0}$ we stop.

Central-high case. If $h_n c \in T^{n,0}$ and $h_n T^{n,0} \ni c$, we consider the cascade of central returns until the first moment of high return. It produces the intervals $T_i^{n,j}$ by pulling $T_i^{n,0}$ back by the central branch of h_n . Then we stop.

Low case. Acting as in the above particular case let us consider cut-off iterates $H_n^{\circ m} T^{n,0}$ of the central interval until the first moment l when

$$H_n^{\circ l} T^{n,0} \cap T^{n,0} \neq \emptyset.$$

Then let us pull T^0 back by $H_n^{\circ l}$. It gives us the central interval $T^{n+1,0}$.

Similarly we will construct a non-central interval $T^{n+1,j}(x) \equiv T_i^{n+1,j}$, $x \in \omega(c)$, as the pull-back of appropriate $T_i^{m,j}$, $m < n$, corresponding to the first moment k when the $\text{orb}(c)$ is separated from the $\text{orb}(x)$:

$$H_n^{\circ k} x \in T_i^{m,j} , \quad H_n^{\circ k} c \notin T_i^{m,j}.$$

Define $h_{n+1} | T^{n+1,j}(x) = H_n^{\circ(k+1)}$.

Central-low case. Let $h_n T^{n,0} c \in T^{n,0}$ and $h_n T^{n,0} \not\supset c$. Then let us consider the cascade of central returns until the first one which is low. It produces the intervals $T_i^{n,j}$ by pulling $T_i^{n,0}$ and $T_i^{m,j}$, $(m < n, i \neq 0)$ back by the central branch of h_n . Now let us define a map

$$F : \left(\bigcup_{m \leq n, i \neq 0} T_i^{m,j} \right) \cup T_0^{n,j(n)} \rightarrow T^0.$$

For $m < n$ set $F|T_i^{m,j} = H_n$.

For $m = n, i \neq 0$, set $F|T_i^{n,j} = H_n^{o(j+1)}$.

Finally set $F|T_0^{n,j(n)} = H_n$.

Now taking cut-off iterates $F^{\circ m}T^{n,j(n)}$ we can construct the intervals $T_i^{n+1,j}$ in the same way as in the low case.

Class $\tilde{\mathcal{T}}$. Observe that the above map F does not belong to class \mathcal{T} : the image of the central interval belongs to $\text{int } T^0$. Such a situation always occurs in the end of a cascade of central returns. In order to handle it we need to introduce a wider class of maps.

Let us consider an interval T^{-1} whose interior contains finitely many disjoint closed intervals T_i^0 , with $T^0 \equiv T_0^0$ containing c and symmetric with respect to it. Let $g : \cup T_i^0 \rightarrow T^{-1}$ be a map with negative Schwarzian derivative and a single critical point c satisfying the following properties:

- 1) g diffeomorphically maps any non-central interval $T_i^0, i \neq 0$ onto T^{-1} ;
- 2) $g|T^0$ is a composition of a quadratic map and a diffeomorphism onto T^{-1} with negative Schwarzian derivative;
- 3) If $g(T^0) \cap T_i^0 \neq \emptyset$ and $gc \notin T_i^1$ then $g(T^0) \supset T_i^1$ (“Markov property”).
- 4) $gc \notin T^0$ (non-central return).

Let $\tilde{\mathcal{T}}$ be the class of such maps.

Observe that the first renormalization of a map of class $\tilde{\mathcal{T}}$ belongs to class \mathcal{T} . Moreover, if the scaling factor $\mu = |T^0|/|T^{-1}|$ is small then the scaling factor of the renormalized map is also small. It follows that the results of §3 are still valid if we start with a map of class $\tilde{\mathcal{T}}$: if $\mu < \delta$ then the scaling factors of renormalized maps go down to 0. Let us find the best δ satisfying this property. Then given any small $\sigma > 0$, there is a map $g \in \tilde{\mathcal{T}}$ such that

$$\mu < \delta(1 + \sigma). \quad (5 - 9)$$

Let us start with such a map, and go through the general construction described above. If this construction did not stop then we would have a map of class \mathcal{T} with arbitrary small initial scaling factor such that the scaling factors of the renormalized maps would not go to 0. Since this is impossible, our construction must stop. Then we come up with a map $H = H_N$ of class $\tilde{\mathcal{T}}$ (with $T' = T^{N,j(N)}$ as the central interval) such that $HT' \supset T'$ (high return). Since the scaling factors of H stay away from 0 (compare the above Remark on the scaling factors), we conclude that

$$\mu < \frac{|T'|}{|T^0|}(1 + \sigma) < \frac{\text{dist}(Hc, c)}{|T^0|}(1 + \sigma). \quad (5 - 10)$$

Push-forward. Let us consider now the interval $B = T^{N,0}$ just constructed. The map $H = f^{o(l+1)}$ is unimodal on B . Moreover, there is an interval $A \supset fB$ which is a monotone pull-back of T^0 by f^l .

