CLOSED ORBITS FOR ACTIONS OF MAXIMAL TORI ON HOMOGENEOUS SPACES

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1. Introduction

Let G be the group of real points of a $\mathbb{Q}$-algebraic group $\mathbf{G}$, and let Γ be an arithmetic subgroup of G (i.e. $\Gamma \cap \mathbf{G}(\mathbb{Z})$ has finite index in both Γ and $\mathbf{G}(\mathbb{Z})$). Any subgroup H of G acts on the homogeneous space G/Γ by

$$h\pi(g) = \pi(hg),$$

where $\pi : G \to G/\Gamma$ is the natural quotient map. Such an action is called a *subgroup action*.

A celebrated conjecture of M. S. Raghunathan, proved in full generality by M. Ratner in the early 1990's, implies that when H is connected and generated by unipotents, all orbit–closures for the subgroup action admit an algebraic description. The resulting reduction of certain dynamical questions to algebraic questions has had many deep applications in number theory and geometry. We refer the reader to [KlShSt] for an up to date survey of these developments.

The most general conjecture regarding dynamics of subgroup actions on homogeneous spaces was formulated by G. A. Margulis [M, Conjecture 1]. A very interesting and highly nontrivial special case of this conjecture is when $H = T$ is a maximal $\mathbb{R}$-split (algebraic) torus in G (that is, if G is realized as a matrix group in $\mathrm{GL}(n, \mathbb{R})$, then T is conjugate in $\mathrm{GL}(n, \mathbb{R})$ to a group of diagonal matrices). In this paper we give an explicit algebraic description of all (topologically) closed orbits for this action. More specifically, all closed orbits are 'standard' – they correspond to $\mathbb{Q}$–subtori in $\mathbf{G}$.

Consider the simplest case in which $G = \mathrm{SL}(2, \mathbb{R})$, $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ and T is the subgroup of positive diagonal matrices. The action of T is then the geodesic flow on the unit tangent bundle to the modular surface $\mathbb{H}^+/\Gamma$. This is a noncompact manifold, with one cusp whose lifts correspond to all rational numbers. Any closed orbit for this flow is either periodic or divergent (an orbit $T\pi(x)$ is *divergent* if the orbit map $t \mapsto t\pi(x)$ is proper, or equivalently, if $\{t_n\pi(x)\}$ leaves compact subsets of G/Γ whenever $\{t_n\}$ leaves compact subsets of T). Since a divergent orbit must go 'into the cusp', a geodesic goes to infinity in

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both directions if and only if one (hence any) of its lifts to $\mathbb{H}^+$ has both endpoints on $\mathbb{Q} \cup \{\infty\}$. Equivalently, $T\pi(g)$ is divergent if and only if $g^{-1}Tg$ is diagonalizable over $\mathbb{Q}$. An orbit $T\pi(x)$ is periodic if and only if $\mathrm{Stab}(\pi(x)) = x^{-1}Tx \cap \mathrm{SL}(2,\mathbb{Z})$ is infinite and thus Zariski dense in $x^{-1}Tx$. Thus $x^{-1}Tx$ is defined over $\mathbb{Q}$ and admits no algebraic homomorphisms to $\mathbb{R}$ which are defined over $\mathbb{Q}$, i.e. $x^{-1}Tx$ is a $\mathbb{Q}$-anisotropic $\mathbb{Q}$-torus.

In 1997, in response to a question of the second-named author, Margulis generalized the description of divergent orbits to the higher–rank case, proving the following theorem (unpublished):

Theorem 1.1 (Margulis)**.** *Let $G = \mathrm{SL}(n,\mathbb{R})$, $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, and let T be the group of all diagonal matrices. Then $T\pi(g)$ is divergent if and only if $g^{-1}Tg$ is a real $\mathbb{Q}$-split $\mathbb{Q}$-torus.*

Earlier work of S. G. Dani [Da] showed that no algebraic description is possible for the action of one–parameter diagonalizable subgroups on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$, or more generally, for actions on G/Γ when $\mathrm{rank}_{\mathbb{Q}}(\mathbf{G}) \geq 2$. Dani discusses trajectories rather than orbits, that is, the action of $T = \{d(t) : t \in \mathbb{R}\}$ is replaced by the action of the semigroup $\{d(t) : t \geq 0\}$. In [Da] actions of multidimensional semigroups or groups were not considered.

In this paper we extend Margulis' theorem to any $\mathbb{Q}$-algebraic group $\mathbf{G}$ (Theorem 1.3 below). Although we essentially use ideas from Margulis' proof for $\mathrm{SL}(n,\mathbb{R})$, the proof in the general case requires additional ideas. For the reader's convenience, we include Margulis' proof in an appendix.

Our main result is:

Theorem 1.2. *Let $\mathbf{G}$ be a $\mathbb{Q}$-algebraic group and let $\mathbf{T}$ be a maximal $\mathbb{R}$-split torus, defined over $\mathbb{Q}$ and containing a maximal $\mathbb{Q}$-split torus $\mathbf{S}$. Let $G = \mathbf{G}(\mathbb{R})$, $T = \mathbf{T}(\mathbb{R})$, $S = \mathbf{S}(\mathbb{R})$, and let Γ be an arithmetic subgroup of G. Then there exists a compact subset $K \subset G/\Gamma$ such that:*

(1) *$T\pi(x) \cap K \neq \emptyset$ for any $x \in G$.*

(2) *One of the following holds:*

(i) $x \in Z_G(S)\mathbf{G}(\mathbb{Q})\mathcal{R}_u(G)$, where $\mathcal{R}_u(G)$ is the unipotent radical of G and $Z_G(S)$ is the centralizer of S in G;

(ii) the 'set of recurrence'

$$\{d \in T : d\pi(x) \in K\}$$

is unbounded in T.

We apply Theorem 1.2 in order to describe the divergent orbits for T on G/Γ.

Theorem 1.3. *Let the notation be as in Theorem 1.2, and let $x \in G$. The following are equivalent:*

(1) *The orbit $T\pi(x)$ is divergent.*
(2) $x \in Z_G(S)\mathbf{G}(\mathbb{Q})\mathcal{R}_u(G)$ *and* $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} = \mathrm{rank}_{\mathbb{R}}\mathbf{G}$.
(3) *There is $u \in \mathcal{R}_u(G)$ such that $(xu)^{-1}\mathbf{T}xu$ is a $\mathbb{Q}$-split $\mathbb{Q}$-torus.*

In particular, if $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} \neq \mathrm{rank}_{\mathbb{R}}\mathbf{G}$ *then G/Γ does not contain divergent orbits for T.*

Our results about divergent orbits also yield information about all *closed* orbits.

Theorem 1.4. *Let the notation be as in Theorem 1.2. Given $x \in G$ we denote by $\mathbf{T}_0$ the connected component of the Zariski closure of $x^{-1}Tx \cap \Gamma$ in $\mathbf{G}$. Then the orbit $T\pi(x)$ is closed if and only if there exists an element $u \in Z_{\mathbf{G}}(\mathbf{T}_0) \cap \mathcal{R}_u(G)$ such that $(xu)^{-1}\mathbf{T}xu$ is defined over $\mathbb{Q}$.*

In the case of reductive groups, Theorems 1.3 and 1.4 immediately imply the following

Corollary 1.5. *Let $\mathbf{G}$ be a reductive $\mathbb{Q}$-group, and let $x \in G$. Then:*

- *$T\pi(x)$ is a closed orbit if and only if the torus $x^{-1}\mathbf{T}x$ is defined over $\mathbb{Q}$;*
- *$T\pi(x)$ is a divergent orbit if and only if the torus $x^{-1}\mathbf{T}x$ is defined over $\mathbb{Q}$ and $\mathbb{Q}$-split.*

As a further application, we deduce from the first statement in Theorem 1.2 by a standard use of Zorn's lemma:

Corollary 1.6. *Any closed T-invariant subset of G/Γ contains a minimal (w.r.t. inclusion) closed invariant subset.*

To conclude this introduction we briefly describe our method. All our results follow from Theorem 1.2, whose proof we now discuss. Assume for simplicity that G is reductive. Following Margulis' argument, to establish the first statement of Theorem 1.2, we use a 'push out' argument: we show that there is a finite subset $F \subset T$ and a neighborhood W of 0 in the Lie algebra $\mathcal{G}$ of G such that for any $g \in G$, there is $t \in F$ such that all elements of $W \cap \mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}}$ are enlarged by applying $\mathrm{Ad}(t)$ (see Proposition 4.1 for a precise formulation). Then, applying successively elements of F, after finitely many steps we obtain $t_0 \in T$ such that $\mathrm{Ad}(t_0 g)\mathcal{G}_{\mathbb{Z}} \cap W = \{0\}$, which implies that $t_0\pi(g)$ is in a compact subset of G/Γ depending only on W.

Note that a statement similar to Proposition 4.1 is established in [KaMa], where it is shown that F as above exists in G. We show that F can be found inside T.

Note also that in the case of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$, using Mahler's compactness criterion, it suffices to prove Proposition A.1, in which the adjoint representation is replaced by the action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$. This greatly simplifies the calculations.

We then show that in case a compact $C \subset T$ is given and $x \notin Z_G(S)\mathbf{G}(\mathbb{Q})$ then the element t_0 obtained above can be shown not to belong to C. In the case $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$, using Mahler's criterion, this boils down to proving Proposition A.2: for $g \notin T\mathrm{SL}(n,\mathbb{Q})$, any finite set L of nonzero vectors in $g\mathbb{Z}^n$, and any neighborhood W of 0 in $\mathbb{R}^n$, there is $t_0 \in T - C$ such that $t_0 \cdot L \cap W = \emptyset$.

In the general case, an analogue of Proposition A.2 in which the action of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$ is replaced with the adjoint representation of G on $\mathcal{G}$, turns out to be false. Remedying this is one of the difficult parts of the proof, and requires a compactness criterion involving what we call 'horospherical subsets' – finite sets of vectors spanning the unipotent radical of a maximal $\mathbb{Q}$-parabolic subalgebra – rather than individual vectors (see Definition 3.4 and Proposition 3.5). We prove Proposition 5.1, which is an analogue of Proposition A.2 for horospherical subsets. Thus it appears that horospherical subsets share some of the advantageous properties of Mahler's compactness criterion for $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$.

The results of this paper were announced in the partially survey paper [To].

