

# JACOBIANS WITH GROUP ACTIONS AND RATIONAL IDEMPOTENTS

ANGEL CAROCCA AND RUBÍ E. RODRÍGUEZ

## CONTENTS

1. Introduction	1
2. Preliminaries	2
3. The construction of primitive rational idempotents	4
4. Rational idempotents invariant under a subgroup	8
5. An application: Group actions on abelian varieties	10
5.1. The case of Jacobians	11
Appendix A. An example	15
Appendix B. Another example: a quasi-Prym	19
References	20

## 1. INTRODUCTION

It is known that every finite group acts on some curve, hence on some Jacobian, and therefore on some principally polarized abelian variety (*p.p.a.v.*). On the other hand, every action of a finite group on an abelian variety induces an isogeny decomposition into factors related to the rational irreducible representations of the given group.

However, the existence of this decomposition is obtained from the theory of group representations, and there has been no general geometric interpretation of the factors.

The object of this paper is to prove some general results about rational idempotents for a finite group  $G$  and deduce from them geometric information about the components that appear in the decomposition of the Jacobian variety  $JW$  of a curve  $W$  with  $G$ -action.

The simplest case of such a decomposition is when the group  $G$  is the group with two elements  $G \cong \mathbb{Z}/2\mathbb{Z} = \langle j : j^2 \rangle$  acting on a curve  $W$ . It is clear that then the Jacobian variety  $JW$  of  $W$  also has an involution  $j$  acting on it and, furthermore, that  $JW$  is isogenous to the product  $B_1 \times B_2$ , where  $B_1 = \text{Image}(1 + j)$  is the  $G$ -invariant part of  $JW$  and  $B_2 = \text{Image}(1 - j)$  is the anti-invariant part. This decomposition was already observed by Wirtinger in [W] and used by Schottky and Jung in [S-J]. Observe that  $B_1$  is isogenous to the Jacobian  $JW_G$  of the quotient curve  $W_G = W/G$  and that  $B_2$  was later called by Mumford [Mum] the Prym variety  $P(W/W_G)$  of the given cover.

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The decomposition of Jacobians (or, more generally, abelian varieties) with group actions has been studied in different settings, with applications to theta functions, to the theory of integrable systems and to the moduli spaces of principal bundles of curves. For other special groups, there are the dihedral group  $D_p$  with  $p$  prime in [Ri], the symmetric group  $S_3$  in [R-R-1], all the subgroups of  $S_4$  in [R-R-2], the alternating group  $A_5$  in [SA-1], some Weyl groups in [K],  $S_4$  and  $WD_4$  in [D-M], the dihedral groups  $D_n$  in [C-R-R] and  $S_5$  in [SA-2] and [L-R-2].

For all these cases except the last two, it was shown that the factors appearing in the decomposition were all of the form of either Jacobians or Prym varieties of intermediate covers. In the last two some of the factors were not of this type, but they could still be identified as (connected components of the origin of) intersections of Prym varieties of intermediate covers.

Mérindol in [Me] (for abelian varieties) and Donagi in [D] (for Jacobians) gave a decomposition in the case when all the irreducible rational representations of the group are absolutely irreducible: i.e., when they stay irreducible when considered as complex representations. In [L-R-1], Lange and Recillas have recently given the general decomposition for any abelian variety admitting a group action.

In the case studied in [Me] and [D], there are formulae giving rational projectors whose images are the required factors in the decomposition. These formulae are constructed based on the fact that the condition imposed on the group  $G$  implies the existence of an essentially unique  $G$ -invariant inner product on each of the vector spaces corresponding to the irreducible representations, and therefore cannot be generalized.

In this work we find the corresponding rational projectors for the general case, as well as describe rational projectors invariant under any given subgroup. From them we deduce the decomposition of any Prym or Jacobian variety of an intermediate cover, in the case of a Jacobian with group action.

These explicit constructions allow geometric descriptions of the factors appearing in the decomposition of a Jacobian with group action. For instance, we give a necessary and sufficient condition for a Prym variety of an intermediate cover to be such a factor.

In Section 2 we recall some basic results on irreducible representations and idempotents in order to fix the notation.

We then describe two methods for the construction of rational idempotents of a finite group  $G$ : one for primitive idempotents in Section 3 and another for idempotents that are bilaterally invariant under any given subgroup of  $G$  in Section 4.

In Section 5 we apply our constructions to the decomposition of abelian varieties and Jacobians with group action.

The paper concludes with two appendices, where we include two examples; both groups have rational irreducible representations which are *not* absolutely irreducible and hence could not have been studied without this general theory. The first one illustrates our constructions of rational idempotents and the second one exhibits a factor in the decomposition of the corresponding Jacobians which is not of the previous known types; that is, it is neither a Jacobian nor a Prym nor an intersection of Prym varieties of intermediate covers.

## 2. PRELIMINARIES

Let  $G$  be a finite group. With relation to representations, we follow the notations in [S] and [C-R]. The known results quoted in this section may be found there as well.

We denote by  $\text{Irr}_F(G)$  the set of irreducible representations of  $G$  over the field  $F$ .

If  $V \in \text{Irr}_{\mathbb{C}}(G)$ , we let  $L$  denote the field of definition of  $V$  and let  $K$  denote the field obtained by adjoining to the rational numbers  $\mathbb{Q}$  the values of the character  $\chi_V$ ; then  $K \subseteq L$  and  $m = m_{\mathbb{Q}}(V) = [L : K]$  is the Schur index of  $V$ .

If  $H$  is any subgroup of  $G$ ,  $\rho_H$  will denote the representation of  $G$  induced by the trivial representation of  $H$ . If we denote by  $\langle U, V \rangle$  the usual inner product of the characters of the representations  $U$  and  $V$ , and by  $V^H$  the subspace of  $V$  fixed under  $H$ , then it follows immediately from the Frobenius reciprocity Theorem that

$$\langle \rho_H, V \rangle = \dim_{\mathbb{C}} V^H .$$

It is known that if  $\text{Gal}(L/\mathbb{Q})$ ,  $\text{Gal}(L/K)$  and  $\text{Gal}(K/\mathbb{Q})$  denote the respective Galois groups, then each representation  $V^\sigma$  conjugate to  $V$  by an element  $\sigma$  in  $\text{Gal}(L/\mathbb{Q})$  is also defined over  $L$  and both  $V$  and  $V^\sigma$  share the same field  $K$ . Furthermore,  $V^\sigma$  is equivalent to  $V$  if and only if  $\sigma$  is in  $\text{Gal}(L/K)$ .

We denote by  $U$  the representation

$$U = \bigoplus_{\tau \in \text{Gal}(L/K)} V^\tau \simeq mV$$

and remark that it is actually defined over  $K$ . The Schur index  $m$  of  $V$  is minimal with respect to this property.

It is also known that the irreducible rational representation  $\mathcal{W}$  of  $G$  associated to  $V$  is characterized by its decomposition into irreducible components over the corresponding fields as follows.

$$(2.1) \quad K \otimes_{\mathbb{Q}} \mathcal{W} \simeq \bigoplus_{\varphi \in \text{Gal}(K/\mathbb{Q})} U^\varphi , \text{ and}$$

$$(2.2) \quad L \otimes_{\mathbb{Q}} \mathcal{W} \simeq \bigoplus_{\sigma \in \text{Gal}(L/\mathbb{Q})} V^\sigma \simeq \bigoplus_{\varphi \in \text{Gal}(K/\mathbb{Q})} (mV)^\varphi$$

The (unique) central idempotent  $e_V$  of  $L[G]$  that generates the simple subalgebra associated to  $V$  and the (unique) central idempotent  $e_{\mathcal{W}}$  of  $\mathbb{Q}[G]$  that generates the simple subalgebra associated to  $\mathcal{W}$  are given as follows.

$$(2.3) \quad e_V = \frac{\dim V}{|G|} \sum_{g \in G} \chi_V(g^{-1}) g , \text{ and}$$

$$e_{\mathcal{W}} = \frac{\dim V}{|G|} \sum_{g \in G} \text{Trace}_{K/\mathbb{Q}}(\chi_V(g^{-1})) g .$$

Note that  $e_V$  is in fact an element of  $K[G]$  and that it coincides with the (unique) central idempotent  $e_U$  of  $K[G]$  that generates the simple subalgebra associated to  $U = mV$ ; furthermore, the following relations hold.

$$e_V = e_{V^\tau} \text{ for all } \tau \in \text{Gal}(L/K)$$

and

$$e_U e_{U^\varphi} = 0 \text{ for all } \varphi \in \text{Gal}(K/\mathbb{Q}) .$$

It is also known that the simple algebras  $L[G]e_V$ ,  $K[G]e_V$  and  $\mathbb{Q}[G]e_W$  may be decomposed into the direct sum of  $n$ ,  $\frac{n}{m}$  and  $\frac{n}{m}$  minimal left ideals, respectively. In other words, there exist primitive idempotents  $\{\ell_1, \dots, \ell_n\}$ ,  $\{k_1, \dots, k_{\frac{n}{m}}\}$  and  $\{f_1, \dots, f_{\frac{n}{m}}\}$  in the respective subalgebras, orthogonal among themselves, such that the following relations hold.

$$(2.4) \quad \begin{aligned} e_V &= \ell_1 + \dots + \ell_n \quad \text{in } L[G]e_V \\ e_V &= k_1 + \dots + k_{\frac{n}{m}} \quad \text{in } K[G]e_V \\ e_W &= f_1 + \dots + f_{\frac{n}{m}} \quad \text{in } \mathbb{Q}[G]e_W \end{aligned}$$

These primitive idempotents are far from being unique, and except for special cases, there are no general formulas for writing down neither the  $k_j$  nor the  $f_j$ .

