Aspects of the long time evolution in General Relativity and Geometrizations of Three-Manifolds

A dissertation Presented

by

Martin Reiris

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

December 2005

State University of New York
at Stony Brook
The Graduate School

Martin Reiris

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Michael Anderson
Professor of Mathematics
Dissertation Director

H. Blaine Lawson, Jr.
Distinguished Professor of Mathematics
Chairman of Dissertation

C. Denson Hill
Professor of Mathematics

Peter van Nieuwenhuizen
Distinguished Professor of Physics,
C.N. Yang Institute for Theoretical Physics, Stony Brook
Outside Member

This Dissertation is accepted by the Graduate School.

Dean of the Graduate School

Abstract of the Dissertation

Aspects of the long time evolution in General Relativity and Geometrizations of Three-Manifolds

by

Martin Reiris

Doctor of Philosophy

in

Mathematics

Stony Brook

2005

We investigate global geometric properties of the ground state of the reduced hamiltonian H in the CMC phase space in General Relativity, for manifolds with $\sigma(M) \leq 0$. Precisely, given a bound Λ for the Bel-Robinson energy i.e. ($Q_0 \leq \Lambda$) there is an $\epsilon(\Lambda)$ such that *H-small data* states, i.e. those with $H - H_{inf} \leq \epsilon(\Lambda)$, are close to a strong geometrization. We provide a new example of small data states realizing a non pure geometrization which completes the list of the expected geometric behaviors. As an application we establish the stability of the pure H ground state (flat cone or Löbell space time) under no restriction in the three dimensional hyperbolic geometry, providing in general a proof that had been obtained by L.Andersson and V.Moncrief for rigid hyperbolic manifolds.

"Toda obra humana es deleznable,
pero su ejecucion no lo es."

Carlyle.

Contents

List of Figures

Acknowledgements

I would like to express my deep gratitude to Michael Anderson for an exceptional advisory. His collaborative friendship, in particular, have always been contemplative of the individual in its human and academic sides. He oriented me too in learning and mastering the profession, and invaluably supported at all times my insertion in the scientific community. Finally my debt is for his remarkable thinking which brought light when inspiration faded, and transformed scientific meditation into a passionate intellectual journey, unpredictably marvelous today and tomorrow.

My many thanks too, to the State University of New York and particularly to all the integrants of the Math Department at Stony Brook which have made a dreamed experience possible through steady encouragement and practical support.

Family and friends have encouraged the project all along these years and have been a permanent source of emotional support. To all of them my sincere gratitude.

Chapter 1

Introduction.

1.1 Summary.

General Relativity[1] is a diffeomorphism (gauge) invariant lagrangian theory on four manifolds and (therefore) a constrained system. As such, position and velocities $(q, \dot{q}) = (g, \dot{g} = -2K)$[2,3] (or briefly (g, K) *states*, where g is a three-metric and K is a symmetric two tensor on a three-manifold M) are not freely specifiable. Namely, they are constrained to lie in the zero set of the *(vacuum) constraint equations*

$$\textit{Energy constraint}: \quad R - |K|^2 + (tr_g K)^2 = 0, \tag{1.1}$$

$$\textit{Momentum constraint}: \quad \nabla.K - d(tr_g K) = 0, \tag{1.2}$$

in the tangent space of the manifold of three-metrics. We will deal with *CMC states*, i.e. states with constant average velocity $k = tr_g K$. If the initial state (g_0, K_0) is CMC, then it is possible to choose k as the time coordinate (at least for short times), a choice that conforms a partial, temporal gauge and that restricts the Einstein flow to lie on the subset of CMC states. For manifolds with sigma constant $\sigma(M) \leq 0$, CMC systems can be reduced [FM1] and the flow gets driven by a reduced and monotonically decreasing time dependent hamiltonian H whose global infimum H_{inf} in the CMC set of states is given by $-(\frac{3}{2}\sigma(M))^{\frac{3}{2}}$. We will study the CMC phase space in a particular sector, namely states with $k = -3$ (*normalized CMC states*) and bounded Bel-Robinson energy $Q_0 \leq \Lambda$. We prove that there is a $\epsilon(\Lambda)$ such that if $H - H_{inf} \leq \epsilon(\Lambda)$ the (g, K) states decompose M (via the thick-thin decomposition) in a strong geometrization, where a strong geometrization is a subdivision of the

[1] We will consider only vacuum cosmological solutions of General Relativity.

[2] $\dot{g} = \mathcal{L}_n g$ is the normal Lie derivative of g.

[3] In the corresponding hamiltonian theory (q, p) states are given by $(g_{ab}, \Pi^{ab} = \sqrt{g}(K^{ab} - kg^{ab}))$. From a geometrical point of view $(q, \dot{q})$ variables, or equivalently (g, K) are more natural to work with and is the choice we will prefer.

three-manifold M into $H = \cup H_i$ and $G = \cup G_i$ parts, with H_i a collection of manifolds admitting complete hyperbolic metrics of finite volume g_{H_i} and G_i a collection of graph manifolds. The H and G pieces are glued together along incompressible T^2 tori in M. We prove also that in the limit when a sequence of states (g_i, K_i) has $Q_0 \leq \Lambda$ and $H \rightarrow H_{inf}$, (g_i, K_i) converges (up to diffeomorphism) to (the unique) flat cone states $(g_{H_i}, -g_{H_i})$ on the manifolds H_i and collapses on the G parts, with the volume and L^2 norm of K going to zero on G. Examples of this phenomena are easy to present in the pure case where the geometrization consists of a H or G piece. We provide a new example where a sequence of states show the emergence of a strong geometrization containing both G and H parts.

In a different section and as an application we prove the stability of the flat cone solution (or Löbell space time) for perturbations with small $W_0^{3,2} \times W_0^{2,2}$ norm of $(g - g_h, K + g_H)$, a result that had been obtained in [AM] for rigid hyperbolic manifolds.

Finally, we discuss possible consequences and applications to long time evolution, as well as new directions of further study.

1.2 Motivation and results

The *sigma constant*, $\sigma(M)$, of a closed manifold M is defined as the supremum of the scalar curvatures of unit volume Yamabe metrics. This important topological invariant divides the set of three-manifolds into three classes, those for which either $\sigma(M) < 0, \sigma(M) = 0$ or $\sigma(M) > 0$. All through the work assume M is a closed and oriented three-manifold with $\sigma(M) \leq 0$. By a CMC *state* (g, K) (constant mean curvature state) we mean a 3-metric g and a second fundamental form K with $k = tr_g K$ constant, satisfying the constraint equations in vacuum. If a cosmological space time (i.e. having a compact oriented Cauchy surface) with a Cauchy surface M with $\sigma(M) \leq 0$ admits a CMC slice with $k < 0$, then there is a unique connected foliation of CMC slices with k taking all values in a open interval inside $(-\infty, 0)$ ([B], [MT], [Re], [AM1]). We will consider here such a scenario. When k increases the volume expands, we will call it the future direction. Every 3-manifold (with $\sigma(M) \leq 0$ or not) is the Cauchy surface for a universe having a slice with $k = -3$, thus there are solutions with CMC foliations in any topology. This is due to the fact that every closed manifold admits a metric g of scalar curvature $R = -6$ and that it together with $K = -g$ form a state (g, K) with $k = -3$. Inside the foliation there is then a natural choice of time, or a natural $3+1$ decomposition. Any trivialization ϕ : *Foliation* $\rightarrow M \times I$ (preserving the leaves) gives rise to a vector field X, the *shift vector field*, with $dk(X)$ constant along the leaves. Conversely a shift vector field (with that property) together with a particular ϕ_k from any leaf into M gives rise to a trivialization. The choice of such a global map makes it possible to define a flow of CMC states (g, K) on a fixed manifold M. We will call to any of such flows (i.e. regardless of the gauge or shift vector being used) the *Einstein flow*. CMC foliations form a particularly relevant (partial, temporal) gauge in cosmological space times with beautiful geometrical aspects. The equations for the Einstein flow in this particular 3+1 decomposition were presented in [FM1], [FM2] using a contact structure on a reduced phase space on which a reduced hamiltonian (time dependent) drives the evolution. Although of conceptual significance in itself, we won't use explicitly that contact structure, details of which can be found in [FM1]. The geometric properties of the reduced hamiltonian will be essential however. Precisely, for a state (g, K), the *reduced hamiltonian* is defined as

$$H = -k^3 Vol(M). \tag{1.3}$$

It is strictly monotonically decreasing along the CMC flow to the future except when it is constant and that happens only when the manifold is of hyperbolic type and the CMC solution is a so called flat cone i.e.: $\mathbf{g} = -dt^2 + t^2 g_H$, where g_H is a hyperbolic metric ([FM1]). We will see later that this fact has a precise geometric significance. Its infimum in the phase space of CMC states is given by $H_{inf} = (-\frac{3}{2}\sigma(M))^{\frac{3}{2}}$ ([FM1]). It is of general relevance to realize whether or not the reduced hamiltonian always decays to it along the evolution. We won't address the question of what may prevent the reduced hamiltonian to decrease in that way, but instead we will investigate the geometric consequences of assuming such a decay actually occurs. In [FM2] some compactified Bianchi models have been studied with the purpose of analyzing the behavior of the reduced hamiltonian and it was verified that it decreases to its infimum (or conjectured infimum, see Comment 2 on the determination of the sigma constant for three-manifolds) and more interestingly, as is explained in detail later, such decay of the reduced hamiltonian is observed to be related with the appearance of certain geometric structures and phenomena. However the approach to the infimum may not be the only reason for the geometric behavior to appear. Indeed, as we will see, to reach such geometric conclusions the reduced hamiltonian is largely insufficient as it weakly controls the geometry, therefore some stronger hypothesis must be imposed. In particular we will see that such is the case when the Weyl tensor decays in a certain way with respect to the CMC foliation. The geometric structures that are involved at a first level on the long time evolution when one apriori knows some form of decay of the Weyl tensor are those of weak and strong geometrizations as first noted in [A1].

DEFINITION 1 *(quoted from* [A1]*) Let M be a closed oriented and connected 3-manifold, with $\sigma(M) \leq 0$. A weak geometrization of M is a decomposition of M, $M = H \cup G$ where H is a finite collection of manifolds admitting complete, connected, hyperbolic metrics of finite volume embedded in M and G is a finite collection of connected graph manifolds embedded in M. The union is along a finite collection of embedded tori $T = \cup_i T_i$, $T = \partial H = \partial G$. A strong geometrization of M is a weak geometrization as above, for which each torus $T_i \in T$ is incompressible in M, i.e. the inclusion of T_i into M induces an injection of fundamental groups.*

It is necessary at this point to clarify the meaning we are giving to *long time*. It is an important open problem ([A1], [Re]) to know whether for manifolds with $\sigma(M) \leq 0$ the range of k is $(-\infty, 0)$

and that if the CMC foliation exhaust the whole space time. If that were the case, by meaning *long time* as the properties of the CMC slices when $k \to 0$, we would unequivocally be referring the results to the *end* of the space time. However it may not be the case, and therefore by *long time* we mean the properties of CMC slices when k approaches the end of its range if the results are put in the perspective of a CMC foliation.