Let $p = (m, j)$ denote a point of the index set

$$\tilde{E} = \{(m, j) \in E : m \leq N \text{ or } m = N, j = 0\}.$$

We start with $0 \equiv (0, 0)$, and by $p+1$ we mean the point of \tilde{E} which lexicographically follows p . (So, we identify the set \tilde{E} ordered lexicographically with an interval of the set of integers $p = 0, \dots, P$).

Let us consider now a sequence of intervals $A^p = f^{\circ k(p)}A \subset T^{p-1}$ $p = 0, \dots, P$ defined as the last interval of the orbit $\{f^{\circ m}A\}_{m=0}^l$ visiting T^{p-1} , $s(p) = k(p) - k(p+1)$. Note that $A^0 = T^0$ and $k(0) = l$, while for $p > 0$, $A^p \subset T^{p-1} \setminus T^p$. Moreover, $f^{\circ s(p)}$ diffeomorphically maps A^{p+1} onto A^p , and the map $f^{\circ(s(p)-1)}|_f A^{p+1}$ has a Koebe space spread over T^{p-1} . Also the Koebe space of $f^{\circ k(P)}A$ is spread over T^{P-1} .

Finally let us mark in A^p the corresponding iterate $a_p = f^{\circ(s(p)+1)}$ of the critical point. Then $f^{\circ s(p)}$ gives a diffeomorphism between corresponding marked intervals. Note also that $a_0 = Hc$.

Distortion estimates. For $1 \leq p \leq P$ let the marked point a_p divide A^p into intervals L^p and G^p with G^p being closer to the critical point than L^p . Set $\kappa_p = |G^p|/|L^p|$.

Somewhat abusing notations let us denote by $\mu_n = |T^n|/|T^{n-1}|$ new scaling factors. Acting now as in the above particular cases taking into account estimate (5-10) we obtain the following analogue of (5-5):

$$\kappa_0 \leq \frac{1}{\mu_1}(1 + o(1)) \quad \text{as } \sigma \rightarrow 0. \quad (5-11)$$

Let us now estimate κ_{p+1} through κ_p . We have (compare (5-6)):

$$|G^p| \leq \frac{\kappa_p}{1 + \kappa_p}|A^p| \leq \frac{\kappa_p}{1 + \kappa_p}(1 - \mu_p). \quad (5-12)$$

Let R^p be the component of $T^{p-1} \setminus A^p$ containing c . Then we can estimate the Poincaré length $P(L^p, R^p)$ as in (5-7):

$$P(L^p, R^p) \leq \log \left(1 + \frac{2\kappa_p}{1 + \mu_p} \right) \leq \log \left(1 + \frac{\kappa_p}{\mu_p} \right). \quad (5-13)$$

By the Schwarz lemma

$$\frac{|fG^{p+1}|}{|fL^{p+1}|} \leq \frac{\kappa_p}{\mu_p}. \quad (5-14)$$

Since non-linearity of $f|_{A^{p+1}}$ is estimated by $\log(1/\mu_{p+1})$,

$$\kappa_{p+1} \leq \frac{\kappa_p}{\mu_p \mu_{p+1}}. \quad (5-15)$$

Estimates (5-11) and (5-15) yield

$$\kappa_p \leq \frac{1}{(\mu_1 \dots \mu_{P-1})^2 \mu_P} (1 + o(1)). \quad (5-16)$$

Set now $G' = A \setminus fB$. Then using (5-14)-like estimate and (5-16) we conclude that

$$\frac{|G'|}{|fB|} \leq \frac{1}{(\mu_1 \dots \mu_P)^2} (1 + o(1)). \quad (5-17)$$

Finally, we can actually improve this estimate as

$$\frac{|G'|}{|fB|} \leq \tau^2 \frac{1}{(\mu_1 \dots \mu_P)^2} = \left(\tau \frac{|T^0|}{|B|} \right)^2 \quad (5-18)$$

with an absolute $\tau < 1$ (see the argument preceding estimate (5-8)).

Complex pull-backs. Take now the disk $D(T^0)$, and pull it back by the branches of H . Then by (5-18) the central domain V_0 based upon the interval B is compactly contained in $D(T^0)$. The non-central domains are compactly contained in $D(T^0)$, pairwise disjoint, and disjoint from V_0 for the same reason as in the particular cases treated above.