A known case ([Me]) is the following. Suppose that the rational irreducible representation  $\mathcal{W}$  is *absolutely irreducible*; that is, it stays irreducible when considered as a complex representation. Then  $L = K = \mathbb{Q}$ ,  $m = 1$  and  $n = \frac{n}{m} = \dim V = \dim \mathcal{W}$ .

In this case one can find the  $\ell_j = k_j = f_j$ 's as follows. The hypothesis on  $\mathcal{W}$  implies that there is a unique (up to multiplication by a positive scalar)  $G$ -invariant inner product on  $\mathcal{W}$ , denoted by  $\langle \cdot, \cdot \rangle$ . Let  $\{w_1, \dots, w_n\}$  denote an orthogonal basis on  $\mathcal{W}$  and define

$$\ell_j = \frac{n}{|G| \|w_j\|^2} \sum_{g \in G} \langle w_j, g(w_j) \rangle g.$$

To show that these  $\ell_j$ 's satisfy the required properties, they are rewritten as follows

$$(2.5) \quad \ell_j = \frac{\dim V}{|G|} \sum_{g \in G} r_{jj}(g^{-1}) g, \quad \text{for } j \in \{1, \dots, n\},$$

where  $(r_{ik}(g))$  are the (rational) coefficients of the matrix of the element  $g$  with respect to the given basis. Then (2.4) is clear and that these  $\ell_j$ 's are orthogonal idempotents is a consequence of the orthogonality relations for characters (see [S]).

For the general case of any complex irreducible representation  $V$  of  $G$ , of dimension  $n$ , one may *define*  $\ell_j$ 's by the right hand side of (2.5), where the  $r_{ik}$ 's are now the coefficients of the given  $L$ -representation of  $G$ . The same orthogonality relations show that these  $\ell_j$ 's are primitive orthogonal idempotents in  $L[G]$  whose sum is  $e_V$ .

In this paper we give a general construction for  $k_j$ 's and  $f_j$ 's.

### 3. THE CONSTRUCTION OF PRIMITIVE RATIONAL IDEMPOTENTS

Let  $V$  be a complex irreducible representation of a finite group  $G$ , with associated fields  $L$  and  $K$  as in Section 2.

**Proposition 3.1.** *Let  $V$  be in  $\text{Irr}_{\mathbb{C}} G$ .*

*Denote by  $n = \dim V$ , by  $e = e_V$  the corresponding central idempotent in  $L[G]$ , and by  $\text{Gal}(L/K) = \{\tau_1 = 1, \dots, \tau_m\}$ , with  $m$  the Schur index of  $V$ .*

*Consider any  $n$  primitive orthogonal idempotents  $\ell_1, \dots, \ell_n$  in the simple algebra  $L[G]e_V$  satisfying (2.4). For instance, the  $\ell_j$  given by (2.5).*

*Let  $M_j = \sum_{h=1}^m L[G]\tau_h(\ell_j)$  be the sum of the corresponding left ideals.*

Then the stabilizer in  $\text{Gal}(L/K)$  of each left ideal  $L[G]\ell_j$  is trivial and the sum in  $M_j$  is direct for each  $j$  in  $\{1, \dots, n\}$ .

Furthermore, the left ideals  $L[G]\tau_h(\ell_j)$  in each  $M_j$  are permuted among themselves by the action of  $\text{Gal}(L/K)$  and for each pair  $j, k$  in  $\{1, \dots, n\}$ , either  $M_j = M_k$  or  $M_j \cap M_k = \{0\}$ .

*Proof.* It is clear that  $\tau(\ell_j)$  is a primitive idempotent in  $L[G]e_V$  for each  $\tau$  in  $\text{Gal}(L/K)$ .

Let  $H$  denote the stabilizer of  $\ell_j$  in  $\text{Gal}(L/K)$ , and let  $\{\tau_1 = 1, \tau_2, \dots, \tau_r\}$  be a right transversal of  $H$  in  $\text{Gal}(L/K)$ .

Consider

$$M = L[G]\ell_j + \dots + L[G]\tau_r(\ell_j).$$

Then  $M$  is a left  $K[G]$ -module contained in  $L[G]e_V$  with  $0 < \dim(M) \leq m \dim_L(V) = \dim_K(U)$ .

But the minimality of  $m$  then implies that  $\dim(M) = m \dim(V)$  and therefore  $H = \{1\}$ ,  $r = m$  and

$$M = M_j = L[G]\ell_j \oplus \dots \oplus L[G]\tau_m(\ell_j)$$

The other results are now clear, since  $M_j \cap M_k$  is a left  $K[G]$ -module contained in  $L[G]e_V$  with dimension at most  $m \dim(V)$ .  $\square$

*Remark 3.2.* At this point one could consider the elements of  $K[G]e_V$  given by  $\tilde{k}_j = \sum_{h=1}^m \sigma_h(\ell_j)$  as possible candidates to be the sought  $k_j$ . However, these elements are not necessarily idempotents, since  $\sigma_h(\ell_j)$  and  $\sigma_k(\ell_j)$  may be non orthogonal to each other (e.g., see the example in Appendix A). Furthermore, even if the  $\tilde{k}_j$  are generating idempotents for  $M_j$  for each  $j$  in  $\{1, \dots, n\}$ , there would still remain the question of choosing  $\frac{n}{m}$  among them in such a way that they add up to  $e_V$ .

Our next result gives an explicit way of finding  $k_j$ 's with all the required properties.

**Theorem 3.3.** *Let  $V$  be in  $\text{Irr}_{\mathbb{C}} G$  and denote  $n = \dim(V)$  and  $m = m_{\mathbb{Q}}(V)$ .*

*Consider any  $n$  primitive orthogonal idempotents  $\ell_1, \dots, \ell_n$  in the simple algebra  $L[G]e_V$  satisfying (2.4).*

*Then there exist  $\frac{n}{m}$  primitive orthogonal idempotents  $\{u_s^1\}_{s=1}^{\frac{n}{m}}$  in different left ideals  $J_s = L[G]\ell_{j_s}$  such that the following results hold.*

- (1) *The element  $u_s^h = \tau_h(u_s^1)$  is a primitive generating idempotent for the left ideal  $\tau_h(J_s) = L[G]\tau_h(\ell_{j_s})$ , for each  $\tau_h$  in  $\text{Gal}(L/K)$  and for each  $s$  in  $\{1, \dots, \frac{n}{m}\}$ .*
- (2) *The  $\{u_s^h\}$  satisfy*

$$u_s^h u_t^l = \begin{cases} u_s^h, & \text{if } l=h \text{ and } t=s; \\ 0, & \text{otherwise.} \end{cases}$$

- (3) *The algebra  $L[G]e_V$  decomposes as the direct orthogonal sum of the left ideals generated by  $\{\sigma_h(u_s^1)\}$  and furthermore we have*

$$e_V = \sum_{s=1}^{\frac{n}{m}} \sum_{h=1}^m u_s^h = \sum_{s=1}^{\frac{n}{m}} \sum_{\sigma \in \text{Gal}(L/K)} \sigma(u_s^1).$$

*Proof.* If we denote each minimal left ideal  $L[G]\sigma_h(\ell_s)$  by  $J_s^h$ , it follows from Proposition 3.1 that  $M_1 = \sum_{h=1}^m J_1^h$  is a direct sum of the given left ideals.

If  $m = n$ , then  $L[G]e_V = M_1$ . If  $m < n$ , then there exists  $\ell_j$  with  $j \neq 1$  such that  $\ell_j \notin M_1$ . By renumbering the  $\ell_j$  if needed, we may assume this is  $\ell_2$ . Again from Proposition 3.1 it follows that  $M_2$  is a direct sum of the corresponding ideals  $J_2^h$  and, furthermore, that the sum  $M_1 + M_2$  of primitive left ideals  $\{J_s^h\}_{\substack{s=1,2 \\ h \in \{1, \dots, m\}}}$  is also direct.