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2. Preliminaries

2.1. **Notation and terminology.** As usual $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ denote the complex, real, rational numbers and integers, respectively. Recall that a $\mathbb{Q}$-*algebraic group* $\mathbf{G}$ (or an algebraic group $\mathbf{G}$ *defined over* $\mathbb{Q}$) is a subgroup of $\mathrm{SL}(n,\mathbb{C})$ defined as the set of common zeros of a system of polynomials in $\mathbb{Q}[X_{11}, X_{12}, \ldots, X_{nn}]$. The set of integer matrices in $\mathbf{G}$ is denoted by $\mathbf{G}(\mathbb{Z})$. Also recall that if k is a subfield of $\mathbb{C}$, a Zariski connected algebraic subgroup of $\mathbf{G}$ is called a *k-split k-subtorus* of $\mathbf{G}$ (or defined over k and k-split subtorus of $\mathbf{G}$, or k-split subtorus of $\mathbf{G}$) if it is conjugate under $\mathrm{GL}(n,k)$ to a subgroup of diagonal matrices of $\mathrm{GL}(n,\mathbb{C})$ (see [Bo1, §8] for a discussion of tori). We denote by $\mathbf{S}$ a maximal $\mathbb{Q}$-split $\mathbb{Q}$-subtorus in $\mathbf{G}$ and by $\mathbf{T}$ a maximal $\mathbb{R}$-split $\mathbb{R}$-subtorus in $\mathbf{G}$, which contains $\mathbf{S}$. The last hypotheses on $\mathbf{T}$ entail no loss of generality, since all maximal k-split tori are conjugate under $\mathbf{G}(k)$ [Bo1, 20.9]. The real rank of $\mathbf{G}$ (notation: $\mathrm{rank}_{\mathbb{R}}\mathbf{G}$) is by definition the same as $\dim \mathbf{T}$, and the $\mathbb{Q}$-rank (notation: $\mathrm{rank}_{\mathbb{Q}}\mathbf{G}$) is the same as $\dim \mathbf{S}$.

We will use boldface letters in order to denote $\mathbb{R}$-algebraic groups and the corresponding uppercase letters to denote the groups of $\mathbb{R}$-rational points of these $\mathbb{R}$-algebraic groups. So, we have $G = \mathbf{G}(\mathbb{R})$, $S = \mathbf{S}(\mathbb{R})$, $T = \mathbf{T}(\mathbb{R})$, etc. The group G is called a *real* $\mathbb{Q}$-algebraic group and S is a *real* $\mathbb{Q}$-split $\mathbb{Q}$-torus.

We denote by $\mathcal{R}_u(\mathbf{G})$ (resp., $\mathcal{R}_u(G)$) the (real points of) the unipotent radical of $\mathbf{G}$.

Given a Lie group H we denote the connected component of the identity in H by H^0.

If H and L are subgroups of G then $N_L(H)$ denotes the normalizer of H in L and $Z_L(H)$ denotes the centralizer of H in L.

Let $\pi : G \to G/\Gamma$ be the natural quotient map. For a closed subgroup H of G, we let

$$\Gamma_H = \Gamma \cap H, \quad \text{and } \pi_H : H \to H/\Gamma_H$$

be the natural quotient map.

For any $x = \pi(g) \in G/\Gamma$ we denote

$$\mathrm{Stab}_H(x) = \{h \in H : hx = x\} = H \cap g\Gamma g^{-1}.$$

The *orbit map* is the injective map

$$h\mathrm{Stab}_H(x) \mapsto h\pi(g).$$

It is proper (i.e., the inverse image of a compact set is compact) if and only if $H\pi(g)$ is closed in G/Γ.

We denote by $\mathcal{G}$ the Lie algebra of G. From now on we fix a *norm* $\|\cdot\|$ on $\mathcal{G}$, and using the exponential map, define a *norm on* T^0 (and on S^0) which will be also denoted by $\|\cdot\|$.

We say that a Lie subalgebra $\mathcal{U}$ of $\mathcal{G}$ is *unipotent* if its corresponding subgroup U is unipotent.

2.2. **$\mathbb{Q}$-Lie algebras and roots.** The Lie algebra $\mathrm{Lie}(\mathbf{G})$ of $\mathbf{G}$ is equipped with a $\mathbb{Q}$-structure which is compatible with the $\mathbb{Q}$-structure of $\mathbf{G}$, i.e. the adjoint representation of $\mathbf{G}$ on $\mathrm{Lie}(\mathbf{G})$ is defined over $\mathbb{Q}$. Clearly $\mathcal{G} = \mathrm{Lie}(\mathbf{G})(\mathbb{R})$. We denote $\mathcal{G}_{\mathbb{Z}} = \mathrm{Lie}(\mathbf{G})(\mathbb{Z})$. Recall that

$\mathrm{Ad}(\Gamma)$ is an arithmetic subgroup of $\mathrm{Ad}(\mathbf{G})$ (cf. [Bo3]). So there is an arithmetic subgroup $\Gamma_0 \subset G$ such that $\mathrm{Ad}(\Gamma_0)\mathcal{G}_{\mathbb{Z}} = \mathcal{G}_{\mathbb{Z}}$. Recall that Γ_1 and Γ_2 are said to be *commensurable* if $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2. In view of Lemma 6.1 below the validity of all assertions we will prove is unaffected by a passage from Γ to a commensurable subgroup. So, we can and will assume that

$$\mathrm{Ad}(\Gamma)\mathcal{G}_{\mathbb{Z}} = \mathcal{G}_{\mathbb{Z}}.$$

For any $x = \pi(g) \in G/\Gamma$, we will let

$$\mathcal{G}_x = \mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}}$$

(which makes sense in view of the above hypothesis).

We denote by $X(\mathbf{T})$ the (abelian) group of $\mathbb{R}$-rational characters of $\mathbf{T}$ and by $X(\mathbf{S})$ the group of $\mathbb{Q}$-rational characters of $\mathbf{S}$ (which coincides with the group of $\mathbb{R}$-rational characters of $\mathbf{S}$) . Characters are written additively and are identified with their derivatives, that is, we think of a character as a linear functional on $\mathcal{T} = \mathrm{Lie}(T)$ and of a $\mathbb{Q}$-character as a linear functional on $\mathcal{S} = \mathrm{Lie}(S)$. A nonzero ($\mathbb{Q}$-) character χ is a ($\mathbb{Q}$-) *root* if there is a nonzero vector $v \in \mathcal{G}$ such that for all $t \in T$ (resp., all $t \in S$) we have

$$\mathrm{Ad}(t)v = e^{\chi(t)}v. \tag{1}$$

The root–space corresponding to χ is the subspace consisting of all vectors v for which (1) holds for all t. It is denoted by $\mathcal{G}_\chi$. If χ is a $\mathbb{Q}$-root then $\mathcal{G}_\chi$ is defined over $\mathbb{Q}$ and therefore is spanned by $\mathcal{G}_\chi \cap \mathcal{G}_{\mathbb{Z}}$.

2.3. **Parabolic subgroups.** More generally, let k be any field, K an algebraically closed field containing k, $\mathbf{G}$ a reductive k-algebraic group, $\mathbf{S}$ a maximal k-split algebraic torus, and $\mathbf{T}$ a maximal subtorus of $\mathbf{G}$ containing $\mathbf{S}$. One can define K-roots and k-roots (cf. [Bo1, §5]). The former are K-rational characters on $\mathbf{T}$ and the latter are k-rational characters on $\mathbf{S}$. We will denote the set of K-roots by Φ_K and the set of k-roots by Φ_k (or simply Φ when k is clear from the context) . A choice of a Borel subgroup containing $\mathbf{T}$ (resp., a minimal parabolic k-subgroup containing $\mathbf{S}$) corresponds to a choice of an order on Φ_K (resp., on Φ). We denote the positive elements of Φ_K (resp.Φ) by Φ_K^+ (resp., by Φ^+). We denote the set of all simple roots in Φ_K^+ (resp., Φ^+) by Π (resp. Δ).

Once Δ is fixed, for every $\alpha \in \Delta$ we define a projection $\pi_\alpha : \Phi \to \mathbb{Z}$ by $\pi_\alpha(\lambda) = n_\alpha$ where $\lambda = \sum_{\beta \in \Delta} n_\beta \beta$.

Let

$$\Phi_0 = \{\alpha \in \Phi : \frac{1}{2}\alpha \notin \Phi\}.$$

It is well known that Φ_0 is a *reduced* root system and the root systems Φ and Φ_0 have the same Weyl chambers, bases of simple roots and Weyl group (cf. [ViO, ch. 4, §2]).

Fixing a Borel subgroup $\mathbf{B} \subset \mathbf{G}$, a subgroup $\mathbf{P} \subset \mathbf{G}$ is called a *standard parabolic* if $\mathbf{B} \subset \mathbf{P}$. The standard ($k$-) parabolic subgroups are described in terms of subsets of Π (resp. Δ) (see [Bo1, §§14.6–14.18]), as follows. For $\Psi \subset \Pi$ (or $\Psi \subset \Delta$), the Lie algebra of the corresponding group $\mathbf{P}_\Psi$ is the sum of $\mathrm{Lie}(Z_G(T))$ (resp. $\mathrm{Lie}(Z_G(S))$) and

$$\bigoplus_{\forall \alpha \in \Psi,\ \pi_\alpha(\chi) \geq 0} \mathcal{G}_\chi.$$

The Lie algebra of $\mathcal{R}_u(\mathbf{P}_\Psi)$ is:

$$\bigoplus_{\exists \alpha \in \Psi,\ \pi_\alpha(\chi) > 0} \mathcal{G}_\chi. \tag{2}$$

The Lie algebra which is the sum of $\mathrm{Lie}(Z_G(T))$ (resp. $\mathrm{Lie}(Z_G(S))$) and

$$\bigoplus_{\pi_\alpha(\chi) = 0} \mathcal{G}_\chi \tag{3}$$

corresponds to a maximal reductive subgroup of $\mathbf{P}_\Psi$ which we call the *Levi factor* of $\mathbf{P}_\Psi$.

A *maximal* standard (k-) parabolic is obtained by choosing $\Psi = \{\alpha\}$ for some $\alpha \in \Pi$ (resp., $\alpha \in \Delta$). It is clear from the above that we have:

$$\mathrm{Lie}(\mathbf{P}_\Psi) = \bigcap_{\alpha \in \Psi} \mathrm{Lie}(\mathbf{P}_{\{\alpha\}}). \tag{4}$$

3. Siegel sets, horospherical subsets and compactness criterion

In this section $\mathbf{G}$ is a connected reductive $\mathbb{Q}$-algebraic group.