It is the metric on the normalized CMC slices that, under some hypothesis, sets the weak or strong geometrization on the long time by the so called *thick-thin decomposition* of the manifold. There are two ways to define a thick-thin decomposition, each being important depending on which norms on the space of metrics are relevant to the problem. In our case as Sobolev norms are those to be used, of direct interest to us is the thick-thin decomposition that involves the volume radius (instead of the injectivity radius).

DEFINITION 2 *The volume radius $\nu(x)$ at x is given by*

$$\nu(x) = sup\{r : \frac{VolB_y(s)}{s^3} \geq \mu, \forall B_y(s) \subset B_x(r)\}, \tag{1.4}$$

where μ is any fixed but positive constant.

DEFINITION 3 *Given ϵ, define the ϵ thick-thin decomposition of a manifold M as $M = M^\epsilon \cup M_\epsilon$ where*

$$M^\epsilon = \{x \in M : \nu(x) \geq \epsilon\},\ M_\epsilon = \{x \in M : \nu(x) \leq \epsilon\}. \tag{1.5}$$

The domain M^ϵ is called the ϵ-thick part and M_ϵ the ϵ-thin part of M.

We will say that a metric g implements the geometrization if $M^\epsilon = H$ and $M_\epsilon = G$ (for some ϵ). In general, normalizing the geometric data (g, K) on a CMC slice as $(\frac{k^2}{9}g, \frac{-k}{3}K)$ we get a state with $k = -3$. The results on the long time evolution on this thesis are about the normalized states of the CMC foliation. We will study the appearance of geometrizations via thick-thin decompositions and of the related phenomena of convergence and collapse along $\mathcal{F}$-structures, on sequences[4] of $W_0^{2,2} \times W_0^{1,2}$ states satisfying three conditions (from now on referred as *conditions* 1,2,3):

1. $tr\ K = k = -3$,
2. $H = -k^3 Vol(M) \downarrow (-\frac{3}{2}\sigma(M))^{\frac{3}{2}}$,
3. $Q_0 = \int_M |E|^2 + |B|^2 dv_g \leq \Lambda$.

Observe that H is scale invariant, i.e. invariant under the scaling $(\lambda^2 g, \lambda K)$ $(\lambda > 0)$, therefore a sequence of non normalized states in a CMC foliation in a cosmological space time satisfies condition 2 if and only if the corresponding sequence of normalized states satisfies it too. In condition 3, Q_0 is

[4] Sequences won't be indexed unless their lack may cause confusion.

the Bel-Robinson energy and E and B are the electric and magnetic components of the Weyl tensor (see later for a definition) with respect to the spatial slice. On non normalized sequences of states in a CMC foliation, condition 3 translates into the following decay in the bound $Q_0 \leq -k\Lambda$. Under these assumptions we will prove the following theorem.

Theorem 1 (The ground state for the reduced hamiltonian) *A sequence of states (g, K) on M satisfying conditions 1, 2 and 3, has a subsequence implementing a unique (up to isotopy) strong geometrization in the following way:*

1. Convergence. *There is a $\delta > 0$ such that for every $x \in M$, $\nu(x) > \delta$. A suitable choice of diffeomorphisms make (g, K) converge in the weak $W_0^{2,2} \times W_0^{1,2}$ topology to the standard initial state for a flat cone $(g_H, -g_H)$, where g_H is a hyperbolic metric.*

2. Collapse. *For every $x \in M$, $\nu(x) \to 0$. g collapses in volume, i.e. $Vol_g(M) \to 0$ with the L^2 norm of Ricc, the $W^{1,2}$ and L^4 norms of K uniformly bounded and the L^2 norm of K going to zero.*

3. Convergence - Collapse. *Say the subsequence g is indexed as g_i. There is a sequence x_i and $\delta > 0$ with $\nu_{g_i}(x_i) \geq \delta$ and there is a sequence y_i with $\nu_{g_i}(y_i) \to 0$. There is also a maximal domain $\Omega \subset\subset M$ such that a suitable choice of diffeomorphisms make (g, K) converge in the weak $W_0^{2,2} \times W_0^{1,2}$ topology to a finite set of initial states for flat cones on complete hyperbolic manifolds of finite volume with cusps with incompressible tori at the ends. There is an $\epsilon_0(\Lambda)$ such that for every $\epsilon \leq \epsilon_0$, M_ϵ is a union of graph manifolds and has an $\mathcal{F}$-structure whose orbits are $C\epsilon^{\frac{1}{3}}$ collapsed, i.e. $diam(\mathcal{O}_x) \leq C\epsilon^{\frac{1}{3}}$. M_ϵ collapses in volume in the sense that* $\lim sup_{g_i} Vol_{g_i}(M_\epsilon)$ *goes to zero as $\epsilon \to 0$. Similarly the* $\lim sup_{g_i}$ *of the L^2 norm of Ricc and the $W^{1,2}$ and L^4 norms of K are uniformly bounded in M_ϵ and the* $\lim sup_{g_i}$ *of the L^2 norm of K in M_ϵ goes to zero as $\epsilon \to 0$.*

`Examples showing the behavior of Theorem 1.` Interestingly, the problem of the determination of the sigma constant is related with the problem of finding sequences of CMC states satisfying conditions 1, 2 and 3 and showing the behavior explained in the possibilities 1, 2 and 3 in Theorem 1. The reason is due to the fact explained before, that $(g_Y, -g_Y)$, where g_Y are Yamabe metrics of scalar curvature $R = -6$ are CMC states. In [A5] a conjecture for the realization of the sigma constant was raised, that (informally) in case $\sigma(M) \leq 0$, asserts that for irreducible oriented and closed 3-manifolds there is a sequence of Yamabe metrics whose scalar curvatures seek $\sigma(M)$, that implements a strong geometrization (pretty much in the same way as in the theorem above). From it one extracts the conjectured value for the sigma constant for irreducible oriented and closed 3-manifolds. Concretely, if the corresponding strong geometrization is $M = (\cup H_i) \cup (\cup_i G_i)$, then $\sigma(M) = 6(\sum Vol_{-1} H_i)^{\frac{2}{3}}$. Assuming the conjectured value, examples can be given. For a sequence showing convergence it is enough to take any sequence $(g_Y, -g_Y)$, where g_Y is Yamabe converging in $W_0^{2,2}$ to g_H, the hyperbolic metric. For Collapse instead, take the metric product $\Sigma_{gen} \times S_l^1$ of a surface of genus $gen > 1$ with a metric of scalar curvature $S = -6$ and S_l^1 of length l. As $l \to 0$ the sequence $(g_l, -g_l)$, (g_l is the product metric), shows the phenomena of collapse along the S^1 fibration. A sequence displaying the third behavior is more complicated to present. In section 3.2 we provide in detail one in which two hyperbolic manifolds complete and of finite volume with a cusp each are glued together along a neck with T^2 symmetry. The family is parameterized by the length of the neck, and as it goes to infinity geometrization takes place and the picture of convergence-collapse appears, a representation of that can be seen schematically on Figure 1.

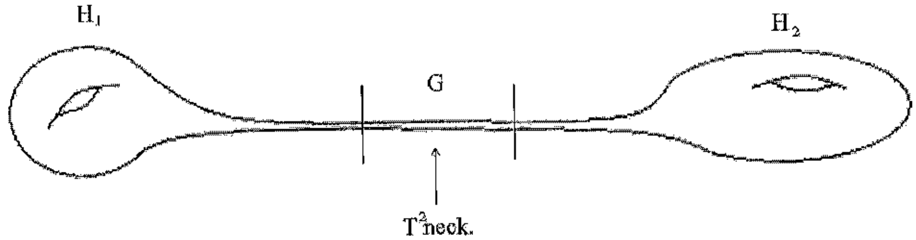

Figure 1.1: Representation of the metric degeneration for sequences of vacuum states in two hyperbolic manifolds joined along their cusps realizing the third kind of behavior in Theorem 1.

It is likely that the states for which the Bel-Robinson energy is close enough to zero and the reduced hamiltonian close enough to its infimum, i.e. those states that can be thought as *small data* for compact universes, decay to their ground states. We state such hope in the following conjecture.

Conjecture 1 *For any 3-manifold M, there exists an ϵ such that if $k = -3$, $-k^3 Vol(g) - (-\frac{3}{2}\sigma(M))^{\frac{3}{2}} \leq \epsilon$ and $Q_0 \leq \epsilon$ then the flow is defined for k in $(-3, 0)$ and it coincides with the maximal globally hyperbolic evolution settles the normalized CMC states into a strong geometrization (the ground state).*

Comment 1.

1. As stated, whether true or not, the conjecture probably won't be solved until some sort of continuation principle in $W^{2,2} \times W^{1,2}$ gets proved, as the control of Q_0 and H on (g, K) is only on $W^{2,2} \times W^{1,2}$ (and therefore nothing would prevent the flow to stop running before k completes the interval $(-3, 0)$). It is possible of course to assume more regularity and state a conjecture with Q_0 replaced by $\mathcal{E} = Q_0 + Q_1$.

2. The evolution of the $\mathcal{F}$ structures forming on cases 2 and 3 of Theorem 1 is the main obstruction to understand the conjecture in cases 2 and 3. On one side it was proved by Ringstöm [R1]

(although in a different gauge than CMC) that the behavior of the $\mathcal{F}$ structures could be almost periodic along slices that approach to be homogeneous. The evolution of such parts is not therefore generally modeled in the long time by the long time evolution of homogeneous initial states.

3. There are a numerous amount of references where the behavior of geometric structures had been identified and studied, some of them are [FM2], [R1], [R2], [R3], [AM], [CB].

We will give a proof of the conjecture for case 1, assuming the extra hypothesis of an initial control of the first order Bel-Robinson energy Q_1, and that the initial manifold is known apriori to be hyperbolic. This last condition can be relaxed to assume that the manifold is atoroidal, this is enough to exclude cases 2 and 3 during the proof.

Theorem 2 (Stability of pure H ground states.) *If M is a hyperbolic manifold and Q_0 is replaced for $\mathcal{E} = Q_0 + Q_1$ then the conjecture holds.*

Comment 2. It is essential to remark that in Theorem 2 we make the assumption that the proof of the Geometrization Conjecture provided by G. Perelman is correct. Assuming that, it is shown in [A2] that Perelman's work implies the conjecture on the realization of the sigma constant on manifolds with $\sigma(M) \leq 0$. Such a result is necessary in the proof of Theorem 2.