If  $M_1 + M_2$  is a proper subset of  $L[G]e_V$ , then we may assume that for  $\ell_3$  the corresponding  $M_3$  has trivial intersection with  $M_1 + M_2$ . Then, as before, the sum  $M_1 + M_2 + M_3$  of minimal left ideals  $\{J_s^h\}_{\substack{s=1,2,3 \\ h \in \{1, \dots, m\}}}$  is direct.

In any case, it is clear that the process terminates after precisely  $\frac{n}{m}$  steps, when we obtain the following direct sum decomposition into minimal left ideals.

$$(3.1) \quad \begin{aligned} L[G]e_V &= J_1^1 \oplus J_1^2 \oplus \dots \oplus J_1^m \\ &\oplus J_2^1 \oplus J_2^2 \oplus \dots \oplus J_2^m \\ &\vdots \\ &\oplus J_{\frac{n}{m}}^1 \oplus J_{\frac{n}{m}}^2 \oplus \dots \oplus J_{\frac{n}{m}}^m . \end{aligned}$$

Hence there exist unique primitive idempotents  $u_s^h$  in  $J_s^h$  such that

$$(3.2) \quad \begin{aligned} e_V &= u_1^1 + u_1^2 + \dots + u_1^m \\ &+ u_2^1 + u_2^2 + \dots + u_2^m \\ &\vdots \\ &+ u_{\frac{n}{m}}^1 + u_{\frac{n}{m}}^2 + \dots + u_{\frac{n}{m}}^m . \end{aligned}$$

It follows that the  $u_s^h$  are all orthogonal to each other, and we claim that

$$u_s^h = \sigma_h(u_s^1)$$

for each  $s$  and  $h$ .

Indeed, if we apply any  $\tau_h \in \text{Gal}(L/K)$  to the identity (3.2), the left hand side is invariant and we obtain

$$\begin{aligned} e_V &= \tau_h(u_1^1) + \tau_h(u_1^2) + \dots + \tau_h(u_1^m) \\ &+ \tau_h(u_2^1) + \tau_h(u_2^2) + \dots + \tau_h(u_2^m) \\ &\vdots \\ &+ \tau_h(u_{\frac{n}{m}}^1) + \tau_h(u_{\frac{n}{m}}^2) + \dots + \tau_h(u_{\frac{n}{m}}^m) . \end{aligned}$$

where each  $\tau_h(u_s^k)$  is a primitive idempotent in some (unique)  $J_s^{k_h} = \tau_h(J_s^k)$ .

In particular,  $\tau_h(u_s^1)$  is in  $\tau_h(J_s^1) = J_s^h$  and then the uniqueness of the decomposition (3.2) for the given ideals (since the right hand side of (3.1) is a direct sum) implies the desired result.  $\square$

*Remark 3.4.* Starting from the  $\ell_j$ , the  $u_s^h$  of Theorem 3.3 may be found explicitly, possibly with some computer help. Our example in Appendix A was built using the GAP program.

The general algorithm is as follows, where we follow the notation in the proof of the Theorem.

- (1) Find a basis  $B_s^h$  for each vector space  $J_s^h$  appearing in (3.1). It actually suffices to find a basis  $B_s^1$  for each vector space  $J_s^1$ , since then a basis for each  $J_s^h$  is given by  $\tau_h(B_s^1)$ .
- (2) Since the sum on the right hand side of (3.1) is direct, it follows that the union  $B$  of all the bases  $B_s^h$  is a basis for  $L[G]e_V$ .

Computing the coordinates of  $e_V$  with respect to the basis  $B$ , we find the desired primitive orthogonal idempotents  $u_s^h$  in each  $J_s^h$ .

Alternatively, once the decomposition (3.1) has been obtained, one may use the method described in [F] to actually find the  $u_s^h$ .

**Corollary 3.5.** *Consider the  $\{u_s^1\}_{s=1}^{\frac{n}{m}}$  constructed in Theorem 3.3 and set*

$$(3.3) \quad k_s = \sum_{\tau \in \text{Gal}(L/K)} \tau(u_s^1)$$

for each  $s$  in  $\{1, \dots, \frac{n}{m}\}$ .

Then  $\{k_s\}_{s=1}^{\frac{n}{m}}$  is a set of primitive orthogonal idempotents in  $K[G]$  such that

$$(3.4) \quad e_V = k_1 + \dots + k_{n/m} .$$

*Proof.* The results follow from the definition of the  $k_s$  and the properties of the  $\tau(u_s^1)$  given by the Theorem.

In particular, the primitivity of each  $k_s$  follows from the fact that the dimension of the  $L$ -vector space  $L[G]k_s = M_s$  is equal to  $m \dim_L(V) = \dim_K(U)$ .  $\square$

**Corollary 3.6.** *Given the idempotents  $\ell_j$ , construct the idempotents  $k_s$  as in Corollary 3.5 and let  $\mathcal{W}$  denote the rational irreducible representation associated to  $V$ .*

For each  $s$  in  $\{1, \dots, \frac{n}{m}\}$  set

$$(3.5) \quad f_s = \sum_{\varphi \in \text{Gal}(K/\mathbb{Q})} \varphi(k_s) = \sum_{\sigma \in \text{Gal}(L/\mathbb{Q})} \sigma(u_s^1) .$$

Then the  $f_s$  are primitive orthogonal idempotents in  $\mathbb{Q}[G]e_{\mathcal{W}}$  and they satisfy

$$(3.6) \quad e_{\mathcal{W}} = f_1 + f_2 + \dots + f_{\frac{n}{m}} .$$

*Proof.* It is clear that each element  $\varphi$  in  $\text{Gal}(K/\mathbb{Q})$  gives rise to primitive idempotents  $\varphi(k_s)$ , each corresponding to the  $K$ -irreducible representation  $U^\varphi$ .

Since the  $k_s$ 's are orthogonal among themselves and their images under different  $\varphi$ 's are associated to non conjugate representations, we obtain the following relations for every  $\varphi_i, \varphi_j$  in  $\text{Gal}(K/\mathbb{Q})$  and for every  $k_s, k_t$ .

$$\varphi_i(k_s) \varphi_j(k_t) = \begin{cases} \varphi_i(k_s) & \text{if } j = i \text{ and } t = s; \\ 0 & \text{otherwise.} \end{cases}$$

Using these relations, it is easy to verify that the  $f_s$ 's are indeed orthogonal idempotents. That their sum is  $e_W$  is an immediate consequence of (3.4) and (2.3).

All that is left to show is that each  $f_s$  is primitive, so assume not; that is, assume there are rational orthogonal idempotents  $p_1$  and  $p_2$  such that  $f_s = p_1 + p_2$ . But then it follows from (3.5) that  $k_s = f_s e_U = p_1 e_U + p_2 e_U$ , and this is a decomposition of the primitive idempotent  $k_s$  as the sum of two elements in  $K[G] e_U$ .

We will now prove that the  $p_j e_U$  are orthogonal idempotents, thus obtaining a contradiction. First, it is clear that  $p_i e_U p_j e_U = p_i p_j e_U$  and therefore the  $p_j e_U$  are orthogonal elements equal to their own squares.

We just need to verify that they are not equal to zero. But if we assume that  $p_j e_U = 0$ , we obtain

$$p_j = p_j e_W = \sum_{\varphi \in \text{Gal}(K/\mathbb{Q})} \varphi(p_j e_U) = 0$$

which is impossible. □

#### 4. RATIONAL IDEMPOTENTS INVARIANT UNDER A SUBGROUP

In this section we find idempotents  $f$  in  $\mathbb{Q}[G]$  which satisfy  $hf = fh$  for all  $h$  in a given subgroup  $H$  of  $G$ . This kind of idempotents will be needed in the study of the decomposition of Jacobian varieties with  $G$ -action to be developed in Section 5.1.

*Remark 4.1.* Let  $V$  denote a complex irreducible representation of a finite group  $G$  and let  $K$  denote the field obtained by adjoining the values of the character  $\chi_V$  of  $V$  to the rational numbers.

Let  $H$  be a subgroup of  $G$  and consider the central idempotent  $e_V = \frac{\dim V}{|G|} \sum_{g \in G} \chi_V(g^{-1}) g$  in  $K[G]$  and the following elements

$$p_H = \frac{1}{|H|} \sum_{h \in H} h \quad , \quad \text{and}$$

$$f_V = p_H e_V \in K[G] .$$

Turull observes the following two facts in the proof of Theorem 1.4 of [T].

- (1)  $p_H$  is an idempotent of  $F[G]$  for any field  $F$  containing  $\mathbb{Q}$  and  $F[G] p_H$  is an  $F[G]$ -module affording the representation  $\rho_H$  of  $G$  induced by the trivial representation of  $H$ , and
- (2) if

$$\langle \rho_H, V \rangle = 1$$

then  $f_V$  is a primitive idempotent in  $K[G]$  and  $K[G] f_V$  is a  $K[G]$ -module affording  $V$ .