§6. Polynomial-like maps.

The following three pages, up to Lemma 6.3, present a self-contained exposition of a generalized version of the Branner-Hubbard theory [BH], [B]. The reader can see the following differences. We adjust the theory to a local setting of generalized polynomial-like maps which gives us a great flexibility in applications (see, e.g., [L2]) (The original theory was all about cubic polynomials with one escaping critical point). We translate it from the original tableau language to the language of pull-backs which nicely corresponds to the one-dimensional discussion of §2. Finally, we state the main rigidity result of the theory in the parameter plane as a lemma on qc conjugacy of polynomial-like maps with the same combinatorics (Lemma 6.3). A direct proof of this lemma was given by J.Kahn.

After this preparation we complete the proof of Theorem B.

Puzzle-pieces, kids and pull-backs. Let us remember that by a *polynomial like map* we mean an analytic branched covering

$$f : \cup_{i=0}^l U_i \rightarrow V$$

where U_i and V are topological disks, $\text{cl } U_i \subset V$. (The Douady-Hubbard polynomial-like maps [DH] correspond to $l = 1$.) The set

$$K(f) = \{x : f^{\circ n} x \in \cup U_i, n = 0, 1, \dots\}$$

of non-escaping points is called the *filled-in Julia set*. By \mathcal{A} we denote a class of polynomial-like maps f with a single non-degenerate critical point $c \in U_0$. This class is a complex counterpart of the class \mathcal{T} of one-dimensional maps.

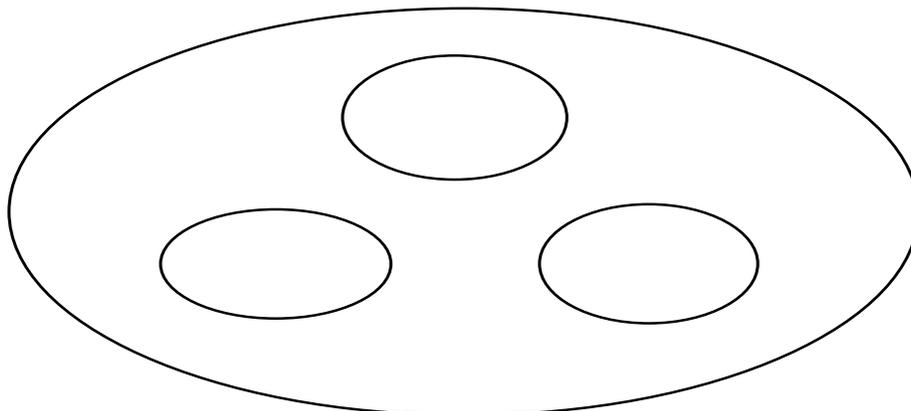


Figure 7

Set $V^{(n)} = f^{-n}V$. The connected components $V_k^{(n)}$ of $V^{(n)}$ are called *puzzle-pieces of depth n* . The puzzle piece of level n containing a point x will be also denoted by $V^{(n)}(x)$. The puzzle-pieces $V_0^{(n)} \equiv V^{(n)}(c)$ containing the critical point are called *critical*. The family of puzzle-pieces is Markov in the following sense: for $n > 0$ $V^{(n)}(x)$ is mapped under f onto a puzzle piece $V^{(n-1)}(fx)$. Moreover, this map is a two-to-one branched covering if $V^{(n)}(x)$ is critical, and a conformal isomorphism otherwise.

Let $f^{\circ m}x \in V_k^{(n)} \equiv W$. Then we can *pull* the puzzle-piece W *back* along the orb $_m(x)$ and come up with the puzzle-piece $V^{(n+m)}(x) \equiv P$. The pull-back is called *univalent* if the map $f^{\circ m} : P \rightarrow W$ is. It is called *quadratic-like* if P is critical and $f^{\circ(m-1)} : fP \rightarrow W$ is univalent.

A critical puzzle-piece $P = V_0^{(n+m)}$ is called a *kid* of $W = V_0^{(n)}$ if it is obtained by the quadratic-like pull back of W along the orb $_m(c)$. If this corresponds to the first return of the critical point back to W , then P is called the *first kid* of W . Repeating this construction we can talk about *grandkids* of the n th generation.

Let us say that f *admits a quadratic-like renormalization* if there is a critical piece $W = V_0^n$ and its first kid $P = V_0^{n+m}$ such that the critical point does not escape P under iterates of $f^{\circ m}$. In such a case $f^{\circ m} : P \rightarrow W$ is a quadratic-like map with a connected Julia set.