Theorem 2 implies the stability of the flat cone in general (in the sense of Conjecture 1), i.e. without any assumption about the hyperbolic manifold. It is possible to give a proof of stability of the flat cone that doesn't make use of Thurston Geometrization; such is the content of Theorem 3 proved in section 3.4. Finally, it has been conjectured in [FM2], [AM], that the inclusion of the reduced hamiltonian would provide a general proof of the stability of the flat cone. The present work confirms that expectation.

Theorem 3 (Stability of the flat cone.) *There is a neighborhood U of initial states around $(g_H, -g_H)$ in the $W_0^{3,2} \times W_0^{2,2}$ topology that under the normalized Einstein flow, every normalized trajectory (with a suitable choice of the shift vector) converges in that topology to $(g_H, -g_H)$.*

The results here are closely related to those in [A1] but in the Sobolev setting. In [A1] it is assumed a L^∞ decay of the norm of the Weyl tensor and its derivative with respect to the CMC foliation. In particular it is proposed the following decay in the bound

$$|R(x)| + t(x)|\nabla R(x)| \leq C t^{-2}(x), \tag{1.6}$$

where $|R|^2 = |E|^2 + |B|^2$ and $|\nabla R|^2 = |\nabla E|^2 + |\nabla B|^2$ and $t(x)$ is the proper time of the CMC slices to a fixed slice. It is proved then that a cosmological space time for which the CMC foliation exhausts the space time to the future, is future geodesically complete and any diverging sequence of slices has a subsequence with $t^{-2}(x)g$ decomposing M into a weak geometrization. Technically, to reach such conclusions it is used the Cheeger-Gromov theory of convergence and collapse of Riemannian manifolds ([A4]) as the conditions above control the 3-curvature in L^{∞}. It is worth mentioning that the quantity $t^{-3}Vol(M)$ was used in [A1] as playing a conceptually similar role to H. L^{∞} norms are not well suited however in the study of hyperbolic equations and as such Anderson's decay conditions would be difficult to deduce in general situations where such a decay is expected. It would be worth to have results guaranteeing the fall into a weak or strong geometrization with hypothesis involving technically workable quantities in general relativity, some that could be deduced analytically without imposing much regularity. In this article we will work with the reduced hamiltonian H and the Bel-Robinson energy of zero and first order. In a Sobolev setting we use an L^2 analog of Cheeger-Gromov theory of convergence and collapse developed in [A4]. The lack of reference to any particular spatial gauge (or choice of shift vector) along the paper is a reflection of the use of that theory in the analysis of metrics on Cauchy surfaces, a theory that is basically on its grounds diffeomorphism (gauge) invariant. Due to this fact, there is some lack of information on the part of the four-metric involving the shift vector, however the conclusions are enough for purposes of stability or evolution of the CMC states (g, K).

Comment 3. Recently, H. Ringström ([R4]) studied the extent to which the apriori hypothesis on equation (1.6) are satisfied on examples. Interestingly he found that on Bianchi VIII models not of NUT type, there is a sequence of time slices t (CMC) for which they don't hold (although there is one sequence for which they do). For us it is of relevance as condition 3 is not satisfied either, i.e. the normalized Bel-Robinson energy diverges. It is worth to emphasize too, that condition 2 holds i.e. the reduced hamiltonian decays to its actual infimum, there is also formation of $\mathcal{F}$ structures and collapse of the volume radius.

1.3 Further study

In this section we informally discuss possible avenues of further research.

The present work embodies into the broader philosophy of studying the gravitational field through intrinsic and global quantities. From a rigidity point of view, it is based on the idea of identifying the global quantities that make a certain geometric configuration (or state) rigid and understanding their evolution near the rigid state. One such example is the pure ground state in a hyperbolic manifold as it is characterized by the rigidity condition $H = H_{inf}$. Other examples of potential interest are the stability of the $k = -3$ hyperboloid's causal future in Minkowski space time. This problem in particular, shares the features of the stability of the pure H ground state where the role (or control) of the reduced hamiltonian may be played by the Bondi energy. Stability of non pure ground states (conjecture 1) is another case of great interest. However in this case, new quantities other than the Bel-Robinson energies and the reduced hamiltonian are needed to have an effective control of the thin parts. A problem of this kind is the one of the stability of the infinite double cusp solution constructed in section 3.2. Related with this is also the stability of the Kasner family in T^3 as it is rigidly characterized by $R = 0$ and $B = 0$. The central problem is to prove that in a situation where the injectivity radius is bounded below the decrease in the energies H, $\tilde{Q}_0$ and $\tilde{Q}_1$ is at least as fast as the decrease of the injectivity radius. In that way one may perform a systematic unwrap of the geometry, to keep the injectivity radius bounded below, ultimately having a geometry converging to local rigid configurations possessing killing fields. Finally in this line of thought is the much more difficult stability of the Kerr family using existent rigidity results. It is a remarkable fact that all these systems posses a monotonic quantity, whose critical points always are related with the rigid configurations. In the black hole case, one may recall the (a posteriori) entropy $S = \frac{1}{4}A$, where A is the area of the event horizon.

Another front line of interest is the study of the long time evolution of solutions where one apriori knows are *strongly singularity free (S.S.F)*

DEFINITION 4 *A CMC cosmological space time with* $\sigma(M) \leq 0$ *is* strongly singularity free *if it is future geodesically complete and the zero and first order Bel-Robinson energies, with respect to the CMC foliation, have a scale invariant bound* Λ *to the future, i.e.*

$$Q_0 \leq -\Lambda k, \tag{1.7}$$

$$Q_1 \leq \Lambda k^2, \tag{1.8}$$

for k *in* $(-3, 0) \cap I$.

The previous definition, is the analogous in this context to that in [H] for non singular solutions of the Ricci flow. The definition is inspired by the work in [A1].

The following result (not proved here), is the first step in the understanding of strongly singularity free space times (see [H], [A1] for related results).

Theorem 4 *In a strongly singularity free space time the maximal CMC interval I contains* $(-3, 0)$, *the CMC foliation covers the space time to the future and the normalized states implement, persistently on the long time, a weak or strong geometrizations. If the reduced hamiltonian decays to its infimum then the geometrization is strong.*

It is conjectured too, that on strongly singularity free space times H must decrease to H_{inf} and thus those space times would always settle down into strong geometrizations.

Conjecture 2 *In a strongly singularity free space time,* $H \downarrow H_{inf}$.

A weaker conjecture however is

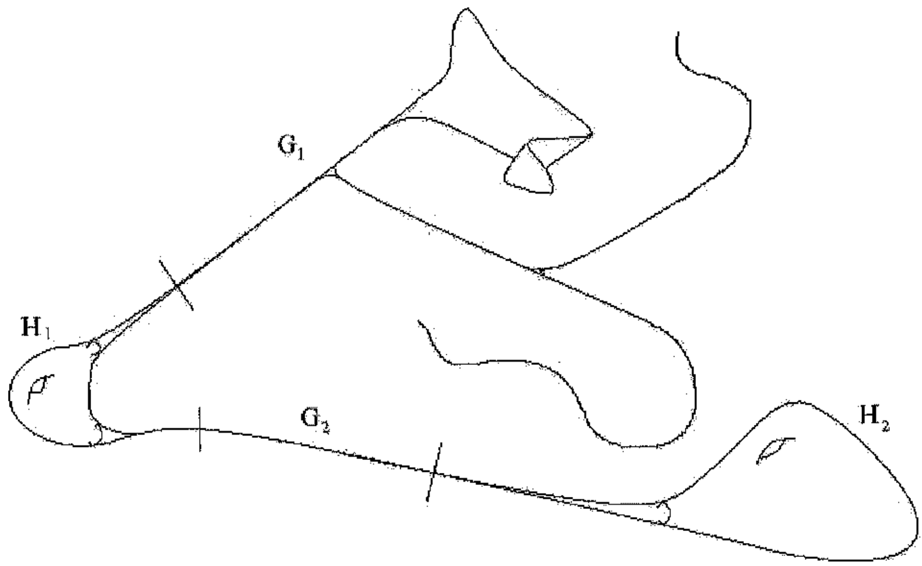

Figure 1.2: Representation of a strongly singularity free cosmological solution after a sufficiently large time and persistently (with the graph pieces becoming thinner) since then.

Conjecture 3 *Say* $(\mathbf{M}, \mathbf{g})$ *is a strongly singularity free space time whose normalized geometry decays into a geometrization with n hyperbolic pieces* H_i. *Then* $H \downarrow 27 \sum_{i=1}^{n} Vol(H_i)$.

Conjecture 2 pictures CMC evolution (on strongly singularity free solutions) as a time dependent hamiltonian system where the spectrum has one ground level H_{inf}, realized by a (degenerate if M is not hyperbolic) state implementing a strong geometrization with H equal twenty seven times the sum of the volumes of the hyperbolic pieces, and higher levels of H, realized by (degenerate) states implementing a weak geometrization, with H equal twenty seven times the volume of the hyperbolic pieces. Conjecture 1 says that the system always decays to its ground state.

There are no examples so far of strongly singularity free space times decomposing the manifold on the long time into a non pure geometrization, i.e. displaying a mixed convergence-collapse picture. As proved in section 3.2.3 the full neck solution is a non compact example of a strongly singularity free space time undergoing a mixed convergence-collapse picture. The fact that the evolution at the ends of the neck is compatible with the evolution of flat cone cusps makes possible to construct an almost solution of the Einstein equation having the same properties of strongly singularity free space times and displaying the same conclusion as in Theorem 1. It is likely it could be perturbed to get a true solution.

It would be interesting too to investigate whether similar results as those presented here hold for the Einstein equations with matter. In particular whether there is a critical density below which the (normalized CMC) universe evolve into a strong geometrization.

Chapter 2

Background

2.1 Topologies and miscellaneous formulae.

`Topologies.`

Define the *strong* $W_0^{p,q}$ topology by fixing a C^∞ metric g_0 and completing the space of C^∞ (p,q)-tensors with respect to the norm

$$\|h\|_0^q = \int_M \sum_{|i|\leq p} |\nabla_{g_0}^i h|_{g_0}^q dv_{g_0}. \tag{2.1}$$

We will use occasionally L^q instead of $W^{0,q}$ to stress that no derivatives are involved. When a Sobolev norm of a geometric tensor constructed out of the state (g,K) is taken with respect to the intrinsic metric g it will be denoted without the subscript 0 i.e. by $W^{p,q}$. We will use sometimes the *weak* $W_0^{3,2} \times W_0^{2,2}$ topology defined with respect to the inner product induced by equation 2.1. It must be understood the strong topology unless explicitly stated.