We will see in Section 5.1 that (1) together with the appropriate generalization of (2) given in Theorem 4.4 turn out to be key points in understanding the isotypical decomposition of a Jacobian variety with  $G$ -action.

*Remark 4.2.* Note that Condition (2) above implies that the Schur index  $m_{\mathbb{Q}}(V)$  of  $V$  is equal to one.

Since it is known that  $m_{\mathbb{Q}}(V)$  divides  $\langle \rho_H, V \rangle$  for every subgroup  $H$  of  $G$  (see [I, Lemma 10.4]), a natural generalization of Condition (2) would be to consider the case when

$$\langle \rho_H, V \rangle = m_{\mathbb{Q}}(V) ,$$

which we do in Corollary 4.5.

The full general case

$$\langle \rho_H, V \rangle > 0$$

is considered in Theorem 4.4.

The following elementary result will be used in the proof of Theorem 4.4. It is an immediate consequence of (2.2) and the Frobenius reciprocity theorem ([S]), but the authors were unable to find it in the literature.

**Lemma 4.3.** *Denote  $\text{Irr}_{\mathbb{Q}}(G) = \{\mathcal{W}_1, \dots, \mathcal{W}_r\}$  and for each  $\mathcal{W}_j$  choose a corresponding  $V_j$  in  $\text{Irr}_{\mathbb{C}}(G)$ . Also let  $H$  denote a subgroup of  $G$ .*

*Then the rational isotypical decomposition of  $\rho_H$  has the following form*

$$(4.1) \quad \rho_H = a_1 \mathcal{W}_1 \oplus \dots \oplus a_r \mathcal{W}_r ,$$

where

$$a_j = \frac{\langle \rho_H, V_j \rangle}{m_j} = \frac{\dim V_j^H}{m_j} .$$

The following Theorem generalizes the second result in Remark 4.1.

**Theorem 4.4.** *Let  $\mathcal{W}$  be a rational irreducible representation of a group  $G$ , and denote by  $e_{\mathcal{W}}$  the associated central idempotent in  $\mathbb{Q}[G]$ . We denote by  $V$  one of the complex irreducible representations of  $G$  appearing in the isotypical decomposition of  $\mathbb{C} \otimes_{\mathbb{Q}} \mathcal{W}$ .*

*For any subgroup  $H$  of  $G$ , let*

$$p_H = \frac{1}{|H|} \sum_{h \in H} h$$

*be the central idempotent in  $\mathbb{Q}[H]$  corresponding to the trivial representation of  $H$ .*

*Then*

$$f_H = p_H e_{\mathcal{W}} = e_{\mathcal{W}} p_H$$

*is an element of the simple algebra  $\mathbb{Q}[G] e_{\mathcal{W}}$  satisfying the following conditions.*

- (1)  $f_H^2 = f_H$ ,
- (2)  $h f_H = f_H = f_H h$  for every  $h$  in  $H$ , and
- (3)  $f_H = 0$  if and only if  $\langle \rho_H, \mathcal{W} \rangle = 0$  if and only if  $\dim V^H = 0$ .

*Furthermore, in the case  $f_H \neq 0$ , the left ideal  $\mathbb{Q}[G] f_H$  generated by the idempotent  $f_H$  is a left  $\mathbb{Q}[G]$ -module affording the representation  $\mathcal{W}$  with multiplicity given by  $\frac{\dim V^H}{m}$ .*

*Proof.* The first statement is clear since  $e_{\mathcal{W}}$  is central in  $\mathbb{Q}[G]$  and both  $p_H$  and  $e_{\mathcal{W}}$  are idempotents.

The second statement is obvious from the definition of  $f_H$ , and it follows from it that  $f_H$  is in  $\mathcal{W}^H$ .

Therefore  $f_H = 0$  if  $0 = \dim_{\mathbb{Q}} \mathcal{W}^H$ .

We also have that

$$\dim_{\mathbb{Q}} \mathcal{W}^H = \langle \rho_H, \mathcal{W} \rangle = \frac{\dim V^H}{m} \langle \mathcal{W}, \mathcal{W} \rangle = [L : \mathbb{Q}] \dim_{\mathbb{C}} V^H$$

where the second equality follows from Lemma 4.3 and the third from (2.2), thus proving that  $\langle \rho_H, \mathcal{W} \rangle = 0$  if and only if  $\dim_{\mathbb{C}} V^H = 0$ .

We now show that if  $\langle \rho_H, \mathcal{W} \rangle \neq 0$  then  $f_H \neq 0$  as follows.

Since

$$f_H = \frac{1}{|H||G|} \sum_{h \in H, g \in G} \chi_{\mathcal{W}}(g^{-1}) h g$$

we compute the coefficient of 1 of  $f_H$  to be

$$\frac{1}{|H||G|} \sum_{h \in H} \chi_{\mathcal{W}}(h) = \frac{1}{|G|} \langle \text{Res}_H(\mathcal{W}), 1_H \rangle_H = \frac{1}{|G|} \langle \mathcal{W}, \rho_H \rangle$$

and therefore  $f_H \neq 0$  if  $\langle \mathcal{W}, \rho_H \rangle \neq 0$ , which finishes the proof of the third statement.

As for the last statement, the first result in Remark 4.1 applied to the case  $F = \mathbb{Q}$  together with the result of Lemma 4.3 imply that the left ideal  $\mathbb{Q}[G] f_H$  is equal to  $\mathbb{Q}[G] p_H \cap \mathbb{Q}[G] e_{\mathcal{W}}$  and it follows from Lemma 4.3 that it is therefore equal to the direct sum of  $\frac{\dim_{\mathbb{C}} V^H}{m}$  minimal left ideals, each of which affords the representation  $\mathcal{W}$ , thus finishing the proof.  $\square$

**Corollary 4.5.** *Continuing with the notation of Theorem 4.4, suppose that there exists a subgroup  $H$  of  $G$  such that*

$$\langle \rho_H, V \rangle_G = m_{\mathbb{Q}}(V) ,$$

*or, equivalently, that there exist a subgroup  $H$  of  $G$  and a rational representation  $\widetilde{\mathcal{W}}$  of  $G$  with*

$$\rho_H = \mathcal{W} \oplus \widetilde{\mathcal{W}} , \text{ and } \langle \mathcal{W}, \widetilde{\mathcal{W}} \rangle = 0 .$$

*Then  $f_H$  is a primitive idempotent in  $\mathbb{Q}[G] e_{\mathcal{W}}$  commuting with every element  $h$  in  $H$  and  $\mathbb{Q}[G] f_H$  affords the representation  $\mathcal{W}$ .*

## 5. AN APPLICATION: GROUP ACTIONS ON ABELIAN VARIETIES

For a group  $G$  acting on an abelian variety  $A$ , Lange and Recillas [L-R-1] have proved the following result, called the *isogeny decomposition* of  $A$  with respect to  $G$ .

**Theorem:** *Let  $G$  be a finite group acting on an abelian variety  $A$ . Let  $\mathcal{W}_1, \dots, \mathcal{W}_r$  denote the irreducible  $\mathbb{Q}$ -representations of  $G$  and  $n_j := \dim_{D_j}(\mathcal{W}_j)$  with  $D_j := \text{End}_G(\mathcal{W}_j)$  for  $j = 1, \dots, r$ . Then there are abelian subvarieties  $B_1, \dots, B_r$  such that each  $B_j^{n_j}$  is  $G$ -stable and associated to the representation  $\mathcal{W}_j$ , and also an isogeny*

$$(5.1) \quad A \sim B_1^{n_1} \times \dots \times B_r^{n_r} .$$

Observe that the integers  $n_j$  in the theorem satisfy  $n_j = \frac{\dim V_j}{m_j}$ , where  $V_j$  is a complex irreducible representation associated to  $\mathcal{W}_j$  and  $m_j = m_{\mathbb{Q}}(V_j)$  is the Schur index of  $V_j$ .

To explain what the varieties  $B_j$  are, observe that the action of  $G$  on the abelian variety  $A$  induces an algebra homomorphism

$$\rho : \mathbb{Q}[G] \rightarrow \text{End}_{\mathbb{Q}}(A).$$

If  $e$  denotes any idempotent of the algebra  $\mathbb{Q}[G]$ , define

$$\text{Image}(e) := \text{Image}(\rho(me)) \subseteq A$$

where  $m$  is some positive integer such that  $me \in \mathbb{Z}[G]$ .  $\text{Image}(e)$  is an abelian subvariety of  $A$ , which certainly does not depend on the chosen integer  $m$ .

The  $B_j$  in the above theorem are then defined by  $B_j = \text{Image}(p_1^j)$ , where

$$e_{\mathcal{W}_j} = p_1^j + p_2^j + \dots + p_{n_j}^j$$

is any decomposition of the central idempotent  $e_{\mathcal{W}_j}$  as a sum of primitive orthogonal idempotents.