The critical point is called *combinatorially recurrent* if its orbit crosses all critical puzzle-pieces.

Two kids lemma. *Assume that $f \in \mathcal{A}$ does not admit a quadratic-like renormalization. Then each critical puzzle-piece has at least two kids.*

Proof. Let us consider a critical puzzle-piece $W = V_0^{(n)}$ and its first kid $P = V_0^{(n+m)}$. Since f does not admit quadratic-like renormalizations, the critical point must escape P under some iterate $f^{\circ mk}$. Let k be the first escape moment; then $f^{\circ km}c \in W \setminus P$. Since the critical point is combinatorially recurrent, we can find the first return moment l of $f^{\circ km}c$ back to W . Then the puzzle-piece $V^{(n-l)}(f^{\circ km}c) \subset W \setminus P$ is obtained by the univalent pull-back of W along the orb $_l(f^{\circ km}c)$. Pulling this piece further along the orb $_{km}(c)$ until it first hits the critical point, we will find the second kid. \square

Let us consider now multiply-connected domains $A^{(n)}(x) = V^{(n)}(x) \setminus V^{(n+1)}$. The mod $(A^{(n)}(x))$ can be defined as the reciprocal of the extremal length of the family of (non-connected) curves separating the outer boundary component of $A^{(n)}(x)$ from all inner ones.

The divergence property. *Let $f \in \mathcal{A}$ does not admit a quadratic-like renormalization. Then for any $z \in K(f)$*

$$\sum_{n=0}^{\infty} \text{mod}(A^{(n)}(z)) = \infty.$$

Proof. Argue as Branner & Hubbard. Let us concentrate on the principle case when the critical point is recurrent, and $z = c$. All critical pieces are descendents of $V = V_0^{(0)}$, and can be graded by generations.

By the previous lemma there are at least 2^n grandkids in n th generation. Since $\text{mod}(A_j^{(n)}) = \text{mod}(V)/2^n$ for any such grandkid, the total sum of moduli over n th generation is at least $\text{mod}(V)$. Hence the total sum of moduli over all descendents is ∞ . \square

Corollary 6.1. *A map $f \in \mathcal{A}$ does not admit a quadratic-like renormalization if and only if the filled-in Julia set is Cantor.*

For this reason we also call maps which don't admit quadratic-like renormalizations *Cantor polynomial-like*. In this case $K(f)$ certainly coincides with the Julia set.

A set $K \subset \mathbf{C}$ is called *removable* if given a neighborhood $U \supset K$, any conformal/qc embedding $\psi : U \setminus K \rightarrow \bar{\mathbf{C}}$ allows conformal/qc continuation across K . The conformal and qc settings in this definition are equivalent (by the Measurable Riemann Mapping Theorem). The following important observation was made by Jeremy Kahn:

Corollary 6.2. *Assume that $f \in \mathcal{A}$ does not admit a quadratic-like renormalization. Then the Julia set $K(f)$ is removable.*

Proof. This follows from the Modular Test on removability (see [SN], §1). \square

Talking about a conjugacy between two polynomial-like maps, we always mean *local* conjugacy in neighborhoods of their filled-in Julia sets.

Lemma 6.3. *Let f and g be two Cantor polynomial-like maps. If they are topologically conjugate by a homeomorphism h then they are qc conjugate by a qc map H which agrees with h on the Julia set.*

Proof. As in Lemma 2.10, let us set $g = \tilde{f}$ and mark the related objects by tilde. Let h be a topological conjugacy. Select an N such that $V^{(N)} \subset \text{Dom}(h)$. Let us consider an isotopy h^t such that $h^0 = h$, h^1 is smooth in a neighborhood of $V^{(N)} \setminus V^{(N+1)}$, $h^1 \circ f = g \circ h^1$ holds in a neighborhood of $\partial V^{(N+1)}$, and $h^t \equiv h$ in a neighborhood of the filled-in Julia set. Since $h^t \equiv h$ near $K(f)$, we can pull this isotopy back to $V^{(N+1)}$ in such a way that for the pull-back $h_1^t \equiv h$ also holds near $K(f)$:

$$\begin{array}{ccc} V^{(N+1)} & \xrightarrow{h_1^t} & \tilde{V}^{(N+1)} \\ f \downarrow & & \downarrow \tilde{f} \\ V^{(N)} & \xrightarrow{h^t} & \tilde{V}^{(N)} \end{array}$$

Then $h_1^t \equiv h^t$ in a neighborhood of $\partial V^{(N+1)}$. Hence we can continue h_1^t to $V^{(N)}$ as h^t .