`The Riemann tensor.`

$$Rm_{ij}{}^{r}{}_{s}V^s = \nabla_i\nabla_j V^r - \nabla_j\nabla_i V^r. \tag{2.2}$$

`Differential operators and algebraic operations on symmetric two-tensors.`

$$div(A)_k = \nabla^i A_{ik} = \nabla.A, \tag{2.3}$$

$$curl(A)_{ab} = \frac{1}{2}(\epsilon_a{}^{cd}\nabla_d A_{cb} + \epsilon_b{}^{cd}\nabla_d A_{ca}), \tag{2.4}$$

$$(A\times B)_{ab} = \epsilon_a{}^{cd}\epsilon_b{}^{cf}A_{ce}B_{df} + \frac{1}{3}(A.B)g_{ab} - \frac{1}{3}tr(A)tr(B)g_{ab}, \tag{2.5}$$

$$(A\wedge B)_a = \epsilon_a{}^{bc}A_b{}^d B_{dc}, \tag{2.6}$$

$$d^{\nabla *}d^{\nabla}(A) + 2div^* div(A) = 2\nabla^*\nabla A + \mathcal{R}(A), \tag{2.7}$$

$$
\begin{aligned}
d^{\nabla}(A)_{ijk} &= \nabla_i A_{jk} - \nabla_j A_{ik}, && (2.8)\\
(d^{\nabla *}M)_{ij} &= -(\nabla_l M^l{}_{ij} - \nabla_l M^l{}_{ji}), && (2.9)\\
(div^* M)_{ij} &= -(\nabla_i M_j + \nabla_j M_i), && (2.10)\\
\mathcal{R}(A)_{ij} &= Ricc_i{}^k A_{kj} + Ricc_j{}^k A_{ki} - 2Rm_{iljm}A^{lm}. && (2.11)
\end{aligned}
$$

We will use a hat above 2-tensors to denote its traceless part, i.e. $\hat{K} = K - \frac{1}{3}(tr_g K)g$.

2.2 The CMC gauge

Definition 5 *A CMC (cosmological) space time* $(\mathbf{M}, \mathbf{g})$ *is a globally hyperbolic cosmological solution having a (compact) Cauchy surface with constant mean curvature* $k \neq 0$.

We choose the future direction in such a way that $k < 0$. The Cauchy development $(\mathbf{M}, \mathbf{g})$ of an initial state (g, K) (not necessarily CMC) with $W_0^{s,2} \times W_0^{s-1,2}$ $(s > 2.5)$ regularity has $W_0^{s,2}$ regularity ([CBY]). All through the work we will consider $W_0^{3,2} \times W_0^{2,2}$ initial states and therefore $W_0^{3,2}$ space times.

`Foliations and the Cauchy problem in` $W_0^{3,2} \times W_0^{2,2}$ `for the CMC gauge.`

When addressing the problem of existence and unicity of Cauchy developments there are two approaches possible. The first is based on the existence of CMC foliations. It is proved in [Ba1], [Ge] that a C^∞ CMC space time $(\mathbf{M}, \mathbf{g})$ has a unique connected maximal foliation of C^∞ CMC slices with the range of k an open subinterval I of $(-\infty, +\infty)$. If $\sigma(M) \leq 0$ then it must be $I \subset (-\infty, 0)$. As a consequence of these facts we have:

1. *CMC gauge.* On the range of the foliation the C^∞ 4-metric can be presented in $M \times I$ as

$$\mathbf{g} = -N^2 dk^2 + g_{ij}(dx^i + X^i dk)(dx^j + X^j dk), \tag{2.12}$$

   N is the *lapse function* and X the *shift vector*. $g(k)$ is a three-metric on the manifold M.

2. *Hamilton (evolution) equations.*

$$
\begin{aligned}
\dot{g} &= -2NK + \mathcal{L}_X g, && (2.13)\\
\dot{K} &= -\nabla\nabla N + N(Ricc + kK - 2K \circ K) + \mathcal{L}_X K, && (2.14)\\
\dot{k} &= 1 = -\Delta N + |K|^2 N. && (2.15)
\end{aligned}
$$

   These equations comprise a strongly well posed (posses existence, unicity and ($W_0^{3,2} \times W_0^{2,2}$) short time stability) system on the set of C^∞ initial states, i.e. the C^∞ Cauchy developments, exist, are unique and short time $W_0^{3,2}$ stable with respect to the $W_0^3 \times W_0^2$ initial conditions but must be C^∞. The *lapse equation* (2.15) comes as a result of taking the trace of the evolution equations (2.13, 2.14).

When proving the stability of the flat cone one may take the approach of working with C^∞ initial conditions and proving long time stability with respect to the $W_0^{3,2} \times W_0^{2,2}$ norms of them. This approach requires however to work with higher orders of the Bel-Robinson energies to guarantee that regularity is kept along the flow if the field is controlled in $W_0^{3,2} \times W_0^{2,2}$. There is however a second approach. It is proved in [AM1] that thought as the equations for a flow of states the Einstein equations, in a CMC-spatially harmonic gauge, are strongly well posed in $C^3(\mathcal{H}^3)$ i.e. the map, from initial conditions in a hyperbolic manifold M into solutions

$$\mathcal{H}^s \to C^k_T(\mathcal{H}^s), \tag{2.16}$$

is a continuous map with a time of existence T which depends continuously on the initial data in $\mathcal{H}^s$. The space $C^k_T(\mathcal{H}^s)$ is defined as

$$C^k_T(\mathcal{H}^s) = \cap_{0\leq j\leq k-1} C^j([0,T];\mathcal{H}^{s-j}). \tag{2.17}$$

`The constraint equations.`

The constraint equations on CMC states are

$$Energy\ constraint: \qquad R - |K|^2 + k^2 = 0, \tag{2.18}$$

$$Momentum\ constraint: \qquad \nabla . K = 0. \tag{2.19}$$

One of the advantages of the CMC gauge is the fact that the constraint equations decouple in a conformal sense. The *conformal* method was developed by Lichnerowicz, Choquet-Bruhat and York ([CBY]) and was used by Isenberg in [I] to parameterize the set of CMC solutions in a compact manifold. It is based on the following fact (observe first that the momentum constraint is equivalent to $\nabla.\hat{K} = 0$), if $\hat{K}^{ij}$ is a trace free symmetric tensor then

$$\nabla_{(\phi^4 g)}.(\phi^{-10}\hat{K}) = \phi^{-10}(\nabla_g.\hat{K}). \tag{2.20}$$

Therefore if $\hat{K}^{ij}$ is a solution to the momentum constraint with the covariant derivative of the metric g, then $\phi^{-10}\hat{K}^{ij}$ is a solution of the momentum constraint with the covariant derivative of the conformally related metric $\phi^4 g$. Once the pair (g, K) solves the momentum constraint the energy constraint can be solved by conformally deforming g as $\phi^4 g$ and solving the Lichnerowicz equation

$$\Delta\phi = \frac{1}{8}R_g\phi - \frac{1}{8}|\hat{K}|^2_g\phi^{-7} + \frac{k^2}{12}\phi^5. \tag{2.21}$$

The solution to the constraints is given by $(\phi^4 g, K)$. It was proved in [I] that if $\hat{K} \neq 0$ and $k \neq 0$ then the Lichnerowicz equation 2.21 admits a unique positive solution. We will use this fact in section

3.2.5.

Scale invariant variables.

We will use scale invariant variables (or normalized states). We will denote them by a tilde above the variable. For a state (g, K), we define $(\tilde{g}, \tilde{K}) = (\frac{k^2}{9}g, -\frac{k}{3}K)$. $(\tilde{g}, \tilde{K})$ are scale invariant in the sense that they remain invariant under the scale transformations $(\lambda^2 g, \lambda K)$ with $\lambda > 0$. With this definition $\tilde{k} = -3$ and $\tilde{N} = \frac{k^2}{9}N$. The logarithmic time is defined as $\sigma = -\ln(-k)$. ∂_σ is a scale invariant derivative.

The reduced hamiltonian.

DEFINITION 6 *The reduced hamiltonian H at the CMC state (g, K) is given by*

$$H = -k^3 Vol(M). \tag{2.22}$$

In terms of the scale invariant variables, H and $\partial_\sigma H$ are

$$H = C_1 \int_M dv_{\tilde{g}}, \tag{2.23}$$

$$\partial_\sigma H = -C_2 \int_M \tilde{N} |\hat{\tilde{K}}|^2 dv_{\tilde{g}}. \tag{2.24}$$

When $\partial_\sigma H = 0$ the second derivative is zero and the third is calculated as

$$\partial^3_\sigma H = -C_3 \int_M \tilde{N}^3 |\hat{\tilde{Ricc}}|^2 dv_{\tilde{g}}. \tag{2.25}$$

C_1, C_2, C_3 are positive constants. Equations (2.24, 2.25) show that H is strictly monotonically decreasing unless the solution is a flat cone and in that case H is constant. The infimum in the space of CMC states is given by $(-\frac{3}{2}\sigma(M))^{\frac{3}{2}}$ ([FM1]). It was proved in [FM1] that H has a strict local minimum at $(g_H, -g_H)$ when M is of hyperbolic type, among normalized CMC states (up to diffeomorphism) in $W^{3,2}_0 \times W^{2,2}_0$. It is conjectured that it is a global minimum (see Comment 2).

2.3 The Bel-Robinson energies.

Weyl fields.

DEFINITION 7 *A Weyl field $W = W_{\alpha\beta\gamma\delta}$, is a traceless four tensor with the symmetries of the Riemann tensor.*

If $\mathbf{g}$ is a solution to the Einstein equations in vacuum then $\mathbf{Ricc} = 0$ and $\mathbf{Rm} = W$ is a Weyl field. If T is a vector field then $\nabla_T W$ is also a Weyl field.

Hodge duals.

$$ {}^*W_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta} \tag{2.26} $$

$$ W^*_{\alpha\beta\gamma\delta} = W_{\alpha\beta}{}^{\mu\nu}\frac{1}{2}\epsilon_{\mu\nu\gamma\delta} \tag{2.27} $$

Electric-magnetic decomposition.