Since there are, in general, many decompositions of  $e_{\mathcal{W}_j}$  of the form just given, it is clear that the  $B_j$  are far from unique, and defined only up to isogeny.

With the explicit construction of primitive orthogonal rational idempotents  $f$  given in Section 3 through Theorem 3.3, Remark 3.4 and Corollaries 3.5 and 3.6, the  $B_j$ 's may now be obtained as images of explicit idempotents, as in the following result.

**Proposition 5.1.** *Let  $G$  be a finite group acting on an abelian variety  $A$ .*

*Then each  $B_j$  in the isotypical decomposition (5.1) of  $A$  with respect to  $G$  may be chosen as the image of any of the corresponding primitive rational idempotents  $f$  constructed in Corollary 3.6.*

**5.1. The case of Jacobians.** For the case of a  $G$ -action on the Jacobian variety  $JW$  of a curve  $W$ , we can be even more explicit.

The main new observation here is that every intermediate geometric object specified by the cover  $W \rightarrow W_G = W/G$ , such as  $JW_H = J(W/H)$  or  $P(W_H/W_N)$  for subgroups  $H \subseteq N \subseteq G$ , has its own isogeny decomposition determined by an appropriate combination of rational representation of  $G$  of the form  $\rho_H$ .

In fact, we will give the isotypical decomposition of any intermediate Jacobian variety  $JW_H$  and of any intermediate Prym variety  $P(W_H/W_N)$ ; we will also provide the rational idempotents whose image is the corresponding  $B_i^{s_i}$  inside the given variety.

From now on we assume that the rational irreducible  $\{\mathcal{W}_1, \dots, \mathcal{W}_r\}$  representations of  $G$  are numbered so that  $\mathcal{W}_1$  is the trivial representation; therefore, the first factor in the isotypical decomposition (5.1) corresponding to a Jacobian  $JW$  with  $G$ -action will always be  $B_1^1 = JW_G$ .

**Proposition 5.2.** *Given a Galois cover  $W \rightarrow W_G$ , consider the associated isotypical decomposition (5.1) of  $JW$  given as*

$$JW \sim JW_G \times B_2^{\frac{\dim V_2}{m_2}} \times \dots \times B_r^{\frac{\dim V_r}{m_r}}$$

where  $V_j$  is a complex irreducible representation associated to  $\mathcal{W}_j$  and  $m_j = m_{\mathbb{Q}}(V_j)$  is the Schur index of  $V_j$ .

Let  $H$  be a subgroup of  $G$  and denote by  $\pi_H : W \rightarrow W_H$  the corresponding quotient map.

Then the corresponding isotypical decomposition of  $JW_H$  is given as follows.

$$(5.2) \quad JW_H \sim JW_G \times B_2^{\frac{\dim V_2^H}{m_2}} \times \dots \times B_r^{\frac{\dim V_r^H}{m_r}},$$

where  $V_j^H$  is the subspace of  $V_j$  fixed by  $H$ .

Furthermore, considering images in  $JW$  of rational idempotents as defined at the beginning of Section 5, and setting  $p_H = \frac{1}{|H|} \sum_{h \in H} h$  and  $f_H^j = p_H e_{\mathcal{W}_j}$  as in Theorem 4.4, the following equalities hold.

$$(5.3) \quad \text{Image}(p_H) = \pi_H^*(JW_H)$$

and if  $\dim V_j^H \neq 0$  then

$$(5.4) \quad \text{Image}(f_H^j) = B_j^{\frac{\dim V_j^H}{m_j}}.$$

*Proof.* First, it is clear that  $\text{Image}(p_H)$  is the connected component containing the origin of the subvariety  $JW^H$  of  $JW$  fixed by  $H$ , and therefore equal to  $\pi_H^*(JW_H)$ , which is of course isogenous to  $JW_H$ .

Secondly, it follows from Theorem 4.4 that the image of each non-zero  $f_H^j$  is contained in  $\text{Image}(p_H)$  and is equal to  $B_j^{\frac{\dim V_j^H}{m_j}}$ .

Finally, the equality

$$p_H = \sum_{j \in \{1, \dots, r\}} f_H^j$$

induces the sought isogeny

$$JW_G \times B_2^{\frac{\dim V_2^H}{m_2}} \times \dots \times B_r^{\frac{\dim V_r^H}{m_r}} \rightarrow JW_H$$

giving the isotypical decomposition of  $JW_H$ . □

*Remark 5.3.* The existence of non negative integers  $s_j$  such that there is a decomposition of the type given in the following corollary is shown in [L-R-1]; in the case when the group  $G$  has only absolutely irreducible representations, the  $s_j$ 's are shown to be equal to  $\dim V_j^H - \dim V_j^N$  in [L-R-2], using the existence of an essentially unique  $G$ -invariant inner product on each  $\mathcal{W}_j = V_j$ , which does not hold when the Schur index is larger than one.

We now give the isotypical decomposition of any intermediate Prym variety.

**Corollary 5.4.** *Given a Galois cover  $W \rightarrow W_G$ , consider the associated isotypical decomposition of  $JW$  given as*

$$JW \sim JW_G \times B_2^{\frac{\dim V_2}{m_2}} \times \dots \times B_r^{\frac{\dim V_r}{m_r}}.$$

*Then for any subgroups  $H \subseteq N \subseteq G$  the corresponding decomposition of  $P(W_H/W_N)$  is given as follows.*

$$P(W_H/W_N) \sim B_2^{s_2} \times \dots \times B_r^{s_r}$$

where

$$s_j = \frac{\dim V_j^H}{m_j} - \frac{\dim V_j^N}{m_j}.$$

*Proof.* The previous Proposition gives the isotypical decompositions for  $JW_H$  and  $JW_N$ , but we also have the natural isogeny  $JW_H \sim JW_N \times P(W_H/W_N)$ .

If we lift all these isogenies to the corresponding equalities inside the tangent space to  $JW$  at the origin, we obtain equalities between sums of subspaces, which may be chosen to be orthogonal among themselves with respect to the Hermitian product induced by the natural polarization on  $JW$  (see [C-R-R] for similar arguments).

In other words, letting  $\pi_H : W \rightarrow W_H$  denote the corresponding cover map, the following decompositions are chosen to be orthogonal decompositions.

$$(5.5) \quad (d\pi_H^*)_0(T_0JW_H) = (d\pi_G^*)_0(T_0JW_G) \oplus T_0(B_2^{\frac{\dim V_2^H}{m_2}}) \oplus \dots \oplus T_0(B_r^{\frac{\dim V_r^H}{m_r}})$$

$$(5.6) \quad (d\pi_N^*)_0(T_0JW_N) = (d\pi_G^*)_0(T_0JW_G) \oplus T_0(B_2^{\frac{\dim V_2^N}{m_2}}) \oplus \dots \oplus T_0(B_r^{\frac{\dim V_r^N}{m_r}})$$

and

$$(5.7) \quad (d\pi_H^*)_0(T_0JW_H) = (d\pi_N^*)_0(T_0JW_N) \oplus T_0(P(W_H/W_N))$$

Now we can replace (5.6) in (5.7) and comparing the result with (5.5), the proof is finished.  $\square$

*Remark 5.5.* Note that  $B_{\mathcal{W}}$  (for  $\mathcal{W}$  different from the trivial representation) is always contained in some  $\pi_H^*(P(W_H/W_N))$ , possibly in more than one.

This is because  $B_{\mathcal{W}}$  is contained in  $\pi_H^*(JW_H)$  for any  $H$  subgroup of  $G$  such that  $\langle \rho_H, \mathcal{W} \rangle \neq 0$ ; in particular, this condition is always satisfied for the trivial subgroup of  $G$ . Moreover, for any  $H$  satisfying  $\langle \rho_H, \mathcal{W} \rangle \neq 0$ , there exist (at least) one subgroup  $N$  of  $G$  containing  $H$  and such that  $\langle \rho_N, \mathcal{W} \rangle = 0$ : for instance  $N = G$ , and for any such subgroup  $N$ ,  $B_{\mathcal{W}}$  will be contained in  $\pi_H^*(P(W_H/W_N))$ .

In fact, in this case we know from the previous Corollary that it will be contained with multiplicity  $\frac{\dim V^H}{m_{\mathbb{Q}}(V)}$ , and from Proposition 5.2 that  $B_{\mathcal{W}}^{\frac{\dim V^H}{m_{\mathbb{Q}}(V)}} \subseteq \pi_H^*(P(W_H/W_N))$  may be obtained as the image of the idempotent  $f_H = p_H e_{\mathcal{W}}$ .