3.1. **Siegel sets.** Let us fix a minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ and a maximal $\mathbb{Q}$-split $\mathbb{Q}$-subtorus $\mathbf{S}$ of $\mathbf{P}$. Denote by M the connected component of the (unique) maximal compact subgroup of $Z_G(S)$, by K a maximal compact subgroup of G and by Δ the set of simple $\mathbb{Q}$-roots of $\mathbf{G}$ corresponding to the choice of $\mathbf{P}$. For every $r > 0$ we denote $B_r = \{x \in \mathcal{G} : \|x\| \leq r\}$. Also for every $\eta > 0$ we put

$$S_\eta = \{s \in S : \forall \alpha \in_{\mathbb{Q}} \Delta,\ \alpha(s) \leq \eta\}. \tag{5}$$

Following [Bo2, 12.3], by a *Siegel set* with respect to K, P and S we mean the set

$$\Sigma = \Sigma_{\eta,\omega} = KS_\eta\omega \tag{6}$$

where ω is a compact neighborhood of e in $M\mathcal{R}_u(P)$.

The Siegel sets are related to the fundamental sets. A subset $\Sigma \subset G$ is *fundamental for an arithmetic subgroup* Γ of G if the following conditions are fulfilled : $K\Sigma = \Sigma$, $\Sigma\Gamma = G$ and, for each $b \in \mathbf{G}(\mathbb{Q})$, the set

$$\{\gamma \in \Gamma : \Sigma b \cap \Sigma\gamma \neq \emptyset\}$$

is finite.

Recall the following classical result [Bo2, Theorem 15.5]:

Theorem 3.1 (Borel and Harish-Chandra)**.** *With the above notations there exists a Siegel set Σ and a finite subset $C \subset \mathbf{G}(\mathbb{Q})$ such that $\Omega = \Sigma C$ is a fundamental subset for Γ in G. Furthermore, Ω is compact if and only if $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} = 0$ and it has finite Haar measure if and only if $\mathbf{G}^0$ does not admit non-trivial $\mathbb{Q}$-characters.*

Note that if A is a precompact subset of $\mathcal{R}_u(P)$ and $\eta > 0$ then $\{sus^{-1} : s \in S_\eta,\ u \in A\}$ is also precompact. Since S centralizes M it follows from (6), the definition of S_η and Theorem 3.1 that the following assertion holds.

Proposition 3.2. *There exist $\eta_0 > 0$, a compact $K_0 \subset G$ and a finite subset $C_0 \subset \mathbf{G}(\mathbb{Q})$ such that*

$$\Sigma_0 = K_0 S_{\eta_0} C_0$$

is fundamental subset for Γ.

The above proposition is needed for the proof of the following

Proposition 3.3. *There exists a compact neighborhood W of 0 in $\mathcal{G}$ such that for every $x \in G/\Gamma$ the subalgebra generated by $W \cap \mathcal{G}_x$ is unipotent.*

Proof. Let P^- be the parabolic subgroup of G opposite to P. Then

$$\mathcal{G} = \mathcal{U} \oplus \mathcal{Z} \oplus \mathcal{U}^- \tag{7}$$

where $\mathcal{U} = \mathrm{Lie}(\mathcal{R}_u(P))$, $\mathcal{U}^- = \mathrm{Lie}(\mathcal{R}_u(P^-))$ and $\mathcal{Z} = \mathrm{Lie}(Z_G(S))$. Note that the decomposition in (7) is defined over $\mathbb{Q}$ and therefore, after replacing $\mathcal{G}_{\mathbb{Z}}$ with a finite index lattice,

$$\mathcal{G}_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}} \oplus \mathcal{Z}_{\mathbb{Z}} \oplus \mathcal{U}^-_{\mathbb{Z}}, \tag{8}$$

where $\mathcal{U}_{\mathbb{Z}},\ \mathcal{Z}_{\mathbb{Z}},\ \mathcal{U}^-_{\mathbb{Z}}$ are the lattices of integer vectors in $\mathcal{U},\ \mathcal{Z},\ \mathcal{U}^-$ respectively.

We choose the norm $\|\cdot\|$ in such a way that

$$\|v\| = \max\{\|u\|, \|z\|, \|u^-\|\} \tag{9}$$

for all $v = u + z + u^- \in \mathcal{G}$, $u \in \mathcal{U}$, $z \in \mathcal{Z}$ and $u^- \in \mathcal{U}^-$.

Since $\mathrm{Ad}(q)\mathcal{G}_{\mathbb{Z}}$ is commensurable with $\mathcal{G}_{\mathbb{Z}}$ if $q \in \mathbf{G}(\mathbb{Q})$, there exists a positive integer n such that

$$\frac{1}{n}\mathcal{G}_{\mathbb{Z}} \supset \mathrm{Ad}(g)\mathcal{G}_{\mathbb{Z}} \tag{10}$$

for all $g \in C_0$.

In view of the compactness of K_0 and (5) there exists a constant $0 < c < 1$ such that

$$\|\mathrm{Ad}(h)w\| \geq c\|w\| \tag{11}$$

if $w \in \mathcal{G}$ and $h \in K_0$, or if $w \in \mathcal{U}^- \oplus \mathcal{Z}$ and $h \in S_{\eta_0}$.

Let $r > 0$ be such that $B_r \cap \mathcal{G}_{\mathbb{Z}} = \{0\}$ and $W = B_\epsilon$ where $\epsilon < \frac{rc^2}{n}$. Let $x = \pi(g)$, where $g = ksq$, $k \in K_0$, $s \in S_{\eta_0}$ and $q \in C_0$, and let $\mathrm{Ad}(g)v \in W$, where $v \in \mathcal{G}_{\mathbb{Z}},\ v \neq 0$. Write $\mathrm{Ad}(q)v = v_1 + v_2$, where $v_1 \in \mathcal{U}^- \oplus \mathcal{Z}$ and $v_2 \in \mathcal{U}^+$. By (8) and (10) we have $nv_i \in \mathcal{G}_{\mathbb{Z}},\ i = 1, 2$. Assume by contradiction that $v_1 \neq 0$. By the choice of r we then have $\|v_1\| \geq r/n$ and in view of (9) and (11),

$$\begin{aligned}
\|\mathrm{Ad}(g)v\| &= \|\mathrm{Ad}(ks)(v_1 + v_2)\| \\
&\geq c\|\mathrm{Ad}(s)(v_1 + v_2)\| \\
&\geq c\|\mathrm{Ad}(s)v_1\| \\
&\geq c^2\|v_1\| \\
&\geq \frac{rc^2}{n} > \epsilon,
\end{aligned}$$

a contradiction. Therefore $v_1 = 0$, i.e. $\mathrm{Ad}(q)v = v_2 \in \mathcal{U}$ for all $v \in W \cap \mathcal{G}_x$, which completes the proof. □

3.2. **Horospherical Subsets.** Let us introduce the following

Definition 3.4. By a *horospherical subset* we mean a minimal (with respect to inclusion) finite subset M of $\mathcal{G}$ which spans a subalgebra conjugate to the unipotent radical of a maximal parabolic $\mathbb{Q}$-subalgebra of $\mathcal{G}$.

Proposition 3.5 (Compactness Criterion)**.** *A subset $A \subset G/\Gamma$ is precompact if and only if there exists a neighborhood W of 0 in $\mathcal{G}$ such that for all $x \in A$, $\mathcal{G}_x \cap W$ does not contain a horospherical subset.*

Proof. Suppose first that A is precompact. Thus there is a compact subset $K \subset G$ such that $A \subset \pi(K)$. From continuity of Ad, there is a neighborhood W of 0 in $\mathcal{G}$ such that $\mathrm{Ad}(K)(\mathcal{G}_{\mathbb{Z}}) \cap W = \{0\}$. In particular for any $x \in A$, $\mathcal{G}_x \cap W = \{0\}$ does not contain a horospherical subset.

Now suppose A is not precompact. In view of Theorem 3.1 and Proposition 3.2 there exists a sequence $\{g_n\}$ such that $\pi(g_n) \in A$, $g_n = k_n s_n f_n$ where $k_n \in K_0$, $s_n \in S_{\eta_0}$ and $f_n \in C_0$, and $g_n \to \infty$. Passing to a subsequence, we find $\alpha \in_{\mathbb{Q}} \Delta$ such that $\alpha(s_n) \to -\infty$, and assume that $f_n = f$ for some fixed $f \in C_0$. Let $\mathbf{P}_\alpha$ be the maximal $\mathbb{Q}$-parabolic corresponding to α, and let $\mathcal{V}_\alpha$ be the Lie algebra of $\mathcal{R}_u(P_\alpha)$. It follows from (2) that $\mathrm{Ad}(s_n)u \to 0$ for every $u \in \mathcal{V}_\alpha$. Let $u_1, \ldots, u_r$ be a basis for $\mathcal{V}_\alpha$ which is contained in $\mathcal{G}_{\mathbb{Z}}$. Multiplying all of the u_i by the common denominator of the coordinates of $\mathrm{Ad}(f^{-1})$, we may assume that $\mathrm{Ad}(f^{-1})u_i \in \mathcal{G}_{\mathbb{Z}}$ for all $i = 1, \ldots, r$. Since K_0 is compact we get

$$\lim_{n\to\infty} \mathrm{Ad}(g_n)(f^{-1}u_i) = 0$$

for $i = 1, \ldots, r$. In particular any neighborhood W of 0 in $\mathcal{G}$ contains the horospherical subset $\{\mathrm{Ad}(g_n)(f^{-1}u_1), ..., \mathrm{Ad}(g_n)(f^{-1}u_r)\}$ for all sufficiently large n. □

4. Pushing out

In the present section we prove the following

Proposition 4.1. *Suppose $\mathbf{G}$ is a reductive $\mathbb{Q}$-group. Then there exist a compact neighborhood W of 0 in $\mathcal{G}$, a constant $c > 1$ and a finite subset F of T^0 such that for every $x \in G/\Gamma$ there is $f \in F$ such that for all $v \in \mathrm{span}(W \cap \mathcal{G}_x)$,*

$$\|\mathrm{Ad}(f)v\| \geq c\|v\|.$$

Let us fix a maximal T-invariant unipotent subgroup $U_0 \subset G$. Let $\mathcal{U}_0$ be the Lie algebra of U_0 and $r = \mathrm{dim}U_0$. Let $\bigwedge^r \mathcal{G}$ be the r-th exterior power of $\mathcal{G}$, $\mathrm{P}(\bigwedge^r \mathcal{G})$ the linear projective space corresponding to $\bigwedge^r \mathcal{G}$ and $\mathrm{Gr}_r(\mathcal{G})$ the Grassmannian subvariety of $\mathrm{P}(\bigwedge^r \mathcal{G})$. $\mathrm{Gr}_r(\mathcal{G})$ is compact and its points corresespond bijectively to the r-dimensional linear subspaces of $\mathcal{G}$. We fix a nonzero vector $a \in \bigwedge^r \mathcal{U}_0$ and denote by $[a]$ the point in $\mathrm{Gr}_r(\mathcal{G})$ corresponding to $\mathcal{U}_0$. The group G acts on $\mathrm{Gr}_r(\mathcal{G})$ via the adjoint representation. Put $X_0 = \mathrm{Ad}(G)[a]$. From the fact that all minimal parabolic subgroups are conjugate, if follows that X_0 is the *space of all maximal unipotent subalgebras of* $\mathcal{G}$. Since $P_0 = \{g \in G : \mathrm{Ad}(g)[a] = [a]\}$ is a (minimal) parabolic subgroup, the quotient G/P_0 is compact and therefore so is X_0. A simple argument using the continuity of the action of G on $\mathrm{Gr}_r(\mathcal{G})$ proves the following

Lemma 4.2. *Let $g \in G$ and $\mathcal{U} \in X_0$. Assume that there exists $c > 1$ such that*

$$\|\mathrm{Ad}(g)v\| \geq c\|v\|$$

for all non-zero vectors $v \in \mathcal{U}$, and let $1 < c' < c$.