Now we can pull h_1^t back in the same way, etc. We will obtain a sequence of isotopies h_n^t such that

- (i) h_n^t agree with h on $K(f)$;
- (ii) h_n^t agrees with h_{n-1}^t on $V^{(N)} \setminus V^{(N+n)}$;
- (iii) $f \circ h_n^t = h_{n-1}^t \circ f$.
- (iv) h_n^1 is smooth outside $V^{(n)}$ with a uniformly bounded qc dilatation (since the pull-backs by conformal maps preserve the dilatation).

Hence we can consider a family of maps $H^t = \lim h_n^t$ where the limit is understood in a pointwise sense. Then H^t is an isotopy outside the Julia set, and $H^1|_{V^{(N)} \setminus K(f)}$ is a smooth qc map.

Moreover, since the isotopy h_n^t is concentrated in $V^{(N+n)}$, it carries all puzzle-pieces $V^{(N+n)}(x)$ to the corresponding pieces $\tilde{V}^{(N+n)}(hx)$. Hence H^1 also carries $V^{(N+n)}(x)$ to $\tilde{V}^{(N+n)}(hx)$. Since the diameters of these pieces shrink down to zero, we conclude that H^1 is continuous across the Julia set, and agrees with h .

By Corollary 6.2, H^1 is actually qc. \square

In conclusion let us mention the following result which links local and global settings of the above theory (compare [DH]):

Straightening Theorem. *Any polynomial-like map of class \mathcal{A} is qc conjugate to a polynomial with one non-escaping critical point.* \square

R-symmetric case. If the topological disks U_i and V are symmetric with respect to the real axis, and f preserves the real axis, we call f the **R-symmetric polynomial-like map**.

Lemma 6.4. *If two R-symmetric polynomial-like maps f and g are topologically conjugate on the real line by then the conjugacy can be continued to the complex plane as well.*

Proof. Let us continue the conjugating homeomorphism h to the complex plane in such a way that it respects the dynamics on ∂V^1 . Now let us pull it back (as in lemmas 2.10 and 6.30), so that the pull-backs $h_n : V \rightarrow V$ agree with h on the real line. Then there is a pointwise limit $H = \lim h_n$ conjugating f and g . Since the puzzle-pieces shrink down to zero, H is a homeomorphism. \square

Lemma 6.5. *A R-symmetric polynomial-like map $f \in \mathcal{A}$ admits a quadratic-like renormalization if and only if its restriction to the real line admits a unimodal renormalization.*

Proof. Indeed, set $J = V \cap \mathbf{R}$ and $J_i^{(n)} = V_i^{(n)} \cap \mathbf{R}$ provided the intersection is non-empty. Then $J_i^{(n)}$ are exactly the intervals of the Markov family \mathcal{M}_J (see §2). Let $V_0^{(n+m)}$ be the first kid of $V_0^{(n)}$. Then the property “orb(c) does not escape $V_0^{(n+m)}$ under iterated $f^{\circ m}$ ” is certainly equivalent to “orb(c) does not escape $J_0^{(n+m)}$ under iterated $f^{\circ m}$ ”. The former property means that f admits a quadratic-like renormalization, while the latter one is equivalent to $f|_{\mathbf{R}}$ admits a unimodal renormalization (see Lemma 2.7). \square

Let us remember that \mathcal{T}_{\min} denotes the class of maps $f \in \mathcal{T}$ with the recurrent c and the minimal critical set $\omega(c)$ which does not admit unimodal renormalizations. Putting together Theorem 2.12 and the last three lemmas, we conclude:

Theorem 6.6. *Two R-symmetric polynomial-like maps of class \mathcal{T}_{\min} with the same combinatorial type (that is $\bar{\chi}_f = \bar{\chi}_g$) are qc conjugate.* \square

Hence these maps are qc conjugate on the real line.

The standard family. Given $p, q \in \mathbf{N}$ and a spin function ϵ , let us consider a standard family of maps of class $\mathcal{T}(p, q, \epsilon)$

$$f^t : \bigcup_{k=-q}^p I_k \rightarrow J$$

defined after Lemma 2.7. The quadratic central branch of f^t depends on t while all non-central linear branches are fixed. By Lemma 2.7 any admissible combinatorial type $\bar{\chi}$ can be realised in such a family. Since the lengths of the intervals I^k can be selected arbitrarily, we can construct a map f with a given combinatorial type and arbitrarily small first scaling factor $\mu_0 = |I_0|/|J|$. By §3, the conclusion of Theorem B holds for such a map.