Given a foliation of $\mathbf{M}$ by compact Cauchy surfaces Σ_t denote by T the future unit normal to Σ_t. Then define

$$ E(W)_{\alpha\beta} = W_{\alpha\mu\beta\nu}T^\mu T^\nu \tag{2.28} $$

$$ B(W)_{\alpha\beta} = {}^*W_{\alpha\mu\beta\nu}T^\mu T^\nu \tag{2.29} $$

Both E and B are trace free with respect to $\mathbf{g}$ and g. From now on assume $W = \mathbf{Rm}$. Then the Gauss-Codazzi equations give

Defining equations for E and B.

$$ \nabla_i K_{jm} - \nabla_j K_{im} = \epsilon_{ij}{}^l B(W)_{lm} = -W_{ijkT} \tag{2.30} $$

$$ Ricc_{ij} - K_{im}K_j^m + K_{ij}k = E(W)_{ij} \tag{2.31} $$

The component of W can be recovered from those of E and B as

$$ W_{iTjT} = E_{ij} \tag{2.32} $$

$$ W_{ijkT} = -\epsilon_{ij}{}^m B_{mk} \tag{2.33} $$

$$ W_{ijkl} = -\epsilon_{ijm}\epsilon_{kln}E^{mn} \tag{2.34} $$

E and B satisfy the following

Divergence equalities.

$$ div(E) = K \wedge B \tag{2.35} $$

$$ div(B) = -K \wedge E \tag{2.36} $$

$E(\nabla_T W)$ and $B(\nabla_T W)$ are given by

$$ E(\nabla_T W) = curl(B) - \frac{3}{2}(E \times K) + \frac{1}{2}kE \tag{2.37} $$

$$ B(\nabla_T W) = -curl(E) - \frac{3}{2}(B \times K) + \frac{1}{2}kB \tag{2.38} $$

(2.39)

Bel-Robinson tensor.

The Bel-Robinson tensor $Q(W)_{\alpha\beta\gamma\delta}$ associated with W is

$$Q(W)_{\alpha\beta\gamma\delta} = W_{\alpha\mu\gamma\nu} W_{\beta\ \delta}^{\ \mu\ \nu} +^* W_{\alpha\mu\gamma\nu} {}^*W_{\beta\ \delta}^{\ \mu\ \nu} \tag{2.40}$$

In terms of E and B the components of Q are given by

$$Q(W)_{TTTT} = |E|^2 + |B|^2 \tag{2.41}$$

$$Q(W)_{iTTT} = 2(E \wedge B)_i \tag{2.42}$$

$$Q(W)_{ijTT} = -(E \wedge E)_{ij} - (B \wedge B)_{ij} + \frac{1}{3}(|E|^2 + |B|^2) g_{ij} \tag{2.43}$$

Divergence.

$$\nabla^\alpha Q_{\alpha\beta\gamma\delta} = 0 \tag{2.44}$$

The Bel-Robinson energies of zero and first order are defined as

Defining equations for Q_0 and Q_1.

$$Q_0 := \int_M |E(W)|^2 + |B(W)|^2 dv_g \tag{2.45}$$

$$Q_1 = \int_M |E(\nabla_T W)|^2 + |B(\nabla_T W)|^2 dv_g \tag{2.46}$$

2.4 L^2-theory of convergence and collapse of Riemannian metrics.

In this section we describe the theory of convergence and collapse of Riemannian manifolds under a uniform bound on the L^2 norm of the curvature as developed in [A4]. In basic terms, it describes the behavior of three-metrics when one controls some metric invariants as the L^2 curvature, the volume or the volume radius. There are three basic behaviors, we describe them below.

Convergence.

Theorem 5 ([A4] Theorem 3.7). *For a fixed $\Lambda > 0$, $\epsilon > 0$ and V_0, the space of Riemannian metrics on M such that*

$$\int_M |Ricc|^2 \leq \Lambda,\ \nu \geq \epsilon,\ Vol(M) \leq V_0, \tag{2.47}$$

is precompact in the weak $W_0^{2,2}$ topology.

Collapse.

Theorem 6 ([A4] Corollary 3.10) *Suppose g_i is a sequence of metrics in M such that*

$$\int_M |Ricc|^2 dv_g \leq \Lambda, \ \nu_{g_i} \to 0. \tag{2.48}$$

Then there is a sequence $\mathcal{F}_i$ of $\mathcal{F}$-structures such that g_i collapses along the $\mathcal{F}_i$ orbits, i.e.

$$diam_{\mathcal{F}_i} \mathcal{O}_x \leq c(\nu_{g_i})^{\frac{1}{3}}. \tag{2.49}$$

Convergence-Collapse.

Theorem 7 ([A4] Theorem 3.19) *Let g_i be a sequence of metrics on M satisfying*

$$Vol_g(M) = 1, \ \int_M |Ricc|^2 dv_g \leq \Lambda \tag{2.50}$$

and there is a sequence of points $x_i \in M$ such that

$$\nu_{g_i}(x_i) \geq \epsilon, \tag{2.51}$$

for some constant $\epsilon > 0$ and sequence $x_i \in M$. Then there is a subsequence $\{i'\}$ of $\{i\}$ and a maximal open set $\Omega \subset\subset M$ such that the sequence $(\Omega, g_{i'})$ converges modulo diffeomorphism, in the weak $W_0^{2,2}$ topology to a countable collection of cusps (connected component of Ω). The thin part M_ϵ of $(M, g_{i'})$ contains a neighborhood V of $M - M^{\epsilon'}$ (for ϵ' sufficiently small) for all $i' \geq i'_0$. There is a sequence of $\mathcal{F}$ structures, $\mathcal{F}_i$ defined on V, which partially collapses under $g_{i'}$; that is, part of $(V, g_{i'})$ converges to neighborhoods of the boundary, or neighborhoods of infinity, of the collection of cusps N_j, while the remaining part collapses along the $\mathcal{F}$-structures $\mathcal{F}_i$.

Miscellaneous results.

The following results are needed. The first is needed to prove that on the (thick parts) of the (degenerate) ground state for the reduced hamitonian the metric is hyperbolic. The second is needed to prove that the tori joining the H with the G pieces in the ground state are incompressible.

Proposition 1 ([A6] Proposition 3.1) *Say M is a compact three-manifold with $\sigma(M) \leq 0$ then*

$$|\sigma(M)|^2 = \inf_{Vol_g(M)=1} \int_M R^2 dv_g. \tag{2.52}$$

Proposition 2 ([A3] Theorem 2.9) *Let M be a complete hyperbolic manifold of finite volume with cusps. Say that to one of the cusps is glued along a compressible torus to a compact graph manifold. Then it is possible to find a complete metric with*

$$Vol_g(M)^{\frac{1}{3}} \int_M R^2 dv_g < Vol_{g_H}(M)^{\frac{1}{3}} \int_M R_{g_H}^2 dv_{g_H} \tag{2.53}$$

without distorting the metric on the other cusps.

Proposition 3 ([A3] pg 156.) *Strong geometrizations are unique up to isotopy, i.e. the tori T_i are unique up to isotopy.*

Chapter 3

Results

3.1 The ground state for the reduced hamiltonian.

The proof of Theorem 1 divides into three lemmas. From Lemma 1 we conclude that a sequence of states satisfying conditions 1, 2, 3 has bounded L^4 norm of $\hat{K} = K - \frac{k}{3}g$ and from Lemma 2 that the L^2 norm of $\hat{K}$ goes to zero. It is direct then, using the defining equations for E and the constraints that the L^2 norm of *Ricc* is bounded and that R converges to $-\frac{2}{3}k^2$ in L^1. Lemma 3 uses these two conclusions to prove using the L^2 theory of convergence and collapse of Riemannian metrics that there is a subsequence having one of the behaviors 1, 2 or 3 in Theorem 1.

Definition 8 *We say that a set of quantities X is controlled at zero by a set of quantities Z if $X \to 0$ when $Z \to 0$ and that Z controls X if a bound on Z gives a bound for X.*

Lemma 1 *(see [A1] for a similar statement in case of L^∞ bounds of E and B). On normalized states ($k = -3$) in $W_0^{2,2} \times W_0^{1,2}$, Q_0 and $Vol(M)$ control the $W^{1,2}$ and L^4 norms of $\hat{K} = K - \frac{k}{3}g$ and in addition they are controlled at zero by $\|\hat{K}\|_{L^2}$ and Q_0.*

Proof: We make use of the formula (2.7) to get

$$\Delta|\hat{K}|^2 = 2|\nabla\hat{K}|^2 + < d^{\nabla *}W^T, \hat{K} > + < \mathcal{R}(\hat{K}), \hat{K} >, \tag{3.1}$$

where, $W^T_{ijk} = \mathbf{Rm}(e_i, e_j, e_k, T)$.

Denote as $\hat{\lambda}_i$ the eigenvalues of $\hat{K}$ in an orthonormal oriented basis (e_1, e_2, e_3). The Gauss equation reads for $i \neq j$, $Rm_{ijij} = \mathbf{Rm}_{ijij} - \lambda_i\lambda_j$ and (in dimension three) $Ricc_{ii} = \frac{s}{2} - Rm_{jkjk}$. With these formulas and the energy constraint compute the terms of $< \mathcal{R}(\hat{K}), \hat{K} >$ as

$$Ricc_{ij}\hat{K}^i{}_l\hat{K}^{lj} = \sum_i Ricc_{ii}\hat{\lambda}_i^2 = \frac{s}{2}|\hat{K}|^2 + \sum_i \hat{\lambda}_i^2\lambda_j\lambda_k - \hat{\lambda}_i^2\mathbf{Rm}_{jkjk}$$

$$\cdots = \frac{1}{2}(|\hat{K}|^2 - \frac{2}{3}k^2)|\hat{K}|^2 + \sum_i \hat{\lambda}_i^3 + \hat{\lambda}_i^2 - \hat{\lambda}_i^2 \mathbf{Rm}_{jkjk}, \tag{3.2}$$

$$\begin{aligned} Rm_{ijkl}\hat{K}^{ik}\hat{K}^{jl} &= \sum_{i\neq j}\hat{\lambda}_i\hat{\lambda}_j Rm_{ijij} = \sum_{i\neq j}\hat{\lambda}_i\hat{\lambda}_j(\mathbf{Rm}_{ijij} - \lambda_i\lambda_j) \\ \cdots &= \sum_{i\neq j}\hat{\lambda}_i\hat{\lambda}_j\mathbf{Rm}_{ijij} - \hat{\lambda}_i^2\hat{\lambda}_j^2 + 2\hat{\lambda}_i\hat{\lambda}_j^2 - \hat{\lambda}_i\hat{\lambda}_j. \end{aligned} \tag{3.3}$$

The two calculations above imply the inequality

$$-(<\mathcal{R}(\hat{K}), \hat{K}> - |\hat{K}|^4) \leq a_1|\hat{K}|^2 + a_2|\hat{K}|^3 + a_3|\hat{K}|^2(|E|^2 + |B|^2)^{\frac{1}{2}}. \tag{3.4}$$