(i) Then there exists a neighborhood W of $\mathcal{U}$ in X_0 such that for all $\mathcal{U}' \in W$ and all $v' \in \mathcal{U}'$

$$\|\mathrm{Ad}(g)v'\| \geq c'\|v'\|;$$

(ii) Assume in addition that $\mathcal{U} = \lim_{n\to+\infty} \mathrm{Ad}(g^n)\mathcal{U}_1$ where $\mathcal{U}_1 \in X_0$. Then for every $c_1 > 1$ there exists $n_1 > 0$ such that

$$\|\mathrm{Ad}(g^n)v_1\| \geq c_1\|v_1\|$$

for all $n > n_1$ and all $v_1 \in \mathcal{U}_1$.

We also need

Lemma 4.3. *Let $\mathcal{U} \in X_0$ and $c > 1$. Then there exists $t \in T^0$ such that*

$$\|\mathrm{Ad}(t)v\| \geq c\|v\|$$

for all $v \in \mathcal{U}$.

Proof. Let $\mathcal{U}_0$ and a be as above. We choose an order on the roots $\Phi_{\mathbb{R}}$ (equivalently, a basis of simple roots Δ), so that $\mathcal{U}_0$ is spanned by the root subspaces corresponding to all positive roots. Let

$$\bigwedge^r \mathcal{G} = \bigoplus V_\lambda \tag{12}$$

be the decomposition of $\bigwedge^r \mathcal{G}$ into a direct sum of weight subspaces, where λ_0 is the highest weight in (12). Then a spans V_{λ_0}. There exists $g \in G$ such that $\mathrm{Ad}(g)\mathcal{U}_0 = \mathcal{U}$. Let $g = unp$, where $u \in U_0$, $n \in N_G(T)$ and $p \in P_0$ be the Bruhat decomposition of g. Denote by w the projection of n into the Weyl group $N_G(T)/Z_G(T)$. Clearly, $v_0 = \mathrm{Ad}(np)a \in V_{w\lambda_0}$.

Let us say that a vector $v \in \bigwedge^r \mathcal{G}$ *dominates* v_0 if $v = v_0 + v_1$, where

$$v_1 \in \check{V} = \bigoplus_{\lambda > \omega\lambda_0} V_\lambda.$$

We now claim that if v dominates v_0 then so does $\mathrm{Ad}(u)(v)$.

To see this, we write $u = u_1 u_2 \cdots u_r$, where each u_i belongs to a root subgroup, that is, $u_i \in \exp(\mathcal{G}_{\chi_i}),\ \chi_i > 0$. By induction on r it suffices to prove our claim in case $r = 1$, that is in case $u = \exp(X)$ with $X \in \mathcal{G}_\chi,\ \chi > 0$. From the representation theory of sl(2) we know that for $v' \in V_\lambda,\ \ \mathrm{ad}_X^k(v') \in V_{\lambda + k\chi}$.

Now we compute:

$$\begin{aligned}\mathrm{Ad}(u)(v_0 + v_1) &= \mathrm{Ad}(\exp(X))(v_0) + \mathrm{Ad}(\exp(X))(v_1)\\ &= v_0 + \sum_{k\geq 1} \frac{1}{k!}\mathrm{ad}_X^k(v_0) + \sum_{k \geq 0}\frac{1}{k!}\mathrm{ad}_X^k(v_1)\\ &= v_0 + v_2,\end{aligned}$$

where $v_2 \in \check{V}$. This proves our claim. In particular we obtain that $\mathrm{Ad}(g)(a) = \mathrm{Ad}(u)(v_0)$ dominates v_0.

Let t be an element from the interior of the Weyl chamber corresponding to $w\Delta$. The highest weight for the action of $\mathrm{Ad}(t)$ on $\bigwedge^r \mathcal{G}$ is $w\lambda_0$. Therefore $[v_0]$ is an attracting fixed point for the induced action of t on $P(\bigwedge^r \mathcal{G})$. The basin of attraction consists of all $[v]$ for which $v \in \bigwedge^r \mathcal{G}$ has a nonzero $V_{w\lambda_0}$ component. Since we have proved that $\mathrm{Ad}(g)(a)$ dominates v_0, it has a nonzero $V_{w\lambda_0}$ component, and hence

$$\lim_{m\to+\infty} \mathrm{Ad}(t^m)(\mathrm{Ad}(g)[a]) = \mathrm{Ad}(n)[a]$$

in X_0. Now the lemma follows from Lemma 4.2 (ii). □

Proof of Proposition 4.1. Using Lemma 4.3, Lemma 4.2 (i), and the compactness of X_0, we obtain for any $c > 1$ a finite subset $F \subset T^0$ such that for each unipotent subalgebra $\mathcal{U} \subset \mathcal{G}$ there exists an $f \in F$ with

$$\|\mathrm{Ad}(f)v\| \geq c\|v\|$$

for all $v \in \mathcal{U}$.

The proposition now follows by taking W as in Proposition 3.3. □

5. Characterization of $Z_G(S)\mathbf{G}(\mathbb{Q})$

The goal of this section is the proof of

Proposition 5.1. *Let $\mathbf{G}$ be a reductive $\mathbb{Q}$-algebraic group, and let $g \in G$ with $\mathrm{Stab}_S(\pi(g))$ finite. Suppose there are compact subset $C \subset \mathcal{G}$ and a real $r > 0$ such that the following holds: for every $d \in S$ with $\|d\| \geq r$ there exists a horospherical subset $M \subset \mathrm{Ad}(g)\mathcal{G}_\mathbb{Z} \cap C$ such that $\mathrm{Ad}(d)M \subset C$. Then $g \in Z_G(S)\mathbf{G}(\mathbb{Q})$.*

The proof of the proposition relies on two propositions about parabolic subgroups, true in the context of any reductive k-algebraic groups, and on a certain rationality criterion.

5.1. **Intersections of Parabolic Subgroups.** In the following two propositions we suppose that $\mathbf{G}$ is a reductive algebraic group defined over an *arbitrary* field k. We use the notation from 2.3.

Proposition 5.2. *For every minimal parabolic k-subgroup $\mathbf{B}$ containing $\mathbf{S}$ we fix a proper parabolic k-subgroup $\mathbf{P}_B$ containing $\mathbf{B}$. Then*

$$\bigcap_{\mathbf{B}} \mathbf{P}_{\mathbf{B}} \subset N_{\mathbf{G}}(\mathbf{S}). \tag{13}$$

Proof. Let $\mathbf{H}$ be the subgroup of $\mathbf{G}$ defined by the left-hand side of (13), and let $\mathcal{H} = \mathrm{Lie}(H)$. Since $N_{\mathbf{G}}(\mathbf{S})^0 = Z_{\mathbf{G}}(\mathbf{S})$ it is enough to prove that $\mathcal{H}$ coincides with $\mathrm{Lie}(Z_G(S))$.

Note that $\mathbf{S} \subset \mathbf{H}$. Hence $\mathcal{H}$ is a sum of root-spaces with respect to S. Suppose by contradiction that λ is a nontrivial root with $\mathcal{G}_\lambda \subset \mathcal{H}$. Let $\mathbf{B}$ be a minimal parabolic k-subgroup, and let $\mathbf{B}^-$ be the opposite minimal parabolic k-subgroup. Then $\mathcal{G}_\lambda$ is contained in the Levi factor of either $\mathrm{Lie}(P_B)$ or $\mathrm{Lie}(P_{B^-})$), otherwise λ would be simultaneously positive and negative with respect to the order determined by $\mathbf{B}$.

It follows from the above and (3) that for any base of simple roots Δ there is $\beta \in \Delta$ such that $\pi_\beta(\lambda) = 0$. But all roots of a given length are conjugate [Hu1, 10.4, Lemma C and 10.3, Theorem]. Therefore there is a basis of simple roots Δ_0 for which λ is a maximal long or a maximal short root in the reduced root system Φ_0. In order to complete the proof of the lemma it is enough to show that in this case $\pi_\beta(\lambda) \neq 0$ for all $\beta \in \Delta_0$. If λ is a maximal long root the fact is proved in [Hu1, 10.4, Lemma A]. Let λ be a maximal short root. Then its dual $\lambda^\vee$ is a maximal long root in the dual root system $\Phi_0^\vee$ [Hu1, Ex.11, p.55]. Applying again [Hu1, 10.4, Lemma A], we get that $\pi_{\beta^\vee}(\lambda^\vee) \neq 0$ for all $\beta^\vee \in \Delta^\vee$. Since $\gamma^\vee = \frac{2\gamma}{(\gamma,\gamma)}$ for any character γ of $\mathbf{S}$, we obtain that $\pi_\beta(\lambda) \neq 0$ for all $\beta \in \Delta_0$, which completes the proof.

□

Proposition 5.3. *Let $\mathbf{B}$ be a minimal parabolic k-subgroup of $\mathbf{G}$ and $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{V}$ such that $\mathbf{P}$ is conjugate to a k-subgroup of $\mathbf{G}$ and $\mathbf{V} \subset \mathbf{B}$. Then:*

(i) $\mathbf{B} \subset \mathbf{P}$ and $\mathbf{P}$ is a k-group.

(ii) If $g \in \mathbf{G}$ and $g\mathbf{V}g^{-1} \subset \mathbf{B}$ then $g \in \mathbf{P}$.

Proof. First note that (i) implies (ii). Indeed, assuming the validity of (i) we obtain that $\mathbf{B} \subset \mathbf{P}$ and $\mathbf{B} \subset g\mathbf{P}g^{-1}$. Therefore $\mathbf{P} = g\mathbf{P}g^{-1}$ and $g \in \mathbf{P}$ (see [Bo1, 14.22(iii) and 11.16]).