Let us put the intervals I_k into J in such a way that they divide J into commensurable parts. Then the pull-back U_0 of the Euclidian disk $D(J)$ by the central branch has a bounded shape (regardless of the combinatorics and the lengths of I_k). All non-central pull-backs U_k are true disks. Hence if I_k are sufficiently small, f is polynomial-like.

So, we have constructed a polynomial-like map $f \in \mathcal{T}_{\min}$ with a given combinatorial type satisfying the conclusion of Theorem B: Poincaré lengths of the gaps go up to ∞ . But clearly this property is qs-invariant (since qs maps carry commensurable adjacent intervals to commensurable ones). Now Theorem 6.5 yields:

Lemma 6.7. *Theorem B holds for any polynomial-like map $f \in \mathcal{T}_{\min}$.* \square

Proof of Theorem B: concluding argument. Let us consider the subclass \mathcal{T}^* of maps $f \in \mathcal{T}$ for which the conclusion of Theorem B is valid. By the real argument of §3 and §4 this subclass includes all maps with the non-minimal critical set, as well as maps with the minimal critical set of unbounded type.

Let us supply \mathcal{T} with a C^1 -topology. (Observe that the classes $\mathcal{T}(p, q, \epsilon)$ corresponding to the different combinatorics on the first level (see §2) stay far away even in C^0 -topology.) By §3, \mathcal{T}^* is an open subspace in \mathcal{T} . Indeed, given an $\epsilon > 0$, the condition that there is an ϵ -small scaling factor μ_n specifies an open set of maps f . But by §3 this condition forces f to belong to \mathcal{T}^* , provided ϵ is sufficiently small.

Let us take now a map $f \in \mathcal{T}_{\min}$ of bounded type. Assume that $f \notin \mathcal{T}^*$. Then by §4 we can select a sequence of renormalized maps $R^{on(k)}f$ C^1 -converging to a map $g \in \mathcal{T}_{\min}$ of Epstein class. By §5, there is a polynomial-like renormalization h of g . By Lemma 6.7, $h \in \mathcal{T}^*$, hence $g \in \mathcal{T}^*$.

Since \mathcal{T}^* is open, $R^{on(k)}f \in \mathcal{T}^*$ for sufficiently large k . Hence $f \in \mathcal{T}^*$ as well, and this is a Contradiction.

The case of a quadratic polynomial: alternative argument. This case can be treated in a more straightforward manner skipping §§4,5. But then we need the Markov family of Yoccoz puzzle-pieces (see [H] or [M2]) and the renormalization construction of [L2].

This construction goes as follows. Let f be a non-tunable quadratic polynomial in the sense of [DH]. (A real quadratic polynomial is tunable if and only it admits a unimodal renormalization with period > 1). Take a critical puzzle-piece $W = V_0^{(N)}$ such that $V_0^{(N-1)} \setminus V_0^{(N)}$ is a non-degenerate annulus. It satisfies the following “nice” property

similar to (2-4):

$$f^{\circ n}(\partial W) \cap \text{cl}W = \emptyset, \quad n = 1, 2, \dots$$

If $\omega(c)$ is a minimal Cantor set then we can pre-renormalize f on W in the same way as it was described in §2 for the real setting. Namely, consider the first return map to W and select only those pieces of its domain which intersect $\omega(c)$. We obtain a polynomial-like map $g : \cup U_i \rightarrow W$. It does not admit a quadratic-like renormalization since f is non-tunable. Hence it is a Cantor polynomial-like map.

Now if we start with a real quadratic polynomial f then Lemma 6.7 for g implies Theorem B for f .

§7. Absence of attractors.

Let f be a quasi-quadratic map topologically exact on $T = [fc, f^{\circ 2}c]$. By [BL1-3] or by [GJ], [K], $f|T$ has a unique measure-theoretic attractor A , that is, an invariant closed set such that $\omega(x) = A$ for Lebesgue almost all $x \in T$. Moreover, either $A = T$ or $A = \omega(c) \ni c$. Our goal is to prove that the former case holds. It is certainly true if c is not recurrent, or if $\omega(c) = T$.

Let us assume that c is recurrent and $\omega(c) \neq T$. Then we can renormalize f on any nice interval I^0 . Moreover, for I^0 sufficiently short the domain of the pre-renormalized map $g = g_1 : \cup I_k^1 \rightarrow I^0$ is not dense, that is

$$\text{int}(I^0 \setminus \cup I_k^1) \neq \emptyset. \quad (7-0)$$

Since g does not admit unimodal renormalizations, the set $K(g)$ of non-escaping points is nowhere dense (see the argument of Lemma 2.3). Our goal is to prove that this set has zero Lebesgue measure.