Integrating (3.1) gives the inequality

$$\int_M 2|\nabla\hat{K}|^2 + |\hat{K}|^4 dv_g \leq \int_M a_1|\hat{K}|^2 + a_2|\hat{K}|^3 + a_3|\hat{K}|^2(|E|^2 + |B|^2)^{\frac{1}{2}} dv_g + a_4 Q_0, \tag{3.5}$$

where $a_i > 0,\ i = 1,2,3,4$ are numerical. Observe that the inequalities

$$\int_M |\hat{K}|^2(|E|^2 + |B|^2)^{\frac{1}{2}} dv_g \leq (\int_M |\hat{K}|^4 dv_g)^{\frac{1}{2}} Q_0^{\frac{1}{2}}, \tag{3.6}$$

$$\int_M |\hat{K}|^3 dv_g \leq (\int_M |\hat{K}|^2 dv_g)^{\frac{1}{2}} (\int_M |\hat{K}|^4 dv_g)^{\frac{1}{2}}, \tag{3.7}$$

transform equation (3.5) into

$$2\|\nabla\hat{K}\|_{L^2}^2 + \|\hat{K}\|_{L^4}^4 - C(\|\hat{K}\|_{L^2} + Q_0^{\frac{1}{2}})\|\hat{K}\|_{L^4}^2 \leq C(\|\hat{K}\|_{L^2}^2 + Q_0), \tag{3.8}$$

that implies the following bound

$$(\int_M 2|\nabla\hat{K}|^2 + |\hat{K}|^4 dv_g)^{\frac{1}{2}} \leq C'(\|\hat{K}\|_{L^2} + Q_0^{\frac{1}{2}}), \tag{3.9}$$

which proves the statement. □

A technical comment is in order. The proof above works say for K smooth. In case (g, K) is in $W_0^{2,2} \times W_0^{1,2}$ smooth (g, K) out (even not satisfying the constraints) proceed in the calculations and then take the limit back to recover equation (3.9).

Lemma 2 *If a sequence of $W_0^{2,2} \times W_0^{1,2}$ states (g, K) satisfies 1, 2 and 3, then $\|\hat{K}\|_{L^2} \to 0$ (i.e. on normalized states with bounded Q_0, $H - H_{inf}$ controls $\|\hat{K}\|_{L^2}$ at zero).*

Proof:

1. $\sigma(M) = 0$. In this case $-k^3 Vol(M) \to 0$, with $k = -3$, so $Vol(M) \to 0$. Then $\int_M |\hat{K}|^2 dv_g \leq Vol(M)^{\frac{1}{2}}(\int_M |\hat{K}|^4 dv_g)^{\frac{1}{2}} \to 0$.

2. $\sigma(M) < 0$. Let g_Y be the unique Yamabe metric of constant scalar curvature $R_Y = -6$ in the conformal class of g. If $g = \phi^4 g_Y$ then ϕ is determined by

$$-\Delta\phi + \frac{R_Y}{8}\phi - \frac{1}{8}\phi^{-3}|\hat{K}|_Y^2 + \frac{1}{12}k^2\phi^5 = 0, \tag{3.10}$$

where $\Delta = \nabla^2$. The maximum principle implies (placing the correct values of $R_Y = -6$ and $k = -3$) that

$$6(\phi_{min}^5 - \phi_{min}) \geq \phi_{min}^{-3}|\hat{K}|_Y^2 \geq 0, \tag{3.11}$$

which makes $\phi \geq 1$. Then observe that

$$-\sigma(M) \leq -R_Y(\int_M 1\ dv_Y)^{\frac{2}{3}}, \tag{3.12}$$

where $dv_Y = dv_{g_Y}$. This gives

$$0 \leq 6^{\frac{3}{2}}(\int_M \phi^6 - 1\ dv_Y) \leq 6^{\frac{3}{2}}\int_M \phi^6 dv_Y - (-\sigma(M))^{\frac{3}{2}} \tag{3.13}$$

$$\cdots = (\frac{2}{3}k^2 Vol(M)^{\frac{2}{3}})^{\frac{3}{2}} - (-\sigma(M))^{\frac{3}{2}}. \tag{3.14}$$

Therefore

$$\int_M (\phi - 1)^k dv_Y \leq C(H - H_{inf}) \tag{3.15}$$

for $k = 1, \ldots, 6$. Integrating equation (3.10), we get

$$6\int_M (\phi^5 - \phi) dv_Y = \int_M \phi^{-3}|\hat{K}|_Y^2\ dv_Y, \tag{3.16}$$

To prove that $\|\hat{K}\|_{L^2}^2 = \int_M \phi^{-2}|\hat{K}|_Y^2\ dv_Y \to 0$ observe that

$$\int_M \phi^{-2}|\hat{K}|_Y^2\ dv_Y = \int_M \phi\phi^{-3}|\hat{K}|_Y^2\ dv_Y$$

$$\cdots = \int_M \phi^{-3}|\hat{K}|_Y^2\ dv_Y + \int_M (\phi - 1)\phi^{-3}|\hat{K}|_Y^2\ dv_Y, \tag{3.17}$$

and

$$\int_M (\phi-1)\phi^{-3}|\hat{K}|_Y^2\ dv_Y = \int_M (\phi-1)\phi^2\phi^{-5}|\hat{K}|_Y^2\ dv_Y$$

$$\cdots \leq (\int_M (\phi-1)^2\phi^4\ dv_Y)^{\frac{1}{2}}(\int_m \phi^{-10}|\hat{K}|_Y^4\ dv_Y)^{\frac{1}{2}}. \qquad (3.18)$$

Lemma 1 implies that $\|\hat{K}\|_{L^4}^4 = \int_M \phi^{-10}|\hat{K}|_Y^4\ dv_Y$ is uniformly bounded. On the other hand note that

$$\int_M (\phi-1)^2\phi^4\ dv_Y \leq (\int_M (\phi-1)^6\ dv_Y)^{\frac{1}{3}}(\int_M \phi^6\ dv_Y)^{\frac{2}{3}}$$

$$\cdots = (\int_M (\phi-1)^6\ dv_Y)^{\frac{1}{3}} Vol(M)^{\frac{2}{3}} \leq C((H-H_{inf})^{\frac{1}{3}} Vol(M)^{\frac{2}{3}}. \qquad (3.19)$$

All together gives the following bound

$$\|\hat{K}\|_{L^2}^2 \leq C((H-H_{inf}) + (H-H_{inf})^{\frac{1}{6}} Vol(M)^{\frac{1}{3}}\|\hat{K}\|_{L^4}^2) \qquad (3.20)$$

□

A technical comment is in order. To avoid issues of existence of Yamabe metrics in the conformal class of a metric in $W_0^{2,2}$, observe that by smoothing (g,K) out, and deforming k to satisfy the energy constraint, we can proceed on the proof and take the limit back to recover equation (3.20).

`Comment 4`. As said on the background section, in case M is of hyperbolic type R reaches a strict local maximum at $Vol_{g_H}(M)^{-\frac{2}{3}} g_H$ among the Yamabe metrics of volume one restricted to a local slice transversal to the orbits of the diffeomorphism group. That maximum coincides with $\sigma(M)$ if the conjectured value of the sigma constant, as is explained in comment 2 is correct. Lemma 2 and Theorem 1 still hold therefore if we restrict the states to lie inside a suitable ball in $W_0^{2,2} \times W_0^{1,2}$ around $(g_H, -g_h)$ and change H_{inf} by $H_{Hyperbolic}$. We will assume this observation when proving Theorem 3.

Lemma 3 *A sequence of metrics g in $W_0^{2,2}$ of volume 1 having the following asymptotics*

$$\int_M |R\hat{i}cc|^2 dv_g \leq \Lambda, \quad \int_M |R-\sigma(M)| dv_g \to 0, \qquad (3.21)$$

where in case $\sigma(M)<0$, Λ is a bound and in case $\sigma(M)=0$, $\Lambda(g)\to 0$, has a subsequence g with one of the following behaviors:

1. Convergence. *There is a $\delta > 0$ such that for every x, $\nu(x) \geq \delta$, and a suitable choice of diffeomorphisms make g converge in the weak $W_0^{2,2}$ topology to a metric of constant curvature $\frac{\sigma(M)}{6}$.*

2. Collapse. *For every* x, $\nu(x) \to 0$, (M,g) *collapses along a sequence of* $\mathcal{F}$*-structures,* M *is a graph manifold and* $\sigma(M) = 0$.

3. Convergence - Collapse. *Say the subsequence* g *is indexed as* g_i. *There is a sequence* x_i *and* $\delta > 0$ *with* $\nu_{g_i}(x_i) \geq \delta$ *and there is a sequence* y_i *with* $\nu_{g_i}(y_i) \to 0$. *There is also a maximal domain* $\Omega \subset\subset M$ *such that a suitable choice of diffeomorphisms make* g *converge in the weak* $W_0^{2,2} \times W_0^{1,2}$ *topology to a finite set of complete hyperbolic manifolds of finite volume with cusps with unique up to isotopy incompressible tori at the ends. There is an* $\epsilon_0(\Lambda)$ *such that for every* $\epsilon \leq \epsilon_0$, M_ϵ *is a union of graph manifolds and has an* $\mathcal{F}$*-structure whose orbits are* $C\epsilon^{\frac{1}{3}}$ *collapsed, i.e.* $diam(\mathcal{O}_x) \leq C\epsilon^{\frac{1}{3}}$. M_ϵ *collapses in volume in the sense that* $\lim sup_{g_i} Vol_{g_i}(M_\epsilon)$ *goes to zero as* $\epsilon \to 0$.

Proof: Let's divide the proof into three parts:

1. $\nu(x) \geq \delta$, $\forall x \in M$, then (Theorem 5) after a suitable choice of diffeomorphisms a subsequence converges weakly into a metric g_∞ of volume one and scalar curvature $\sigma(M)$ in $W_0^{2,2}$, therefore $\int_M R_\infty^2 dv_{g_\infty} = |\sigma(M)|^2$. g_∞ is then a metric minimizing the functional $\int_M R^2 dv_g$ on metrics of volume one, therefore is a critical metric for it. In case $\sigma(M) < 0$, $R_\infty < 0$ and so (by the Euler Lagrange equation for the square of the scalar curvature functional) has constant sectional curvature $\frac{\sigma(M)}{6}$. If $\sigma(M) = 0$ then $Ricc = 0$ and is therefore of constant sectional curvature $\frac{\sigma(M)}{6}$ too.

2. $\nu(x) \to 0$, $\forall x \in M$. Then (Theorem 6) g collapses along a sequence of $\mathcal{F}$-structures, $\sigma(M) = 0$ and the manifold is a graph manifold.