In order to prove (i), denote by $\mathbf{U}$ the unipotent radical of $\mathbf{B}$, by $\mathbf{B}_0$ a Borel subgroup of $\mathbf{G}$ such that $\mathbf{UV} \subset \mathbf{B}_0 \subset \mathbf{B}$, and by $\mathbf{T}$ a maximal subtorus of $\mathbf{B}_0$. There exists a unique parabolic k-subgroup $\mathbf{P}_0$ such that $\mathbf{B} \subset \mathbf{P}_0$ and $\mathbf{P} = h\mathbf{P}_0h^{-1}$ where $h \in \mathbf{G}$. Let $\mathbf{V}_0$ be the unipotent radical of $\mathbf{P}_0$. Let $h = bwu$ (where b and $u \in \mathbf{B}_0$, $w \in N_{\mathbf{G}}(\mathbf{T})$ be the Bruhat decomposition of h. (Further on, we will identify w with its projection into the Weyl group W_K corresponding to $\mathbf{T}$.) Since $\mathbf{V}_0$ is a normal subgroup of $\mathbf{B}_0$,$\mathbf{V} \subset \mathbf{B}_0$ and $\mathbf{V} = h\mathbf{V}_0h^{-1}$, we get that $w\mathbf{V}_0w^{-1} \subset \mathbf{B}_0$. Assume for the moment that $w \in \mathbf{P}_0$. Then $h \in \mathbf{P}_0$, i.e. $\mathbf{P} = \mathbf{P}_0$. In particular, $\mathbf{P} \supset \mathbf{B}$ and $\mathbf{P}$ is a k-group.

So, our proposition follows from the next lemma.

Lemma 5.4. *Let*

$$w\mathbf{V}_0w^{-1} \subset \mathbf{B}_0,$$

where $\mathbf{B}_0$ *is a Borel subgroup of* $\mathbf{G}$*,* $w \in W_K$ *and* $\mathbf{V}_0$ *is the unipotent radical of a parabolic subgroup* $\mathbf{P}_0$ *containing* $\mathbf{B}_0$*. Then* $w \in \mathbf{P}_0$*.*

Proof. Using the fact that $N_{\mathbf{G}}(\mathbf{P}_0) = \mathbf{P}_0$ is suffices to show that $\mathrm{Ad}(w)(\mathrm{Lie}(\mathbf{P}_0)) = \mathrm{Lie}(\mathbf{P}_0)$, and in view of the description of the standard parabolic subgroups (see §2, (2), (4)), it is enough to prove this when $\mathbf{P}_0$ is maximal. In this case there exists $\alpha \in \Pi$ such that the Lie algebra of $\mathbf{V}_0$ is the span of the $\mathcal{G}_\chi$ with $\pi_\alpha(\chi) > 0$.

Let W_0 be the Weyl group of the Levi subgroup of $\mathbf{P}_0$, i.e. W_0 is generated by all reflections in Φ_K corresponding to $\beta \in \Pi,\ \beta \neq \alpha$.

Recall that w can be written uniquely as $w = xw'$ where $w' \in W_0$ and $x \in W_K$ is such that $x(\beta) > 0$ for all $\beta \in \Pi,\ \beta \neq \alpha$ [Hu2, Proposition 1.10 and Lemma 1.6]. Since w' normalizes $\mathbf{V}_0$, there is a root λ with $\pi_\alpha(\lambda) > 0$ and such that $w'(\lambda) = \alpha$. By the assumption on w,

$$x(\alpha) = w(\lambda) > 0$$

and therefore $x(\Phi_K^+) = \Phi_K^+$, i.e. $x = 1$ and $w \in \mathbf{P}_0$.

□

5.2. **Rationality Criterion.** Because of lack of reference we will prove the following apparently known rationality criterion.

Proposition 5.5. *Let* $\mathbf{V}$ *be a* $\mathbb{Q}$*-algebraic variety and* $\mathbf{W}$ *be an* $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$*-invariant closed algebraic subvariety of* $\mathbf{V}$*. Then* $\mathbf{W}$ *is defined over* $\mathbb{Q}$*.*

Proof. We fix a subfield $\mathbb{C}_t$ of $\mathbb{C}$ such that $\mathbb{C}$ is an algebraic extension of $\mathbb{C}_t$ and $\mathbb{C}_t$ is a transcendental extension of $\mathbb{Q}$ (cf.[L, ch.10, §1, Theorem 1]). Let $\mathbb{C}[\mathbf{V}]$ be the ring of all regular functions of $\mathbf{V}$ and I be the ideal of all functions in $\mathbb{C}[\mathbf{V}]$ vanishing on $\mathbf{W}$. Since $\mathbf{W}$ is invariant under the action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{C}_t)$, $\mathbf{W}$ is defined over $\mathbb{C}_t$, i.e. I is generated by $I \cap \mathbb{C}_t[\mathbf{V}]$ (see [Bo1, AG§14]). Let $f \in I \cap \mathbb{C}_t[\mathbf{V}]$. We can represent f in the following way, $f = a_1 f_1 + a_2 f_2 + \cdots + a_m f_m$, where $f_i \in \mathbb{Q}[\mathbf{V}]$ and a_i are linearly independent over $\mathbb{Q}$ complex numbers from $\mathbb{C}_t$. It is enough to prove that $f_i \in I$ for all i. We will prove this by induction on m. The claim is trivial if $m = 1$. So, let $m > 1$. Without loss of generality we will assume that $a_1 = 1$. There exist finitely many algebraically independent over $\mathbb{Q}$ elements $X_1, X_2, \dots, X_n$ in $\mathbb{C}_t$ such that $\{a_2, \dots, a_m\} \subset \mathbb{Q}(X_1, X_2, \dots, X_n)$. Since $\mathbb{C}_t$ has infinite transcendental degree over $\mathbb{Q}$, there exists $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ such that the elements $\{X_1, X_2, \dots, X_n, {}^\sigma X_1, {}^\sigma X_2, \dots, {}^\sigma X_n\}$ are algebraically independent over $\mathbb{Q}$.

We claim that ${}^\sigma a_2 - a_2, {}^\sigma a_3 - a_3, \dots, {}^\sigma a_m - a_m$ are linearly independent over $\mathbb{Q}$. Indeed, otherwise there are $\lambda_2, \dots, \lambda_m \in \mathbb{Q}$ such that

$$\lambda_2 a_2 + \cdots + \lambda_m a_m = \lambda_2 \, {}^\sigma a_2 + \cdots + \lambda_m \, {}^\sigma a_m.$$

The left hand side is a rational function of the X_i's, the right hand side is a rational function of the ${}^\sigma X_j$'s, and $X_1, \dots, X_m$, ${}^\sigma X_1, \dots, {}^\sigma X_m$ are algebraically independent. Therefore $\lambda_2 a_2 + \cdots + \lambda_m a_m \in \mathbb{Q}$. This contradicts the assumption that $a_1, \dots, a_m$ are linearly independent over $\mathbb{Q}$, proving the claim.

In view of the proposition assumption, ${}^\sigma f - f = ({}^\sigma a_2 - a_2) f_2 + ({}^\sigma a_3 - a_3) f_3 + \cdots + ({}^\sigma a_m - a_m) f_m$ belongs to I. Now by the induction hypothesis $f_2, \dots, f_m \in I$, hence $f_1 \in I$, which completes the proof. □

Proof of Proposition 5.1. Let $\mathbf{B}$ be a minimal parabolic $\mathbb{Q}$-subgroup containing $\mathbf{S}$ and $\mathcal{B}$ be its Lie algebra. We claim that $\mathcal{B} \cap \mathrm{Ad}(g)\mathcal{G}_\mathbb{Z}$ contains a horospherical subset M. Indeed, let $d \in S$ be an element in the interior of the positive Weyl chamber correpsonding to $\mathbf{B}$. This means that for any $v \notin \mathcal{B}$ we have

$$\lim_{n \to +\infty} \mathrm{Ad}(d^n) v = \infty. \tag{14}$$

For every n with $\|d^n\| \geq r$ there is a horospherical subset $M_n \subset \mathrm{Ad}(g)\mathcal{G}_\mathbb{Z} \cap C$ with $\mathrm{Ad}(d^n) M_n \subset C$. Since $\mathrm{Ad}(g)\mathcal{G}_\mathbb{Z} \cap C$ is finite, the family of subsets $\{M_n : n \in \mathbb{N}\}$ is finite. Therefore there is a horospherical subset $M \subset \mathrm{Ad}(g)\mathcal{G}_\mathbb{Z} \cap C$ such that $\mathrm{Ad}(d^n) M \subset C$ for infinitely many $n \in \mathbb{N}$. Now, using (14), we get that $M \subset \mathcal{B}$, as claimed.

Let $\mathcal{V}$ denote the subalgebra generated by M, and let $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Since $\mathcal{B}$ is defined over $\mathbb{Q}$, ${}^{\sigma}M \subset \mathcal{B}$. On the other hand, $\mathrm{Ad}(g^{-1})M \subset \mathcal{G}_{\mathbb{Z}}$ and σ acts trivially on $\mathcal{G}_{\mathbb{Z}}$. Therefore, ${}^{\sigma}M = \mathrm{Ad}({}^{\sigma}gg^{-1})(M)$. So, $\mathrm{Ad}({}^{\sigma}gg^{-1})\mathcal{V} \subset \mathcal{B}$. Denote by $\mathbf{P_B}$ the normalizer of $\mathcal{V}$ in $\mathbf{G}$. Then, in view of Proposition 5.3, $\mathbf{B} \subset \mathbf{P_B}$, $\mathbf{P_B}$ is defined over $\mathbb{Q}$, and ${}^{\sigma}gg^{-1} \in \mathbf{P_B}$ for every minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{B}$. Using Proposition 5.2 we get that ${}^{\sigma}gg^{-1} \in N_{\mathbf{G}}(\mathbf{S})$. Hence, ${}^{\sigma}(g^{-1}\mathbf{S}g) = g^{-1}\mathbf{S}g$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Therefore $g^{-1}\mathbf{S}g$ is defined over $\mathbb{Q}$ (Proposition 5.5). It follows from Proposition 6.4 below that $Sg\Gamma$ is closed. Since $\mathrm{Stab}_S(\pi(g))$ is finite (by assumption) we get that $g^{-1}\mathbf{S}g$ is a maximal $\mathbb{Q}$-split torus. By [Bo1, Theorem 20.9] there exists an $h \in \mathbf{G}(\mathbb{Q})$ such that $g^{-1}\mathbf{S}g = h^{-1}\mathbf{S}h$, i.e. $g \in N_G(S)\mathbf{G}(\mathbb{Q})$. Recall that $(N_G(S) \cap \mathbf{G}(\mathbb{Q}))Z_G(S) = N_G(S)$ (cf. [BoT, Theorem 5.3]). Therefore, $g \in Z_G(S)\mathbf{G}(\mathbb{Q})$. □

6. Proof of Theorem 1.2

We first make some reductions toward the proof of the theorem. The following standard proposition justifies passing from Γ to any commensurable subgroup:

Lemma 6.1. *Let Γ and Γ' be commensurable subgroups of G, $\pi' : G \to G/\Gamma'$ the natural quotient map, and H be a closed subgroup of G. Then the following hold:*

1. *For any subset $A \subset G$, $\pi(A) \subset G/\Gamma$ is precompact (respectively, closed) if and only if $\pi'(A) \subset G/\Gamma'$ is precompact (respectively, closed).*
2. *For any $g \in G$ the orbit $H\pi(g) \subset G/\Gamma$ is divergent (respectively, closed) if and only if the orbit $H\pi'(g) \subset G/\Gamma'$ is divergent (respectively, closed).*

Proof. It is easy to deduce the lemma from the following standard facts:

- If a set $X \subset G/\Gamma$ is compact then there is a compact subset $X_0 \subset G$ such that $X = \pi(X_0)$.
- If Γ_1 is a finite index subgroup of Γ then Γ_1 contains a finite-index subgroup Γ_2 which is normal in Γ.