Combinatorics of the first return maps. Let $I^0 \supset I^1 \supset I^2 \supset \dots$ be the sequence of the first kids of I^0 . Let us construct inductively the return maps f_n to I^{n-1} . Let $\mathbf{L}^n = \cup L_i^n$ be the domain of definition of the f_n where L_i^n are intervals with L_0^n as the central interval, $\mathbf{L}_*^n = \cup_{i \neq 0} L_i^n$. Denote by \mathcal{L}^n the family of these intervals, and by \mathcal{L}_*^n the family of non-central intervals. Let $G^n = \text{int}(I^{n-1} \setminus \mathbf{L}^n)$ be the union of gaps. All these sets are c -symmetric for $n > 1$.

To start induction observe that $\mathbf{L}^1 = \cup I_k^1$, $f_1 = g_1$. Moreover, G^1 is non-empty by (7-0). Let us denote by \mathcal{G}_*^n the free semigroup generated by \mathcal{L}_*^n . For a word $\gamma \in \mathcal{G}_*^n$ denote by the same letter γ the interval whose itinerary through \mathbf{L}^n is given by γ . Let l be the length of γ . Let us consider the subsets γ_r and γ_e of this interval such that $f_n^{\circ l} \gamma_0^n = I^n$ and $f_n^{\circ l} \gamma_e = G^n$, that is, the former subset goes to the central interval, while the latter one escapes through the gaps set G^n . Finally, let $\partial \mathcal{G}^n$ mean the set of infinite words in letters \mathcal{L}_*^n which we identify with the corresponding f_n -invariant Cantor set. Since this set does not contain the critical point, it is hyperbolic and has zero measure. Hence the sets $\cup \gamma_e$ and $\cup \gamma_r$ cover almost completely the set \mathbf{L}_*^n . Let

$$E^n = G^n \bigcup_{\gamma \in \mathcal{G}_*^n} \gamma_e, \quad R^n = I^n \bigcup_{\gamma \in \mathcal{G}_*^n} \gamma_r$$

be the full sets of returning and escaping points. Because of the above remark, their union has full measure in I^{n-1} . Moreover, we have a *transition map* F_n from R^n to the central interval I^n which maps diffeomorphically each interval γ_e onto I^n .

Now, in order to construct the return map f_{n+1} of the next level, just consider the pull-back \mathbf{L}^{n+1} of E^n by the central branch $f_n|_{I^n}$, and define $f_{n+1} = F_n \circ f_n|_{\mathbf{L}_{n+1}}$.

Density estimates.

Lemma 7.1. *Let $n > 1$. Then for any word $\gamma \in \mathcal{G}_*^n$*

$$\frac{|\gamma_e|}{|\gamma_r|} \geq \frac{1}{2} \frac{|G^n|}{|I^n|}.$$

Proof. Let l be the length of the word γ . Then $f_n^{\circ l}$ diffeomorphically maps γ onto I^{n-1} . By the Minimum Principle, there is a component L of $I^{n-1} \setminus J^n$ such that

$$\inf_L Dg_n^{\circ(-l)} \geq \sup_{I^n} Dg_n^{\circ(-l)}.$$

Let $H = G^n \cap L$. By the symmetry, $|H| = (1/2)|G^n|$. Hence

$$\frac{|\gamma_e|}{|\gamma_r|} \geq \frac{|g_n^{\circ(-l)} H|}{|g_n^{\circ(-l)} I^n|} \geq \frac{|H|}{|I^n|} = \frac{1}{2} \frac{|G^n|}{|I^n|}.$$

□

Lemma 7.2. *Let U be any interval such that $\partial U \subset \partial \mathbf{L}^n \cup \partial I^{n-1}$ and $U \cap I^n = \emptyset$. Then*

$$\frac{\text{dens}(E^n|U)}{\text{dens}(R^n|U)} \geq \frac{1}{2} \frac{|G^n|}{|I^n|}.$$

Proof. There is a family \mathcal{L} of intervals $\gamma \in \mathcal{G}_*^n$ such that we have the following coverings up to sets of measure zero:

$$(U \cap E^n) \setminus G^n = \bigcup_{\gamma \in \mathcal{L}} \gamma_e \pmod{0}, \quad U \cap R^n = \bigcup_{\gamma \in \mathcal{L}} \gamma_r \pmod{0}.$$

Apply now the previous lemma. □

Let us state an elementary lemma about the quadratic map $\phi : x \mapsto (x - c)^2$.