**Corollary 5.6.** *Given a Galois cover  $W \rightarrow W_G$ , consider the associated isotypical decomposition of  $JW$  given as*

$$JW \sim JW_G \times B_2^{\frac{\dim V_2}{m_2}} \times \dots \times B_r^{\frac{\dim V_r}{m_r}} .$$

*Assume there exist subgroups  $H \subset N \subseteq G$  and a rational irreducible representation  $\mathcal{W}$  of  $G$  such that*

$$(5.8) \quad \rho_H = \mathcal{W} \oplus \rho_N .$$

*Then*

$$P(W_H/W_N) \sim B_{\mathcal{W}} .$$

*Conversely, if for some  $\mathcal{W}$  rational irreducible representation of  $G$  there are subgroups  $H \subset N \subseteq G$  such that  $P(W_H/W_N) \sim B_{\mathcal{W}}$ , then (5.8) holds.*

*Proof.* Recall that the rational irreducible representations of  $G$  are denoted by  $\{\mathcal{W}_1 = 1, \dots, \mathcal{W}_r\}$  and  $V_j$  denotes a complex irreducible representation associated to  $\mathcal{W}_j$ .

For subgroups  $H \subset N \subseteq G$ , consider the non-negative integers given by  $s_j = \frac{\dim V_j^H}{m_j} - \frac{\dim V_j^N}{m_j}$  for  $j \in \{2, \dots, r\}$  and observe that it follows from Lemma 4.3 that

$$\rho_H - \rho_N = \bigoplus_{j=2}^r s_j \mathcal{W}_j .$$

But it follows from Corollary 5.4 that  $P(W_H/W_N) \sim B_{\mathcal{W}}$  if and only if the unique  $s_j$  corresponding to  $\mathcal{W}$  equals one and all the other  $s_j$  equal zero, thus finishing the proof.  $\square$

**Corollary 5.7.** *Given a Galois cover  $W \rightarrow W_G$ , consider the associated isotypical decomposition of  $JW$  given as*

$$JW \sim JW_G \times B_2^{\frac{\dim V_2}{m_2}} \times \dots \times B_r^{\frac{\dim V_r}{m_r}} .$$

*Assume there are subgroups  $H$ ,  $N_1$  and  $N_2$  of  $G$  such that  $H$  is a subgroup of  $N_k$  for each  $k$ , and*

$$\rho_H = \rho_{N_k} \oplus \mathcal{W} \oplus \mathcal{W}_k$$

*with  $\mathcal{W}$ ,  $\mathcal{W}_1$  and  $\mathcal{W}_2$  rational representations of  $G$  such that*

$$\langle \mathcal{W}, \mathcal{W}_j \rangle = \langle \mathcal{W}_1, \mathcal{W}_2 \rangle = 0$$

*for  $j = 1, 2$  and  $\mathcal{W}$  irreducible.*

*Then, if  $(X)^0$  denotes the connected component of  $X$  containing the origin, we have*

$$(5.9) \quad B_{\mathcal{W}} \sim (P(W_H/W_{N_1}) \cap P(W_H/W_{N_2}))^0 .$$

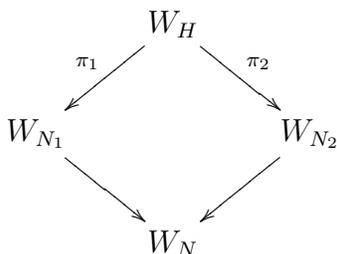
It is clear that a similar result holds if there is a finite number of subgroups  $N_k$  containing  $H$  such that the representations  $\rho_H - \rho_{N_k}$  all have a common rational representation.

*Remark 5.8.* The previous corollaries can be used to give explicit geometric decompositions of Jacobians with group actions in many cases, in the manner illustrated with the examples given in the appendices.

Besides obtaining these isotypical decompositions, there also some interesting new types of abelian subvarieties that emerge from the picture as  $B_{\mathcal{W}}$ 's: the first examples worked out, such as  $D_p$  in [Ri],  $S_3$  in [R-R-1], the Klein group,  $D_4$ ,  $A_4$  and  $S_4$  in [R-R-2] and  $A_5$  in [SA-1], were all Prym or Jacobian varieties of intermediate covers. In the case of the groups  $D_n$  in [C-R-R] and also for  $S_5$  in [SA-2] and [L-R-2], a new type appeared: some of the  $B_{\mathcal{W}}$ 's were nor Jacobian nor Prym varieties, but (connected components of the origin of) intersections of Prym varieties, of the form

$$B_{\mathcal{W}} = \left( \bigcap_{k \in \{1, \dots, s\}} P(W_H/W_{N_k}) \right)^{\circ}$$

where the subgroups  $H \subset N_k \subseteq G$  are as in Corollary 5.7. This last situation may be equivalently described (see [L-R-2]) in the following way in the case when  $r = 2$ . Consider the following diagram of curves and covers



with  $N = \langle N_1, N_2 \rangle$  and where  $H$ ,  $N_1$  and  $N_2$  satisfy the hypothesis of Corollary 5.7.

Then  $B_{\mathcal{W}}$  is the (common) orthogonal complement of  $\pi_k^*(P(W_{N_k}/W_N))$  inside  $P(W_H/W_{N_l})$  for  $k \neq l$ , where orthogonality is taken with respect to the Hermitian product induced by the canonical Hermitian product induced by the natural polarization on  $JW_H$ .

Now a further new type appears, by taking into account the Schur index (all examples mentioned above have Schur index equal to one for every complex irreducible representation). As may be seen in the example in Appendix B, there is a  $B_{\mathcal{W}}$  that is not of any of the above types. However, it is the image of an explicit rational idempotent and, as such, it may be seen to be the orthogonal complement of 2 Prym varieties inside another Prym variety.

*Remark 5.9.* In the study of the decompositions of Jacobians in the case of the special groups mentioned in the previous Remark, some interesting isogenies have arisen. They were found with ad-hoc methods or with restrictions on the acting representations, such as the group having only absolutely irreducible representations.

It is clear now that these isogenies are due to equalities between sums and differences of representations of  $G$  of the type  $\rho_H$  for adequate subgroups  $H$  of  $G$ ; for instance, the equality

$$\rho_S - \rho_R = \rho_X - \rho_G$$

for  $G = S_4$  and respective subgroups  $X = S_3$ ,  $R = D_4$  and  $S$  a Klein non normal subgroup, gives rise to an isogeny between  $P(W_S/W_R)$  and  $P(W_X/W_G)$ , which when studied in detail (see [R-R-2]) gives an alternate proof of the Recillas trigonal construction [Rec].

Since we now know that the representations  $\rho_H$  for the subgroups  $H$  of  $G$  determine (with no restrictions on  $G$ ) the isotypical decompositions of all the geometric intermediate pieces in the full diagram of curves and covers of a curve with  $G$ -action, it is likely that new interesting isogenies will appear.

### APPENDIX A. AN EXAMPLE

Consider the group of order 80 given as follows

$$G = \langle x, y : x^{20}, y^8, x^{10}y^4, y^{-1}xyx^{-3} \rangle$$

and its irreducible representation  $V$  of degree four whose character  $\chi_V$  is given in the following table on each of the fourteen conjugacy classes of  $G$ .

identity	$x$	$x^{19}$	$y$	$x^{10}y^3$	$x^2$	$xy$	$x^{11}y^3$	$y^2$	$x^{10}y^2$	$xy^2$	$x^4$	$x^{10}$	$x^5$
4	$k$	$-k$	0	0	1	0	0	0	0	0	-1	-4	0

where  $k = \sqrt{-5}$ . Then

$$K = \mathbb{Q}(k) \text{ and } \text{Gal}(K/\mathbb{Q}) = \langle \varphi(k) = -k \rangle .$$

Furthermore,  $V$  is defined over

$$L = K(l) = \mathbb{Q}(\sqrt{10} + \sqrt{-2}) \text{ and } \text{Gal}(L/K) = \langle \tau(l) = -l \rangle ,$$

where  $l = \sqrt{-2}$ , as seen from the following matrix representation of  $V$ .

$$x = \begin{pmatrix} \frac{2}{5}\sqrt{-5} & -\frac{1}{5}\sqrt{-5} & \frac{1}{5}\sqrt{-5} & -\frac{2}{5}\sqrt{-5} \\ \frac{3}{5}\sqrt{-5} & \frac{1}{5}\sqrt{-5} & 0 & -\frac{1}{5}\sqrt{-5} \\ 0 & \frac{1}{5}\sqrt{-5} & \frac{2}{5}\sqrt{-5} & -\frac{3}{5}\sqrt{-5} \\ \frac{1}{5}\sqrt{-5} & -\frac{1}{5}\sqrt{-5} & \frac{3}{5}\sqrt{-5} & 0 \end{pmatrix}$$

and

$$y = \begin{pmatrix} -1 + \sqrt{-2} & 0 & -1 - \sqrt{-2} & 1 \\ -2 & 1 & -\sqrt{-2} & \sqrt{-2} \\ -1 & \sqrt{-2} & -\sqrt{-2} & -1 \\ -1 & -1 & 1 - \sqrt{-2} & 0 \end{pmatrix}$$