We omit the details. □

The following will be useful in reducing the proofs of Theorems 1.2 and 1.3 to the case that $\mathbf{G}$ is reductive.

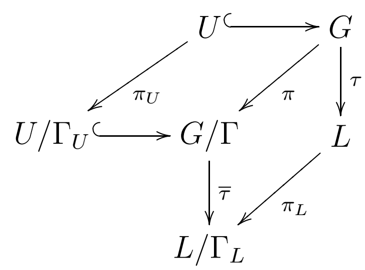
Proposition 6.2 (Levi decomposition over $\mathbb{Q}$)**.** *Let $\mathbf{G}$ be a connected $\mathbb{Q}$-algebraic group, and let $\mathbf{H}$ be its reductive $\mathbb{Q}$-algebraic subgroup.*

Then there is a reductive $\mathbb{Q}$*-subgroup* $\mathbf{L}$ *containing* $\mathbf{H}$ *such that* $\mathbf{G}$ *is a semidirect product defined over* $\mathbb{Q}$ *of* $\mathbf{L}$ *and* $\mathcal{R}_u(\mathbf{G})$*.*

Proof. See [BoS], Proposition 5.1. □

Applying Proposition 6.2 with $\mathbf{H} = \mathbf{T}$, and using the fact that when $\mathbf{N}$ is a unipotent $\mathbb{Q}$-group, Γ_N is cocompact in N [Ra, Chap. 3], we deduce:

Proposition 6.3. *Let* $\tau : \mathbf{G} \to \mathbf{L}$ *be the natural map, and let* $\mathbf{U} = \mathcal{R}_u(\mathbf{G})$*. Possibly after passing to a finite index subgroup, we have the following commutative diagram:*



In particular:

- $\Gamma_L\Gamma_U$ *is of finite index in* Γ*.*
- G/Γ *carries the structure of a fiber bundle, with* L/Γ_L *as base and* U/Γ_U *as a compact fiber.*
- *The orbit* $T\pi(x)$ *is divergent in* G/Γ *if and only if the orbit* $T\pi_L(\tau(x))$ *is divergent in* L/Γ_L*.*

We also record the following well–known fact.

Proposition 6.4. [Bo2, Proposition 7.7] *Suppose* $\mathbf{H}$ *is a reductive* $\mathbb{Q}$*-subgroup of* G*. Then there is a* $\mathbb{Q}$*-representation* $\rho : \mathbf{G} \to \mathbf{GL}(\mathbf{V})$ *and* $v \in \mathbf{V}(\mathbb{Q})$ *such that* $\mathbf{H} = \{g \in \mathbf{G} : \rho(g)v = v\}$*. In particular,* $\rho(\mathbf{G}(\mathbb{Z}))v$ *is closed in* V *and* $H\pi(e)$ *is a closed orbit.*

Proof of Theorem 1.2. Assume first that $\mathbf{G}$ is a reductive group. Let W, c and F be the same as in the formulation of Proposition 4.1. Let $W_0 \subset \mathcal{G}$ be a ball centered at 0 and contained in $\bigcap_{f\in F} \mathrm{Ad}(f)W$. It is easy to see that:

(*) If $v \in \mathcal{G}$ and $v \notin W$ then $\mathrm{Ad}(f)v \notin W_0$ for any $f \in F$.

Let K be the closure of

$$\{x \in G/\Gamma : \mathcal{G}_x \cap W_0 \text{ does not contain a horospherical subset}\}.$$

In view of Proposition 3.5, K is compact. We will prove that K satisfies the conclusions of Theorem 1.2.

For every real $r > 0$ we denote by T_r the ball of radius r in T^0 centered at 1 and by C_r the smallest closed ball in $\mathcal{G}$ centered at 0 which contains

$$W \cup \bigcup_{d \in T_r^{-1} F} \mathrm{Ad}(d)(W).$$

(Recall that $\mathcal{G}$ and T^0 are endowed with norms. See §2.1.)

Let us fix an element $x \in G/\Gamma$. Note that only the following three mutually exclusive cases are possible :

(a) $\mathrm{Stab}_S(x)$ is finite and there exists $r > 0$ such that for every $d \in S^0$ with $\|d\| > r$, $\mathcal{G}_x \cap C_r$ contains a horospherical subset M such that $\mathrm{Ad}(d)M \subset C_r$;

(b) $\mathrm{Stab}_S(x)$ is finite and for every $r > 0$ there exists $d(r) \in S^0$ with $\|d(r)\| > r$ such that $\mathrm{Ad}(d(r))M \not\subseteq C_r$ for every horospherical subset M of $\mathcal{G}_x \cap C_r$;

(c) $\mathrm{Stab}_S(x)$ is infinite.

Now let us prove that $K \cap Tx \neq \emptyset$. Let $r > 0$. In each of the cases (a), (b) and (c) we will construct inductively a finite sequence $d_0, d_1, \ldots, d_n$ in T^0. (In fact, the sequence we construct will depend on r, only in case (b).) We put $d_0 = 1$ in the cases (a) and (c) and we put $d_0 = d(r)$ in the case (b). If $d_0 x \in K$ our sequence consists only of d_0, i.e. $n = 0$. Now assume that $d_0, \ldots, d_i,\ i \geq 0$, have already been chosen. Let $\tilde{d}_i = d_i d_{i-1} \cdots d_0$. Using Proposition 4.1, we fix an element $d_{i+1} \in F$ such that for every $w \in W \cap \mathrm{Ad}(\tilde{d}_i)(\mathcal{G}_x)$ we have

$$\|\mathrm{Ad}(d_{i+1})w\| \geq c\|w\|. \tag{15}$$

It follows from (*) and (15) that if w_i (resp. w_{i+1}) is a shortest nonzero vector in $W_0 \cap \mathrm{Ad}(\tilde{d}_i)(\mathcal{G}_x)$ (resp. $W_0 \cap \mathrm{Ad}(\tilde{d}_{i+1})(\mathcal{G}_x)$) then $\|w_{i+1}\| \geq c\|w_i\|$. Therefore, there exists an index n with the property: d_n is the first element in our sequence such that $W_0 \cap \mathrm{Ad}(\tilde{d}_n)(\mathcal{G}_x)$ does not contain a horospherical subset (equivalently, n is the first natural number for which $\tilde{d}_n x \in K$). This proves that $Tx \cap K \neq \emptyset$.

Now to complete the proof let us consider the cases (a), (b) and (c) separately. In case (a), applying Proposition 5.1 (with $C = C_r$), we get that $g \in Z_G(S)\mathbf{G}(\mathbb{Q})$. Thus (a) implies (i). It remains to show that (b) implies (ii) and that (c) also implies (ii).

For case (b), since $\tilde{d}_n x \in K$ and $r > 0$ is arbitrary it is enough to show that $\|\tilde{d}_n\| > r$. If $n = 0$ then $\tilde{d}_n = d(r)$ and there is nothing to prove. Let $n > 0$ and assume by contradiction that $\tilde{d}_n \in T_r$. By the choice of n, $W_0 \cap \mathrm{Ad}(\tilde{d}_{n-1})(\mathcal{G}_x)$ contains a horospherical subset M. Note that $\tilde{d}_{n-1}^{-1} = \tilde{d}_n^{-1} d_n \in T_r^{-1} F$. Therefore, $M_1 = \mathrm{Ad}(\tilde{d}_{n-1}^{-1})M$ is a horospherical subset of $\mathcal{G}_x \cap C_r$. In view of the choice of d_0 and the definition of C_r, we have that $\mathrm{Ad}(d_0)M_1 \nsubseteq W$. On the other hand, it follows from (*), (15) and the choice of the d_i's that if $v \in \mathcal{G}_x$ and $\mathrm{Ad}(d_0)(v) \notin W$ then $\mathrm{Ad}(\tilde{d}_{n-1})v \notin W_0$. Therefore, $M = \mathrm{Ad}(\tilde{d}_{n-1})M_1$ is not a subset of W_0. Contradiction.

Finally, suppose (c) holds. Then there exists an element of infinite order $t \in \mathrm{Stab}_S(x)$. The sequence

$$\{\tilde{d}_n t^k : k \in \mathbb{Z}\}$$

is unbounded in T and satisfies $\tilde{d}_n t^k x \in K$ for all k. This completes the proof of the theorem in case $\mathbf{G}$ is reductive.

Now let $\mathbf{G}$ be an arbitrary $\mathbb{Q}$-algebraic group and $\mathbf{T}$ and $\mathbf{S}$ be as in the formulation of the theorem. Let $\mathbf{G}' = \mathbf{G}/\mathcal{R}_u(\mathbf{G})$, $\phi : \mathbf{G} \to \mathbf{G}'$ be the natural $\mathbb{Q}$-rational homomorphism, $\Gamma' = \phi(\Gamma)$, $\mathbf{T}' = \phi(\mathbf{T})$ and $\mathbf{S}' = \phi(\mathbf{S})$. We will also use the notation ϕ for the restricted map $\phi : G \to G'$. The homomorphism ϕ induces a natural surjective G-equivariant map $\psi : G/\Gamma \to G'/\Gamma'$. Let K' be a compact subset of G'/Γ' which satisfies the conclusions of the theorem for the reductive group $\mathbf{G}'$ and the tori $\mathbf{S}'$ and $\mathbf{T}'$. Since $\Gamma \cap \mathcal{R}_u(G)$ is a cocompact lattice of $\mathcal{R}_u(G)$, the map ψ is proper. Furthermore, $\phi(\mathbf{G}(\mathbb{Q})) = \mathbf{G}'(\mathbb{Q})$ (see 6.2). This implies readily that the compact $K = \psi^{-1}(K')$ has the required properties.