3. There is a $\delta > 0$ and sequences x_i and y_i such that $\nu(x_i) > \delta$ and $\nu(y_i) \to 0$, then (Theorem 7) there is a maximal domain $\Omega \subset\subset M$ such that a subsequence g_i converges weakly in $W_0^{2,2}$ to a non necessarily complete metric g_∞ with a countable number of cusps. Suppose $Vol_{g_\infty}(M) = 1$, i.e. there is no loss of volume on the thin parts. Then necessarily g_∞ is a critical metric (of constant scalar curvature $\sigma(M) < 0$) for the functional $\int_\Omega R^2 dv_g$ on variations of compact support, for otherwise for i big enough it would be possible to deform g_i with volume one to have $\int_M R^2 dv_g < |\sigma(M)|^2$. Again, it must be of constant sectional curvature $\frac{\sigma(M)}{6}$. As proved below a hyperbolic cusp is necessarily complete, and because the volume is bounded above, the limit metric in every cusp is a complete hyperbolic metric of finite volume. There are therefore a finite number of them. To prove that $\lim_{\epsilon\to 0} limsup_{g_i} Vol(M_\epsilon) = 0$ observe that if not there must be a sequence $\epsilon_j \to 0$ and g_j such that $Vol_{g_j}(M_{\epsilon_j}) > \gamma$ and $|Vol_{g_j}(M^{\epsilon_j}) - Vol_{g_\infty}(\Omega)| \to 0$. For ϵ_j small enough M_{ϵ_j} admits an $\mathcal{F}$ structure which is $c\epsilon_j^{\frac{1}{3}}$ collapsed, therefore each g_j can be deformed to collapse the volume of M_{ϵ_j} with bounded L^∞ curvature without changing much g_j in M^{ϵ_j} ([A4]). That would imply there are metrics with $Vol(M)^{\frac{1}{3}} \int_M R^2 dv_g$ as much close

to $\int_\Omega R^2_{g_\infty} dv_{g_\infty} < |\sigma(M)|^2$ as we like. Proposition 2 shows the tori on the cusps must be incompressible and Proposition 3 that they are unique up to isotopy.

`Hyperbolic cusps are complete.` Let s be an incomplete geodesic in Ω. Fix $p \in s$. Let S_2 be a transversal geodesic 2-simplex in Ω having p in its interior. For $x \in s$ (in the incomplete direction and close to p) consider the 3-simplex $S_3(x)$ formed by all geodesics joining x with a point in S_2. Observe that because (Ω, g_∞) is hyperbolic and s has finite length (in the incomplete direction) every $y \in \partial S_3(x)$ has a cone $C_3(y)$ inside of size bounded below[1]. Now as x approaches the end of s, there is a sequence of $\epsilon \to 0$, a sequence of points $\bar{x}_\epsilon \in \partial S_3(x)$ and a sequence $g_{i(\epsilon)}$ with $\nu_{g_\epsilon(x_\epsilon)} = \epsilon$ and having a cone of size bounded below inside M_ϵ. The blow up limit of the pointed space $(M, x_\epsilon, \frac{1}{\epsilon^2} g_\epsilon)$ has $\nu(x) = 1$ and is complete, flat, having a cone of size (α, ∞) inside, therefore must be R^3 which is a contradiction.

□

Proof of Theorem 1.

Proof: By lemma 1, one has $\|\hat{K}\|_{W^{1,2}} \to 0$ and $\|\hat{K}\|_{L^4}$ bounded. Writing the defining equation for E in terms of $\hat{K}$ and subtracting from it the energy constraint times $\frac{g}{3}$, we get

$$\hat{Ricc} - \hat{K} \circ \hat{K} + \frac{1}{3} k\hat{K} + \frac{1}{3}|\hat{K}|^2 g = E, \tag{3.22}$$

where $\hat{Ricc} = Ricc - \frac{R}{3} g$. Squaring and integrating gives

$$\int_M |\hat{Ricc}|^2 dv_g \leq \Lambda, \tag{3.23}$$

and integrating the energy constraint

$$\int_M |R - \frac{2}{3}k^2| dv_g \to 0. \tag{3.24}$$

Normalizing the state (g, K) to have volume one, i.e. looking at the new metric $\bar{g} = Vol_g(M)^{-\frac{2}{3}} g$ and new second fundamental form $\bar{K} = Vol_g(M)^{-\frac{1}{3}} K$ we get

$$\int_M |\hat{\bar{Ricc}}|^2 dv_{\bar{g}} = Vol_g(M)^{\frac{1}{3}} \int_M |\hat{Ricc}|^2 dv_g \leq Vol_g(M)^{\frac{1}{3}} \Lambda, \tag{3.25}$$

$$\begin{aligned} \int_M |\bar{R} - \frac{2}{3}\bar{k}^2| dv_{\bar{g}} &= \int_M |\bar{R} - \frac{2}{3} Vol_g(M)^{\frac{2}{3}} k^2| dv_{\bar{g}} \\ &= Vol_g(M)^{-\frac{1}{3}} \int_M |R - \frac{2}{3}k^2| dv_g. \end{aligned} \tag{3.26}$$

[1]Given a point x in a Riemannian manifold (M, g) a cone of size (α, l) $(l < inj_x g)$ in M is the image under the exponential map of a cone of size (α, l) (segments from x in $T_x M$ having length l and forming an angle α with a given segment)

If $\sigma(M) < 0$ then $\int_M |\bar{R} - \frac{2}{3}\bar{k}^2| dv_g \to 0$. If $\sigma(M) = 0$ then $Vol_g(M) \to 0$ and

$$Vol_g(M)^{-\frac{1}{3}} \int_M |R - \frac{2}{3}k^2| dv_g \leq Vol_g(M)^{\frac{1}{6}} (\int_M |\hat{K}|^4 dv_g)^{\frac{1}{2}} \to 0. \tag{3.27}$$

Therefore in this case too, $\int_M |\bar{R} - \frac{2}{3}\bar{k}^2| dv_g \to 0$. Finally, note that $\frac{2}{3} Vol_g(M)^{\frac{2}{3}} k^2 \to -\sigma(M)$ and therefore

$$\int_M |\bar{R} - \sigma(M)| dv_{\bar{g}} \to 0. \tag{3.28}$$

Applying Lemma 3, for the volume normalized states we obtain the geometric conclusions. The claims for the second fundamental form follow easily because (informally) on the thin part, $-k^3 Vol(M_{Thin}) \to 0$, and $|K|^2 = |\hat{K}|^2 + \frac{k^2}{3}$, when integrated (on the thin parts), shows that $\|K\|_{L^2}$ (on the thin parts) approaches zero. □

Observe that if $\Lambda \to 0$, as would be the case if $\tilde{Q}_0 \to 0$ then the convergence on the thick parts is strong.

3.2 Non pure ground states.

Say H_1 and H_2 are complete hyperbolic manifolds of finite volume, with cusps C_1 and C_2 (say for concreteness there are only two). We will glue the flat cone states on the hyperbolic pieces through a state in a torus neck G. See figure 1.1 for a clear picture. The procedure consists in finding a CMC initial state in the torus neck, being, up to a given error, compatible to the initial flat cone states at the place on the cusps where the gluing is going to take place. Secondly the metrics are glued and a transverse traceless tensor with respect to the new metric is found. We will follow a direct construction of it, although a more general construction is possible using the method of Lichnerowicz-Choquet Bruhat-York, this way is simpler to control. Finally a conformal perturbation of the metric gives the desired initial state. We will use a theorem of Isenberg (see the background section) that guarantees that in our situation the Lichnerowicz equation has a solution, and make use of the equation itself together with the pointwise estimates for the transverse traceless tensor to get pointwise estimates on the solution. The CMC state in the torus neck depends on a parameter that basically measures the length of the neck (between standard parts). The gluing can be realized the same way to any state in the family. The result is the family of initial states that displays the third kind of behavior in Theorem 1.

The construction is organized as follows. In sections 3.2.1, 3.2.2, 3.2.3 we analyze a polarized torus neck and its main features. The non polarized case follows along similar lines. In section 3.2.1 we find a solution to the Einstein equations in $R \times R \times T^2$, from which in sections 3.2.2 and 3.2.3 we construct slices on which the evolution shows a convergence-collapse picture and the emergence of hyperbolic cusps (on the normalized geometry). We repeat the same procedure in section 3.2.4 for the non polarized case. In section 3.2.5 we find a CMC foliation and perform the gluing with estimates for the solution.

3.2.1 The geometry on a torus's neck, the polarized case.

In this section we find a particular solution to the Einstein equations on $\mathbb{R} \times \mathbb{R} \times T^2$. In section 3.2.5 we find a CMC foliation whose normalized states are the ones needed on G to join H_1 and H_2. Flat cone states in a hyperbolic cusp are $(g_H, -g_H)$ with g_H the metric on the cusp $R \times T^2$

$$g_H = dx^2 + e^{2x} g_T, \tag{3.29}$$

with g_T a flat metric in the 2-torus.

On $R \times R \times T^2$ we look for the (polarized) T^2 symmetric metric in the gauge where it looks like

$$g = e^{2a}(-dt^2 + dx^2) + Re^{2W} d\theta_1^2 + Re^{-2W} d\theta_2^2. \tag{3.30}$$

The functions a, R, W depend on (t, x). Define the coordinates $(-, +) = (t-x, t+x)$. Derivatives with respect to $-$ and $+$ will be denoted with a subscript $+$ or $-$. The Einstein equations for such a metric are

$$\frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial t^2} = 0, \tag{3.31}$$

$$\frac{\partial}{\partial t}(R\frac{\partial}{\partial t}W) - \frac{\partial}{\partial x}(R\frac{\partial}{\partial x}W) = 0, \tag{3.32}$$

$$2\frac{R_\pm}{R}a_\pm = \frac{R_{\pm\pm}}{R} - \frac{1}{2}(\frac{R_\pm}{R})^2 + 2W_\pm^2. \tag{3.33}$$

As we want flat cone states at the end of $\mathbb{R} \times T^2$ we make the ansatz $R(x,t) = R_0(e^{2(t+x)} + e^{2(t-x)})$ that solves the wave equation (3.31). The equation for W (3.32) is the Euler-Lagrange equation for the lagrangian

$$L(t, \partial_t W, \partial_x W) = \int R(\partial_t W)^2 - R(\partial_x W)^2 dx. \tag{3.34}$$

We make the choice $W(x,t) = W_1 + W_0 \arctan e^{2x}$, which is the general form for the W stable solutions, i.e. those W that with fixed values at the boundary (infinity in this case) minimize the potential $V = \int R(x,0)(\partial_x W)^2 dx$. All what is missing is to find out a and then understand the geometry. Observing that

$$2(W_\pm)^2 = \frac{W_0^2}{2\cosh^2 2x}, \tag{3.35}$$

equations (3.33) are

$$2\frac{R_\pm}{R}a_\pm = \frac{R_{\pm\pm}}{R} - \frac{1}{2}(\frac{R_\pm}{R})^2 + \frac{W_0^2}{2\cosh^2 2x}. \tag{3.36}$$