Lemma 7.3. *Let X be a c -symmetric measurable set in a c -symmetric interval I . Then*

$$\text{dens}(X|I) \geq \frac{1}{2} \text{dens}(\phi X|\phi I).$$

□

Let us remember the notation $\mu_n = |I^n|/|I^{n-1}|$ for the scaling factors. Let $\delta_n = \text{dens}(G^n|I^{n-1})$. If U is any interval such that $\partial U \subset \partial \mathbf{L}^n \cup \partial I^{n-1}$ (but perhaps $U \supset I^n$) then Lemma 7.2 implies

$$\frac{\text{dens}(E^n|U)}{\text{dens}(R^n|U)} \geq \frac{1}{2} \delta_n \frac{1 - \mu_n}{\mu_n}. \tag{7-1}$$

Pulling this back by $f_n|I^n = h \circ \phi$ with the h of bounded distortion we obtain the following recurrent estimate:

$$\delta_{n+1} \geq a \delta_n \frac{1 - \mu_n}{\mu_n} \tag{7-2}$$

with an absolute a . It follows that $\delta_{n+1} \geq 2\delta_n$, provided $\mu_n \leq \bar{\mu}$ is sufficiently small.

If $\mu_m > \bar{\mu}$ is not too small then by the results of §3 we are in the tail of a cascade of central returns. Let $m+q-1$ be the first non-central return level of this cascade, $q \geq 1$. Let H be the component of $I^{m+q-1} \setminus I^{m+q}$ containing the critical value $f_{m+q}c$. Let us consider the intervals $V = H \cap f_{m+q}I^{m+q}$ and $U = V \cup I_k^{m+q}$ where $f_{m+q}c \in I_k^{m+q}$. Then U satisfies the assumptions of Lemma 7.2. On the other hand, by §3, the Poincaré length $P(I_k^{m+q}|H)$ is very small. We conclude that

$$\text{dens}(G^{m+q}|V) \geq (1 - \epsilon) \text{dens}(G^{m+q}|U) \quad (7-3)$$

with a very small ϵ .

Further, $f^q : U \rightarrow f^qU$ is a map of bounded distortion. Indeed, since $|I^{m+q}|/|I^m| \geq \tau\bar{\mu}$, $\tau > 0$, stays away from 0, f_m has a bounded non-linearity on the interval $I^m \setminus I^{m+q}$. Since the intervals f^kU , $k = 0, \dots, m$, are pairwise disjoint, we have the bounded distortion of f^q . Hence there is an absolute $a > 0$ such that

$$\text{dens}(G^{m+q}|U) \geq a \text{dens}(G^m|f^qU) \quad (7-4)$$

Now Lemma 7.2 and estimates (7-3), (7-4) yield

$$\text{dens}(G^{m+q}|V) \geq a\delta_m \quad (7-5)$$

(as usually in analysis, an absolute constant a may have different values in different estimates).

Now let us pass to the next level using Lemma 7.3. Let $W \subset I^{m+q}$ be the pull back of V by $f_{m+q}|I^{m+q}$. We conclude that

$$\text{dens}(G^{m+q+1}|W) \geq a\delta_m \quad (7-6)$$

with yet another $a > 0$. On the other hand, by §3 $|I^{m+q+1}|/|W|$ is very small (exponentially small in terms of κ). Hence

$$\frac{\delta_{m+q+1}}{\mu_{m+q+1}} \geq \frac{|G^{m+q+1}|}{|I^{m+q+1}|} \geq \frac{\text{dens}(G^{m+q+1}|W)}{\text{dens}(I^{m+q+1}|W)} > A\delta_m$$

with a big A (for a big m). Passing now to the next level using (7-2), we conclude

Theorem 7.4. *The densities $\delta_n = \text{dens}(G^n|I^{n-1})$ of gap sets stay away from 0, provided n is not in the tail of a long cascade of central returns or immediately after the cascade. Moreover, these densities grow at least exponentially with $\kappa(n)$.*

Concluding argument. Assume that the set $K(g)$ of non-escaping points has positive measure. Let $X = \{x \in K(g) : \omega(x) \ni c\}$. By [BL3], $\text{dens}(X|I^n) = 1$ (The argument: take a density point $x \in X$ and consider the first moment l when $g^l x \in I^n$. Then the corresponding pull-back $T \ni x$ is mapped under g^l onto I^n with a bounded distortion.)

On the other hand, Theorem 7.4 says that $\text{dens}(K(g)|I^n) \rightarrow 0$ for an appropriate subsequence of levels. This contradiction completes the proof of theorem A.

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