7. Proofs of theorems 1.3 and 1.4

Proof of Theorem 1.3. Using the fact that the maximal $\mathbb{Q}$-split $\mathbb{Q}$-tori in $\mathbf{G}$ are conjugate under $\mathbf{G}(\mathbb{Q})$, we obtain that (2) and (3) are equivalent. Let us prove the equivalence of (1) and (2).

Using Proposition 6.3 we may assume in proving the theorem that $\mathbf{G}$ is reductive. Suppose first that $x \in Z_G(S)\mathbf{G}(\mathbb{Q})$ and $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} = \mathrm{rank}_{\mathbb{R}}\mathbf{G}$. Then $S = T$, and $\mathbf{T}_0 = x^{-1}\mathbf{T}x$ is defined over $\mathbb{Q}$ and $\mathbb{Q}$-split. By Proposition 6.4 $T_0\Gamma$ is closed, and hence the orbit map $T_0/T_0 \cap \Gamma \to G/\Gamma$ is proper. Since $\mathbf{T}_0$ is $\mathbb{Q}$-split, and using the fact that $\chi(T_0 \cap \Gamma) \subset \{\pm 1\}$ for any $\mathbb{Q}$-rational character χ on T_0, we see that $T_0 \cap \Gamma$ is finite. Hence $T_0\pi(e)$ is divergent. Therefore so is $T\pi(x) = xT_0\pi(e)$.

Now suppose that $T\pi(x)$ is divergent. Let $\mathbf{H} = Z_{\mathbf{G}}(\mathbf{S})$. Since alternative (ii) in the second statement of Theorem 1.2 does not hold, we have $x \in H\mathbf{G}(\mathbb{Q})$. Let us write $x = hq$ where $h \in H$ and $q \in \mathbf{G}(\mathbb{Q})$. Since Γ and $q\Gamma q^{-1}$ are commensurable, we get that $T\pi(h)$ is also divergent in G/Γ. Since H is a reductive $\mathbb{Q}$-group, $H\pi(e)$ is closed in

G/Γ and therefore $T\pi_H(h)$ is divergent in H/Γ_H. Note that $\mathbf{H}$ is an almost direct product over $\mathbb{Q}$ of $\mathbf{S}$ and a $\mathbb{Q}$-anisotropic subgroup $\mathbf{H}'$. Put $\mathbf{T} = \mathbf{S}\mathbf{T}'$, where $\mathbf{T}'$ is a maximal $\mathbb{R}$-split torus of $\mathbf{H}'$. Write $h = ys$ where $y \in H'$ and $s \in S$. Then $T'\pi_{H'}(y)$ is divergent in $H'/\Gamma_{H'}$. By Theorem 3.1 (a simpler proof is available), $H'/\Gamma_{H'}$ is compact and hence $T'\pi_{H'}(y)$ is a compact divergent orbit. This can only occur if $\mathbf{T}'$ is finite, thus $\mathbf{S} = \mathbf{T}$. This completes the proof. □

Proof of Theorem 1.4. Let $\mathbf{H} = Z_{\mathbf{G}}(\mathbf{T}_0)$. Since $H\Gamma$ is closed by [Ra, Lemma 1.14], the inclusion $H/\Gamma_H \to G/\Gamma$ is proper. The subgroup $\mathbf{T}_0$ is a central $\mathbb{Q}$-torus in $\mathbf{H}$. If $\mathbf{H_1} \subset \mathbf{G}$ is an inclusion of algebraic groups, defined over a field k, we say that there is a k-complement to $\mathbf{H_1}$ in $\mathbf{G}$ if there is a k-subgroup $\mathbf{H_2} \subset \mathbf{G}$ such that $\mathbf{G} = \mathbf{H_1}\mathbf{H_2}$ and $\mathbf{H_1} \cap \mathbf{H_2}$ is finite. Using [Bo1, §8], there is a $\mathbb{Q}$-complement to $\mathbf{T}_0$ in the center of $\mathbf{H}$, and using [Bo1, §7] there is a $\mathbb{Q}$-complement to the center of $\mathbf{H}$ in $\mathbf{H}$. Hence there is a $\mathbb{Q}$-complement $\mathbf{M}$ to $\mathbf{T}_0$ in $\mathbf{H}$. We get the following commutative diagram:

$$\begin{array}{ccccc} M & \hookrightarrow & H & \hookrightarrow & G \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle \pi} \\ M/\Gamma_M & \hookrightarrow & H/\Gamma_H & \hookrightarrow & G/\Gamma \end{array}$$

Note that the inclusion $M/\Gamma_M \to H/\Gamma_H$ is also proper. Furthermore, we claim that

$$\mathcal{R}_u(\mathbf{M}) \subset \mathcal{R}_u(\mathbf{G}).$$

Indeed, it follows from the fact that $\mathbf{T}_0$ is a central torus in $\mathbf{H}$ that $\mathcal{R}_u(\mathbf{M}) = \mathcal{R}_u(\mathbf{H})$. Let $\mathbf{G}_1$ be a Levi factor in $\mathbf{G}$ containing $\mathbf{T}$. Clearly $\mathbf{H} = Z_{\mathbf{G}}(\mathbf{T}_0) \supset Z_{\mathbf{G}_1}(\mathbf{T}_0)Z_{\mathcal{R}_u(\mathbf{G})}(\mathbf{T}_0)$. Moreover, since $\mathbf{T} \subset \mathbf{H}$, $\mathrm{Lie}(\mathbf{H})$ is a sum of root spaces, and so are $\mathrm{Lie}(Z_{\mathbf{G}_1}(\mathbf{T}_0))$ and $\mathrm{Lie}(Z_{\mathcal{R}_u(\mathbf{G})}(\mathbf{T}_0))$. It follows that $\mathrm{Lie}(\mathbf{H}) = \mathrm{Lie}(Z_{\mathbf{G}_1}(\mathbf{T}_0)) \bigoplus \mathrm{Lie}(Z_{\mathcal{R}_u(\mathbf{G})}(\mathbf{T}_0))$ and hence

$$\mathbf{H}^0 = Z_{\mathbf{G}_1}(\mathbf{T}_0)^0 Z_{\mathcal{R}_u(\mathbf{G})}(\mathbf{T}_0)^0.$$

Since $Z_{\mathbf{G}_1}(\mathbf{T}_0)$ is reductive, we must have $\mathcal{R}_u(\mathbf{H}) = Z_{\mathcal{R}_u(\mathbf{G})}(\mathbf{T}_0)^0$, proving the claim.

We write

$$\mathbf{T}' = \mathbf{M} \cap x^{-1}\mathbf{T}x.$$

Suppose first that $T\pi(x)$ is closed. Then so is $x^{-1}Tx\pi(e)$, where $x^{-1}Tx \subset H$. Therefore so also is $T'\pi_M(e)$, in M/Γ_M. By definition of $\mathbf{T_0}$, $T' \cap \Gamma_M$ is finite, and hence $T'\pi_M(e)$ is divergent in M/Γ_M. Applying Theorem 1.3 we obtain that there is $u \in \mathcal{R}_u(\mathbf{M})$ such that $u^{-1}\mathbf{T}'u$ is defined over $\mathbb{Q}$. Since $\mathbf{T_0}$ is defined over $\mathbb{Q}$, so is $\mathbf{T_0}u^{-1}\mathbf{T}'u = (xu)^{-1}\mathbf{T}xu$.

Conversely, suppose that $(xu)^{-1}\mathbf{T}xu$ is defined over $\mathbb{Q}$. Then $T'\pi_M(u)$ is divergent in M/Γ_M and hence by Proposition 6.3, $T'\pi_M(e)$ is divergent in M/Γ_M. This implies that $T_0T'\pi_H(e) = x^{-1}Tx\pi(e)$ is closed in H/Γ_H, and hence $T\pi(x)$ is closed in G. □

8. Examples and Open Questions

In this section $\mathbf{G}$ is always a *semisimple* $\mathbb{Q}$-algebraic group.

Example 1. First we define a quaternion division algebra Δ over $\mathbb{Q}$ as follows. Put $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $j = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$, $k = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$, and let 1 denote the identity matrix in $M_2(\mathbb{R})$.

Then it is easy to see by direct computation that $\Delta = \mathbb{Q}.1 + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ is a division algebra over $\mathbb{Q}$ and $\mathcal{O} = \mathbb{Z}.1 + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ is a ring. Since $\Delta \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$ the matrix algebra $M_2(\Delta)$ is naturally imbedded in $M_4(\mathbb{R})$ and $M_2(\Delta) \otimes_{\mathbb{Q}} \mathbb{R} \cong M_4(\mathbb{R})$. Denote by $\mathbf{G}$ the $\mathbb{Q}$-algebraic group defined by $\mathbf{G}(\mathbb{Q}) = \{x \in M_2(\Delta) : \det(x) = 1\}$. Then $\mathbf{G}(\mathbb{R}) = \mathrm{SL}_4(\mathbb{R})$. Note that $\mathcal{G}_{\mathbb{Z}} = \{a \in M_2(\mathcal{O}) : \mathit{trace}(a) = 0\}$ and $\Gamma = \mathrm{SL}_4(\mathbb{R}) \cap M_2(\mathcal{O})$ is an arithmetic subgroup of $\mathbf{G}(\mathbb{R})$.

In view of the above identifications the set of all matrices

$$d(x) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x^{-1} & 0 \\ 0 & 0 & 0 & x^{-1} \end{pmatrix}, x \in \mathbb{R}^*,$$

represents the group $\mathbf{S}(\mathbb{R})$ where $\mathbf{S}$ is a maximal $\mathbb{Q}$-split $\mathbb{Q}$-subtorus of $\mathbf{G}$. Let

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \alpha = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Denote $g(x) = \sigma.d(x)$. Then

$$\mathrm{Ad}(g(x))\alpha = \begin{pmatrix} 0 & -x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{G}.$$

Let $O(x_0)$ be the orbit $S\pi(g(x))$. Since $\mathrm{Ad}(S)$ centralizes $\mathrm{Ad}(g(x))\alpha$, it follows from a generalization of the Mahler compactness criterion (cf. [Ra, Chapter 10] or [Bo2, §8]) that if K is a compact in G/Γ then there exists a positive ϵ depending on K such that $O(x) \cap K = \emptyset$ for all $0 < |x| < \epsilon$. Therefore if we act on G/Γ with S instead of T

then in contrast to Theorem 1.2 there is no compact $K \subset G/\Gamma$ which intersects all S-orbits. The example also shows that the conclusion of Proposition 4.1 is generally false for the action of S instead of T.