Then the Schur index  $m_{\mathbb{Q}}(V) = 2$  and we should find  $\frac{\dim V}{m} = 2$  primitive orthogonal idempotents  $k_1$  and  $k_2$  in  $K[G]e_V$  and also two primitive orthogonal idempotents  $f_1$  and  $f_2$  in  $\mathbb{Q}[G]e_{\mathcal{W}}$  satisfying the following equalities.

$$(A.1) \quad e_V = k_1 + k_2 = \frac{1}{20}(\text{identity} - x^{10})(4\text{identity} + (-k)x + x^2 + (-k)x^3 + (-1)x^4 + x^6 + (-k)x^7 + (-1)x^8 + (-k)x^9)$$

$$(A.2) \quad e_{\mathcal{W}} = e_V + e_{V\varphi} = f_1 + f_2 = \frac{1}{10}(\text{identity} - x^{10})(4\text{identity} + x^2 - x^4 + x^6 - x^8) .$$

We first compute the four primitive orthogonal idempotents  $\ell_j$  whose sum is  $e_V$  as given by (2.5). It can be verified that  $\ell_j$  and  $\tau(\ell_j)$  are non orthogonal. We illustrate by writing down  $\ell_1$ , given as follows.

$$\begin{aligned} \ell_1 = \frac{1}{100}(\text{identity} - x^{10}) & (5(\text{identity} + y + x^2 + x^{14}y^3 + x^2y^3 + x^4y) \\ & + 10(x^{12}y^2 + x^6y^2) + (4k)(x^{13}y^3 + x^{17}y) + (2k)(x^{19} + x^{13}) + k(x^{15} + x^{17}) \\ & + (k + kl)(x^{11}y^3 + x^{11}y) + (k - kl)(x^3y + x^{15}y^3) + (k - 2kl)(x^{19}y + x^7y^3) \\ & + (k + 2kl)(x^{15}y + x^{19}y^3) + (3k + 3kl)(x^3y^2) + (3k - 3kl)(x^5y^2) \\ & + (4k + 2kl)(x^{17}y^2) + (4k - 2kl)(x^{11}y^2) + (2kl)(x^9y^2) + (10l)(x^{14}y^2) \\ & + (5 + 5l)(y^2 + x^{16}y^3 + x^6y) + (5 - 5l)(y^3 + x^{18}y^2 + x^{18}y)) \end{aligned}$$

Applying the construction detailed in Remark 3.4 we obtain the new primitive orthogonal idempotents  $u_1^1$  in  $L[G]\ell_1$  and  $u_2^1$  in  $L[G]\ell_2$  respectively, as follows.

$$\begin{aligned}
 u_1^1 = \frac{1}{80} & (\text{identity} - x^{10}) (4(\text{identity} + x^2) + 3(x^{18} + x^4) + 6(x^2y + x^{18}y^3) \\
 & + 2k(x^{15} + x^{13}y^3 + x^{17}y + x^{17}) + k(x^{13} + x^{19}) + 2l(x^{14}y^2) + 2kl(x^9y^2) \\
 & + (2 + 2l)(x^{10}y^2) + (2 - 2l)(x^8y^2) + (4 + 3l)(x^4y) + (4 - 3l)(y) \\
 & + (4 + l)(x^{16}y^3) + (4 - l)y^3 + (1 + 4l)(x^6y) + (1 - 4l)(x^{18}y) \\
 & + (1 + 3l)(x^{14}y^3 + x^{16}y^2) + (1 - 3l)(x^2y^2 + x^2y^3) + (2k - kl)(x^{15}y + x^{15}y^3) \\
 & + (2k + kl)(x^{11}y^3 + x^{19}y) + (k - 2kl)(x^{13}y) + (k + 2kl)(xy) \\
 & + (k + kl)(x^9y^3 + x^7y^2) + (k - kl)(xy^2 + x^{17}y^3))
 \end{aligned}$$

$$\begin{aligned}
 u_2^1 = \frac{1}{160} & (\text{identity} - x^{10}) (8(\text{identity}) + 4k(x^5) + l(x^{14}y^2) + (3kl)(x^{17}) \\
 & + kl(x^{19}y^2 + x^{13}y^2) + (10 - 5l)(x^{14}) + (8 - 2l)(x^{10}y) + (8 - 6l)(x^{10}y^3) \\
 & + (8 + 5l)(x^{14}y) + (8 + 3l)(x^6y^3) + (4 + 5l)(x^6 + x^{12}) + (4 + 4l)(y^2) + (4 + l)(x^{18}y^2) \\
 & + (4k - kl)(x^3y^3) + (4k + kl)(xy^3 + x^7y + x^{11}) + (4k + 3kl)(x^9y) \\
 & + (4k - 2kl)(x^5y^3 + x^5y) + (12 + l)(x^{12}y) + (12 - l)(x^8y^3) + (2 + 5l)(x^4y^3) \\
 & + (2 - 5l)(x^8) + (2 + l)(x^6y^2) + (2 - l)(x^{12}y^2) + (2 - 7l)(x^{12}y^3 + x^8y) \\
 & + (2 + 9l)(x^{16}y) + (2k + kl)(x^{11}y^2 + x^{17}y^2) + (2k - kl)(x^{19}) \\
 & + (2k - 3kl)(x^7y^3 + x^3y) + (2k + 3kl)(x^{13} + x^{11}y + x^{19}y^3))
 \end{aligned}$$

Then the  $k_s$  are obtained applying Corollary 3.5; we illustrate with  $k_1$  as follows.

$$\begin{aligned}
 k_1 = u_1^1 + \tau(u_1^1) = \frac{1}{40} & (\text{identity} - x^{10}) (4(\text{identity} + x^2 + x^4y \\
 & + y + x^{16}y^3 + y^3) + 3(x^{18} + x^4 + x^2y + x^{18}y^3) \\
 & + 2(x^{10}y^2 + x^8y^2) + (x^6y + x^{18}y + x^{14}y^3 + x^{16}y^2 + x^2y^2 + x^2y^3) \\
 & + 2k(x^{15} + x^{13}y^3 + x^{17}y + x^{17} + x^{15}y + x^{15}y^3 + x^{11}y^3 + x^{19}y) \\
 & + k(x^{13} + x^{19} + x^{13}y + xy + x^9y^3 + x^7y^2 + xy^2 + x^{17}y^3))
 \end{aligned}$$

Now the  $f_s$  are obtained applying Corollary 3.6, as follows.

$$\begin{aligned}
 f_1 = k_1 + \varphi(k_1) & = u_1^1 + \tau(u_1^1) + \varphi(u_1^1) + \varphi(\tau(u_1^1)) \\
 & = \frac{1}{20} (\text{identity} - x^{10}) (4(\text{identity} + x^2 + x^4y + y + x^{16}y^3 + y^3) \\
 & + x^6y + x^{18}y + x^{14}y^3 + x^{16}y^2 + x^2y^2 + x^2y^3 \\
 & + 3(x^{18} + x^4 + x^2y + x^{18}y^3) + 2(x^{10}y^2 + x^8y^2))
 \end{aligned}$$

$$\begin{aligned}
f_2 &= k_2 + \varphi(k_2) = u_2^1 + \tau(u_2^1) + \varphi(u_1^2) + \varphi(\tau(u_1^2)) \\
&= \frac{1}{20}(\text{identity} - x^{10}) (4(\text{identity} + x^{10}y + x^{10}y^3 + x^{14}y + x^6y^3)) \\
&\quad + 5x^{14} + 6(x^{12}y + x^8y^3) + 2(x^6 + x^{12} + y^2 + x^{18}y^2) \\
&\quad + x^4y^3 + x^8 + x^6y^2 + x^{12}y^2 + x^{12}y^3 + x^8y + x^{16}y)
\end{aligned}$$

Note that the factor  $(\text{identity} - x^{10})$  appearing in both  $f_1$  and  $f_2$  indicates that their images are contained in  $P(W/W_{\langle x^{10} \rangle})$ , and indeed one can verify that

$$\langle \rho_{\{1\}}, V \rangle = 4 \quad \text{and} \quad \langle \rho_{\langle x^{10} \rangle}, V \rangle = 0 .$$

But another computation shows that

$$\langle \rho_{H=\langle xy^2 \rangle}, V \rangle = 2$$

and it follows from Proposition 5.2 that  $B_{\mathcal{W}} = \text{Image}(p_H e_{\mathcal{W}})$  is contained in  $\pi_H^*(JX_H)$  with multiplicity one, where  $p_H e_{\mathcal{W}}$  is given as follows.

$$(A.3) \quad p_H e_{\mathcal{W}} = \frac{1}{20}(\text{identity} - x^{10})(\text{identity} + xy^2) (4\text{identity} + x^2 - x^4 + x^6 - x^8)$$

In order to give the isotypical decomposition of a Jacobian  $JW$  with  $G$ -action we first give some more information about the group  $G$ .