It is clear that if $x \in Z_G(S)\mathbf{G}(\mathbb{Q})$ then $S\pi(x)$ is divergent. With Example 2 below, we show that if $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} \neq \mathrm{rank}_{\mathbb{R}}\mathbf{G}$ then, in contrast with Corollary 1.5, it might exist divergent orbits $S\pi(x)$ such that $x \notin Z_G(S)\mathbf{G}(\mathbb{Q})$ (equivalently, such that $x^{-1}\mathbf{S}x$ is not a $\mathbb{Q}$-split $\mathbb{Q}$-torus).

First let us prove the following lemma.

Lemma 8.1. *With the above notation assume that* $\mathbf{T}$ *is a* $\mathbb{Q}$*-torus. Then*

$$N_G(S)\mathbf{G}(\mathbb{Q}) \cap N_G(T) \subset N_G(S).$$

Proof. Let $w = nq$, where $w \in N_G(T)$, $n \in N_G(S)$ and $q \in \mathbf{G}(\mathbb{Q})$. Then

$$w^{-1}\mathbf{S}w = q^{-1}\mathbf{S}q \subset \mathbf{T}.$$

Hence $w^{-1}\mathbf{S}w$ is a maximal $\mathbb{Q}$-split subtorus in $\mathbf{T}$. Since $\mathbf{T}$ has only one maximal $\mathbb{Q}$-split subtorus [Bo2, 8.15] we get that $\mathbf{S} = w^{-1}\mathbf{S}w$. □

Example 2. Consider the rational quadratic form

$$f = x_1x_5 + x_2^2 - 3x_3^2 + x_4^2.$$

Let $\mathbf{G} = \mathbf{SO}(f)$ be the the orthogonal group defined over $\mathbb{Q}$ corresponding to f. Clearly, $\mathrm{rank}_{\mathbb{R}}\mathbf{G} = 2$, and by a standard argument (see [Se, Chapter 4]) $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} = 1$. We fix a maximal $\mathbb{Q}$-subtorus $\mathbf{T}$ in $\mathbf{G}$ such that $\mathbf{T} = \mathbf{S} \times \mathbf{S}'$, where $\mathbf{S} = \mathbf{SO}(x_1x_5)$ and $\mathbf{S}' = \mathbf{SO}(x_2^2 - 3x_3^2)$. Note that $\mathbf{S}$ is a maximal $\mathbb{Q}$-split subtorus of $\mathbf{G}$ and $\mathbf{S}'$ is a maximal $\mathbb{Q}$-anisotropic subtorus of $\mathbf{G}$. Since the Weyl group $W_{\mathbb{R}}$ with respect to the $\mathbb{R}$-split torus $\mathbf{T}$ acts irreducibly on the space of $\mathbb{R}$-characters of $\mathbf{T}$ [Hu1, §10.4 Lemma B] and $\mathbf{S}$ and $\mathbf{S}'$ are kernels of $\mathbb{R}$-characters, there exists a $w \in N_G(T)$ such that $wSw^{-1} \nsubseteq S \cup S'$. In view of the above lemma $w \notin N_G(S)\mathbf{G}(\mathbb{Q})$. Since $\mathbf{T}$ is defined over $\mathbb{Q}$ the orbit $T\pi(e)$ is closed and homeomorphic to T/Γ_T. But

$$T/\Gamma_T \cong S/\Gamma_S \times S'/\Gamma_{S'},$$

$S \cap \Gamma$ is finite, and $\pi_T(wSw^{-1}) \subset T/\Gamma_T$ is a closed non-compact subgroup. Hence the orbit $S\pi(w) \subset G/\Gamma$ is divergent although $w \notin N_G(S)\mathbf{G}(\mathbb{Q})$.

8.1. **Questions.** In view of Theorem 1.3 we have a satisfactory description of all divergent S-orbits if $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} = \mathrm{rank}_{\mathbb{R}}\mathbf{G}$. Let $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} \neq \mathrm{rank}_{\mathbb{R}}\mathbf{G}$. Comparing Example 2 with [Da, Theorem 6.1] it remains possible that all divergent orbits for S admit a simple description. In order to formulate a precise question we first make a definition generalizing the one in [Da]:

Definition 8.2. Let D be a subgroup of G, and let $g \in G$. We say that the orbit $D\pi(g)$ is a *degenerate divergent orbit* if there is a finite set of representations $\rho_i : \mathbf{G} \to \mathbf{GL}(\mathbf{V_i}),\ i = 1, \ldots, r$, defined over $\mathbb{Q}$, and $v_i \in \mathbf{V_i}(\mathbb{Q})$, such that for any divergent sequence $\{d_n\} \subset D$ there is a subsequence $\{d_{n_k}\}$ and $i \in \{1, \ldots, r\}$ such that

$$\lim_{k\to\infty} \rho_i(d_{n_k} g)v_i = 0.$$

It is easy to see that a degenerate divergent orbit is divergent. Note that the definition in [Da] is more restrictive as it describes explicitly the representations which occur.

We now ask:

Question 1. Is every divergent orbit for the action of S on G/Γ a degenerate divergent orbit?

We have seen that if $\dim S < \dim T$ then there are no divergent orbits for T, where T is a maximal $\mathbb{R}$-split torus. This raises:

Question 2. Suppose D is an $\mathbb{R}$-split torus with $\dim S < \dim D$. Are there any divergent orbits for D?

Appendix A. Proof of Margulis' result

We expose Margulis' proof of Theorem 1.1. In this section $G = \mathrm{SL}(n, \mathbb{R}),\ \Gamma = \mathrm{SL}(n, \mathbb{Z})$, and T is the group of diagonal matrices in G.

We begin with two facts about the action of T on $\mathbb{R}^n$.

Proposition A.1. *There is a ball $W \subset \mathbb{R}^n$, centered at 0, a finite set $F \subset T$, and $c > 1$ such that for every $g \in G$ there is $f \in F$ such that for every $w \in g\mathbb{Z}^n \cap W$ we have:*

$$\|fw\| \geq c\|w\|.$$

Proof. Every $g \in G$ has determinant equal to 1 and therefore preserves the volume element in $\mathbb{R}^n$. It follows that there is a small enough neighborhood W of 0 such that for every g, $\mathrm{span}(W \cap g\mathbb{Z}^n)$ is a proper linear subspace of $\mathbb{R}^n$. So it suffices to show that there is a finite $F \subset T$ and $c > 1$ such that for every proper linear subspace $V \subset \mathbb{R}^n$ there is $f \in F$ such that for all $v \in V$,

$$\|fv\| \geq c\|v\|.$$

By the compactness of the Grassmannian variety it suffices to show that for every proper subspace $V \subset \mathbb{R}^n$ there is $t \in T$ such that for every nonzero $v \in V$ we have $\|tv\| > \|v\|$. This is a simple exercise. □

Proposition A.2. *If $g \in G$ and $g \notin T\mathrm{SL}(n, \mathbb{Q})$ then for any neighborhood W of 0 in $\mathbb{R}^n$, any finite $J \subset g\mathbb{Z}^n - \{0\}$ and any compact $C \subset T$, there is $t \in T - C$ such that*

$$tJ \cap W = \emptyset.$$

Proof. Let $\{\mathbf{e}_i,\ i = 1, \ldots, n\}$ be the standard basis of $\mathbb{R}^n$. It is easy to verify that if $g \notin T\,\mathrm{SL}(n, \mathbb{Q})$ then there is some i such that

$$\mathbb{R}\mathbf{e}_i \cap g\mathbb{Z}^n = \{0\}.$$

Let $\alpha_i(t)$ be the diagonal matrix with $e^{-(n-1)t}$ in the i-th diagonal entry and e^t in all other diagonal entries. Then for any nonzero $w \in g\mathbb{Z}^n$, we have

$$\alpha_i(s)w \to_{s \to \infty} \infty.$$

Thus for all large enough s, we will have

$$\alpha_i(s)J \cap W = \emptyset.$$

□

Proof of Theorem 1.1. It is well-known (see, for example, Proposition 6.4) that $T\pi(g)$ is divergent if $g \in T\,\mathrm{SL}(n, \mathbb{Q})$. Suppose $g \notin T\,\mathrm{SL}(n, \mathbb{Q})$. We will find a compact $K \subset G/\Gamma$ such that for every compact $C \subset T$, there is $t \in T - C$ such that $t\pi(g) \in K$, contradicting divergence.

Let W, F, c be as in Proposition A.1. Suppose with no loss of generality that $1 \in F$, and let

$$W_0 = \bigcap_{f \in F^{-1}} fW.$$

By compactness, W_0 is a neighborhood of 0. It satisfies

$$\forall f \in F^{-1}, \quad fv \notin W \Longrightarrow v \notin W_0 \tag{16}$$

Define

$$K = \pi(\{g \in G : g\mathbb{Z}^n \cap W_0 = \{0\}\}).$$

By Mahler's compactness criterion, K is a compact subset of G/Γ.

Let $J = g\mathbb{Z}^n \cap C^{-1}W$, and using Proposition A.2, let t_0 be an element of $T - C$ such that $t_0 J \cap W = \{0\}$. Define inductively a sequence $t_0, t_1, \ldots$ as follows. If $t_0, \ldots, t_k$ have already been chosen, let $\tilde{t_k} = t_k t_{k-1} \cdots t_0$ and using Proposition A.1 let t_{k+1} be such that

$$w \in W \cap \tilde{t}_k g\mathbb{Z}^n \implies \|t_{k+1}w\| \geq c\|w\|.$$

It follows from (16) that

$$W_0 \cap \tilde{t}_{k+1} g\mathbb{Z}^n \subset t_{k+1}(W_0 \cap \tilde{t}_k g\mathbb{Z}^n)$$

and therefore, for large enough k, we will have $\tilde{t}_k g\mathbb{Z}^n \cap W_0 = \{0\}$. Let k be the smallest index for which this is true. Clearly $\tilde{t}_k \pi(g) \in K$, and it remain to show that $\tilde{t}_k \notin C$. If $k = 0$ this follows from the choice of t_0. Suppose $k \geq 1$ and $\tilde{t}_k \in C$. By minimality of k, there is a nonzero vector $v \in W_0 \cap \tilde{t}_{k-1} g\mathbb{Z}^n$. By (16), $v = \tilde{t}_{k-1} v_0$ for some nonzero $v_0 \in W_0 \cap g\mathbb{Z}^n$ and $\tilde{t}_j v_0 \in W_0$ for $j = 0, 1, \ldots, k-1$. In particular $t_0 v_0 \in W_0$. Also by (16), $t_k v \in W$. So $\tilde{t}_k v_0 = t_k v \in W$ and hence $v_0 \in C^{-1}W$. Thus $v_0 \in J$ and $t_0 v_0 \in W_0$, contradicting the choice of t_0. □

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