The character table for  $G$  is given as follows

	identity	$x$	$x^{19}$	$y$	$x^{10}y^3$	$x^2$	$xy$	$x^{11}y^3$	$y^2$	$x^{10}y^2$	$xy^2$	$x^4$	$x^{10}$	$x^5$
$\chi_{V_1}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_{V_2}$	1	-1	-1	-1	-1	1	1	1	1	1	-1	1	1	-1
$\chi_{V_3}$	1	-1	-1	1	1	1	-1	-1	1	1	-1	1	1	-1
$\chi_{V_4}$	1	1	1	-1	-1	1	-1	-1	1	1	1	1	1	1
$\chi_{V_5}$	1	-1	-1	$-w_4$	$w_4$	1	$w_4$	$-w_4$	-1	-1	1	1	1	-1
$\chi_{V_6}$	1	-1	-1	$w_4$	$-w_4$	1	$-w_4$	$w_4$	-1	-1	1	1	1	-1
$\chi_{V_7}$	1	1	1	$-w_4$	$w_4$	1	$-w_4$	$w_4$	-1	-1	-1	1	1	1
$\chi_{V_8}$	1	1	1	$w_4$	$-w_4$	1	$w_4$	$-w_4$	-1	-1	-1	1	1	1
$\chi_{V_9}$	2	0	0	0	0	-2	0	0	$-2w_4$	$2w_4$	0	2	-2	0
$\chi_{V_{10}}$	2	0	0	0	0	-2	0	0	$2w_4$	$-2w_4$	0	2	-2	0
$\chi_{V_{11}}$	4	1	1	0	0	-1	0	0	0	0	0	-1	4	-4
$\chi_{V_{12}}$	4	-1	-1	0	0	-1	0	0	0	0	0	-1	4	4
$\chi_{V_{13}}$	4	$-k$	$k$	0	0	1	0	0	0	0	0	-1	-4	0
$\chi_{V_{14}}$	4	$k$	$-k$	0	0	1	0	0	0	0	0	-1	-4	0

where  $w_n$  denotes a primitive  $n$ -th root of unity and  $k = \sqrt{-5} = w_{20} + w_{20}^9 - w_{20}^{13} - w_{20}^{17}$ .

Note that the irreducible rational representations of  $G$  are  $V_j$ , for  $j$  in  $\{1, 2, 3, 4, 11, 12\}$ , and  $V_5 \oplus V_6$ ,  $V_7 \oplus V_8$ ,  $V_9 \oplus V_{10}$ ,  $2(V_{13} \oplus V_{14})$ . The representation  $V$  given at the beginning of this section is  $V_{13}$ .

Consider the following subgroups of  $G$ :  $H_1 = \langle x^2, xy \rangle$ ,  $H_2 = \langle x^2, y \rangle$ ,  $H_3 = \langle y^2, x \rangle$ ,  $H_4 = \langle x^2, xy^2 \rangle$ ,  $H_5 = \langle x \rangle$ ,  $H_6 = \langle x^4, xy^2 \rangle$ ,  $H_7 = \langle x^3y^2, xy \rangle$ ,  $H_8 = \langle xy \rangle$ ,  $H_9 = \langle xy^2 \rangle$ ,  $H_{10} = \langle xy^2, x^{10} \rangle$ . Then we have the following result.

**Theorem A.1.** *Let  $W$  denote a compact Riemann surface with  $G$ -action and set*

$$B_1 = P(W_{H_8}/W_{H_7}) \cap P(W_{H_8}/W_{H_1}) , \text{ and}$$

$$B_2 = P(W_{H_9}/W_{H_{10}}) \cap P(W_{H_9}/W_{H_6}) .$$

*Then the isotypical decomposition of the Jacobian variety  $JW$  with respect to  $G$  is given as follows*

$$\begin{aligned} JW \sim JW_G \times P(W_{H_1}/W_G) \times P(W_{H_2}/W_G) \times P(W_{H_3}/W_G) \\ \times P(W_{H_4}/W_{H_3}) \times P(W_{H_5}/W_{H_3}) \times P(W_{H_6}/W_{H_4})^2 \\ \times P(W_{H_7}/W_G)^4 \times B_1^4 \times B_2^2 \end{aligned}$$

*with associated rational representations on the right hand side given as follows.*

$$\begin{aligned} V_1 \oplus V_2 \oplus V_3 \oplus V_4 \\ \oplus (V_5 \oplus V_6) \oplus (V_7 \oplus V_8) \oplus (V_9 \oplus V_{10}) \\ \oplus V_{12} \oplus V_{11} \oplus 2(V_{13} \oplus V_{14}) \end{aligned}$$

*Proof.* The proof is obtained by first computing the corresponding representations  $\rho_H$  for all (conjugacy classes of) subgroups  $H$  of  $G$ , then looking for each irreducible rational representation  $\mathcal{W}$  in terms of the form  $\rho_H - \rho_N$ , for  $H \subset N \subseteq G$ , and finally applying either Corollary 5.6 or Corollary 5.7.

The first two steps can be implemented very quickly with a simple computer program (see [Ro] for a GAP version).  $\square$

## APPENDIX B. ANOTHER EXAMPLE: A QUASI-PRYM

Consider the following group, a  $\mathbb{Z}/3\mathbb{Z}$  extension of the quaternion group of order 8,

$$G = \langle x^4, y^4, z^3, y^{-1}xyx, z^{-1}xzy^{-1}, z^{-1}yz(xy)^{-1} \rangle .$$

Then  $G$  has a complex irreducible representation  $V$  of degree two such that  $K = \mathbb{Q}$  and  $L = \mathbb{Q}(w_3)$ , with  $w_3$  a primitive cubic root of 1. Therefore  $m_{\mathbb{Q}}(V) = 2$  and  $L \otimes_{\mathbb{Q}} \mathcal{W} = 2V$ .

Since  $\frac{\dim V}{m_{\mathbb{Q}}(V)} = 1$ , the factor corresponding to  $\mathcal{W}$  in the isotypical decomposition of a Jacobian  $JW$  with  $G$ -action will be of the form  $B_{\mathcal{W}}^1$ .

It is clear that in this case

$$\begin{aligned} e_{\mathcal{W}} &= e_V \\ &= \frac{1}{12} (\text{id} - x^2) (2 \text{id} - (z + xz + yz + xyz) - (z^2 + x^3z^2 + y^3z^2 + x^3yz^2)) \end{aligned}$$

is a primitive rational idempotent.

Furthermore, it is known (c.f., [B-Z, pp. 177]) that when  $m_{\mathbb{Q}}(V) = \dim V$  then

$$\langle \rho_H, V \rangle \neq 0 \text{ if and only if } H = \{1\} .$$

This fact together with the explicit expression for  $e_{\mathcal{W}}$  given above imply that

$$B_{\mathcal{W}} = \text{Image}(e_{\mathcal{W}}) \subset P(W/W_{(x^2)}) \cap P(W/W_{(z)}) .$$

One can also verify the following identities

$$(B.1) \quad \begin{aligned} \rho_1 - \rho_{\langle x^2 \rangle} &= \mathcal{W} \oplus 2\mathcal{W}_1 \\ \rho_1 - \rho_{\langle z \rangle} &= \mathcal{W} \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \\ \rho_{\langle z \rangle} - \rho_{\langle z, x^2 \rangle} &= \mathcal{W}_1 \end{aligned}$$

where  $\mathcal{W}_1$  and  $\mathcal{W}_2$  are rational representations of  $G$  that are mutually disjoint and also disjoint from  $\mathcal{W}$ .

Furthermore  $\mathcal{W}_1$  is irreducible, and it follows from Corollary 5.4 that the isotypical decomposition of the (connected component of the origin of the) intersection of the two Prym varieties is the following.

$$(P(W/W_{\langle x^2 \rangle}) \cap P(W/W_{\langle z \rangle}))^0 \sim B_{\mathcal{W}} \times B_{\mathcal{W}_1} .$$

Some further computations allow us to exclude other possibilities and therefore show that  $B_{\mathcal{W}}$  is not any of the known types mentioned in Remark 5.8; that is, it cannot be isolated neither as a Prym variety nor as the intersection of Prym varieties nor as the orthogonal complement of a Prym inside another Prym variety.

However, the relations (B.1) show that  $B_{\mathcal{W}}$  is the orthogonal complement of  $B_{\mathcal{W}_1}^2$ , which is isogenous to  $P(W_{\langle z \rangle}/W_{\langle z, x^2 \rangle})^2$ , inside  $P(W/W_{\langle x^2 \rangle})$ .

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FACULTAD DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, CASILLA 306-22, SANTIAGO, CHILE

*E-mail address:* acarocca@mat.puc.cl

FACULTAD DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, CASILLA 306-22, SANTIAGO, CHILE

*E-mail address:* rubi@mat.puc.cl