Dividing by $R_{\pm}/R$ and then adding and subtracting both equations we get

$$\partial_x a = -(\frac{1}{2} + \frac{W_0^2}{2}) \tanh 2x, \tag{3.37}$$

$$\partial_t a = \frac{3}{2} + \frac{W_0^2}{2}, \tag{3.38}$$

which after integration give

$$a(x,t) = a(0) - (\frac{1}{2} + \frac{W_0^2}{2})\frac{1}{2}\ln\cosh 2x + (\frac{3}{2} + \frac{W_0^2}{2})t. \tag{3.39}$$

3.2.2 States's evolution in a torus neck.

We will see that it depends on who looks at the geometry the different descriptions of the evolution they tell. First for those observers who forcedly move keeping their x-coordinate constant and moving uniformly forward in time t, the normalized three geometry, normalized by $e^{(\frac{3}{2}+\frac{W_0^2}{2})t}$, collapses along the tori fibers into a one dimensional geometry on the real line and of finite length

$$g_\infty = e^{a(0)-(\frac{1}{2}+\frac{W_0^2}{2})\frac{\ln\cosh 2x}{2}} dx^2. \tag{3.40}$$

However for those observers who freely fall in space along time like geodesics the normalized three geometry will be seen to evolve into a hyperbolic cusp:

$$g_\infty = dx^2 + R_0 e^{2W_{\pm\infty}} e^{2x} d\theta_1^2 + R_0 e^{-2W_{\pm\infty}} e^{2x} d\theta_2^2. \tag{3.41}$$

Actually there are two sets of free falling observers, those for positive x and those for negative x. Both will observe the normalized three geometry, exponentially in time becoming hyperbolic cusps, and as it turns out then, there are two different cusps one on the left and one on the right and in between the geometry is collapsing, as will be made precise in what follows.

`Free falling observers.`

We will assume an insignificant approximation that in no way will change the global picture, nor the precise statements that follow on the evolution of the exact geometry.

Say $x \geq 10$, there the metric along the radius is almost like

$$e^{2((\frac{3}{2}+\frac{W_0^2}{2})t-(\frac{1}{2}+\frac{W_0^2}{2})x)}(-dt^2 + dx^2). \tag{3.42}$$

If we make s denote the geodesic's parameter, or proper time, then it can be calculated that, independently of the initial velocity, the coordinates $(t(s), x(s))$ of the geodesics behave according to

$$-(\frac{1}{2} + \frac{W_0^2}{2})t + (\frac{3}{2} + \frac{W_0^2}{2})x = \frac{1}{2}\ln\frac{3+W_0^2}{1+W_0^2} + o(\frac{1}{s}), \tag{3.43}$$

$$-(\frac{1}{2}+\frac{W_0^2}{2})x+(\frac{3}{2}+\frac{W_0^2}{2})t=\ln s+\frac{1}{2}\ln\frac{(3+W_0^2)(1+W_0^2)}{2}+o(\frac{1}{s}). \tag{3.44}$$

What this formulae tells us is that the coordinates

$$t'=-(\frac{1}{2}+\frac{W_0^2}{2})x+(\frac{3}{2}+\frac{W_0^2}{2})t, \tag{3.45}$$

$$x'=-(\frac{1}{2}+\frac{W_0^2}{2})t+(\frac{3}{2}+\frac{W_0^2}{2})x, \tag{3.46}$$

are the natural coordinate system constructed by a free falling set of observers. In these new coordinates and after choosing $a(0)=\frac{1}{2}\ln(-(\frac{1}{2}+\frac{W_0^2}{2})^2+(\frac{3}{2}+\frac{W_0^2}{2})^2)$

$$g=e^{2t'}(-dt'^2+dx^2)+R_0e^{2(\frac{\pi}{2}W_0+W_1)}(e^{2(t'+x')}+e^{\frac{2}{2+W_0^2}(t'-x')})d\theta_1^2+\ldots \tag{3.47}$$

$$\ldots+R_0e^{-2(\frac{\pi}{2}W_0+W_1)}(e^{2(t'+x')}+e^{\frac{2}{2+W_0^2}(t'-x')})d\theta_2^2. \tag{3.48}$$

After making $W_{+\infty}=\frac{\pi}{2}W_0+W_1$ and normalizing by $e^{2t'}$ we see that the observers observe a local three geometry exponentially falling to the hyperbolic three cusp

$$g=dx^2+R_0e^{2W_{+\infty}}e^{2x}d\theta_1^2+R_0e^{-2W_{+\infty}}e^{2x}d\theta_2^2 \tag{3.49}$$

3.2.3 A convergence-collapse picture

Let us describe now a foliation of Cauchy hypersurfaces (labeled by $s\geq 1$) in where to see the picture of convergence-collapse. For any s the hypersurface will be (first zone) $\{(t,x),\ -(\frac{1}{2}+\frac{W_0^2}{2})\ln s+(\frac{3}{2}+\frac{W_0^2}{2})t=s,\ |\,x\,|\leq\ln t\}$, (second zone) $\{(t,x),\ s=t'=-(\frac{1}{2}+\frac{W_0^2}{2})x+(\frac{3}{2}+\frac{W_0^2}{2})t,\ x\geq\ln s\}$ and (third zone) $\{(t,x),\ s=t''=(\frac{1}{2}+\frac{W_0^2}{2})x+(\frac{3}{2}+\frac{W_0^2}{2})t,\ x\leq-\ln s\}$. After normalizing the three metrics by the common factor e^{-2s} the limit of the three metrics are: i) on the first zone

$$g_\infty=d\tilde{x}^2, \tag{3.50}$$

the infinite-length one dimensional geometry on the real line, and ii) on the second zone

$$g_\infty=dx^2+R_0e^{2W_{+\infty}}e^{2x}d\theta_1^2+R_0e^{-2W_{+\infty}}e^{2x}d\theta_2^2, \tag{3.51}$$

on the whole $\mathbb{R}\times T^2$, and similarly for the third zone. A schematic picture is given in figure 3.1.

3.2.4 Geometry in a torus's Neck, the non-polarized case.

In this section we follow the same procedure as in section 3.2.1. We find first a solution to the Einstein equations with non polarized T^2 symmetry and then make a necessary change of variables at the ends.

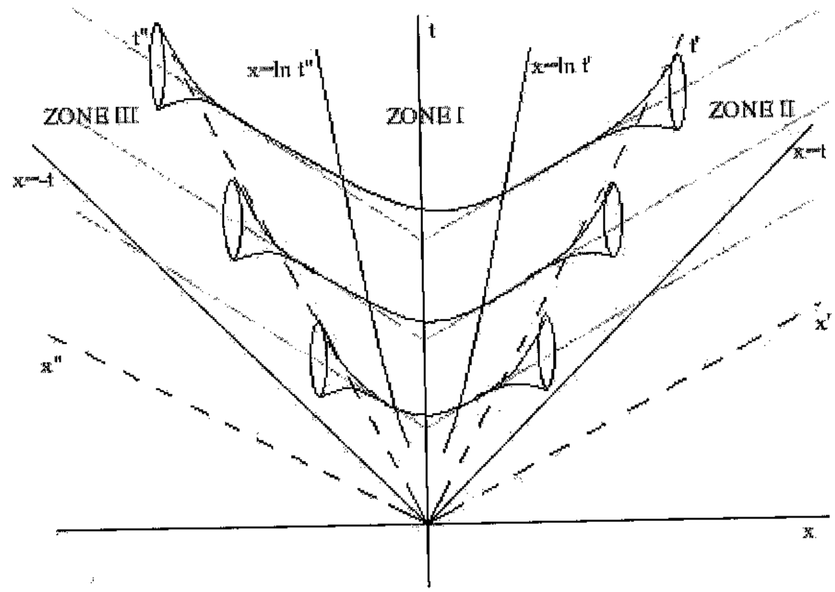

Figure 3.1: Schematic figure for the evolution of the normalized geometry.

On $\mathbb{R} \times \mathbb{R} \times T^2$ we look for a non polarized T^2 symmetric metric in the gauge where it looks like

$$g = e^{2a}(-dt^2 + dx^2) + R(e^{2W} + q^2 e^{-2W})d\theta_1^2 - Rqe^{-2W} 2d\theta_1 d\theta_2 + Re^{-2W} d\theta_2^2, \tag{3.52}$$

where $a,\ R,\ W$ depend on (t, x) or $(u, v) = (-, +) = (t - x, t + x)$. The Einstein equations reduce to

$$R_{+-} = 0, \tag{3.53}$$

$$2\frac{R_{++}}{R} - (\frac{R_+}{R})^2 + 4W_+^2 + q_+^2 e^{-4W} - 4a_+ \frac{R_+}{R} = 0, \tag{3.54}$$

$$2\frac{R_{--}}{R} - (\frac{R_-}{R})^2 + 4W_-^2 + q_-^2 e^{-4W} - 4a_- \frac{R_-}{R} = 0, \tag{3.55}$$

$$(RW_-)_+ + (RW_+)_- + Rq_+ q_- e^{-4W} = 0, \tag{3.56}$$

$$(Re^{-4W} q_+)_- + (Re^{-4W} q_-)_+ = 0. \tag{3.57}$$

We first make the ansatz $R(x, t) = R_0 e^{2t} \cosh(2x)$. Then we solve for time independent W and q which realize arbitrary flat metrics on the two tori at the ends, i.e. which have prescribed asymptotics $q_\infty,\ q_{-\infty},\ W_\infty,\ W_{-\infty}$. Finally solve for a and make a necessary change of variables at the ends.

Solving for time independent W and q.

The equation 3.57 forces q' to satisfy

$$q' = \frac{2ce^{4W}}{\cosh(2x)}, \tag{3.58}$$

and together with equation 3.56 forces W to be

$$W'' + 2\tanh(2x)W' = \frac{-2c^2e^{4W}}{\cosh^2(2x)}, \tag{3.59}$$

where c is any constant. The strategy to find solutions for W and q having arbitrary values at the ends is fix c first and find W having the arbitrary ending values $W(\infty) = W_\infty$ and $W(-\infty) = W_{-\infty}$. Then make c vary keeping fixed the end conditions for W and prove that we can reach at some c the prescribed asymptotic values for q, $q(\infty) = q_\infty$ and $q(-\infty) = q_{-\infty}$. That will be accomplished by proving that varying c from some value of c toward zero the integral of equation 3.58 that defines q gives (having $q_{-\infty}$ as the lower limit of integration) all possible asymptotic values for q_∞.

Although equation 3.59 is highly non linear it can be integrated exactly. We note that it is equivalent (unless W is constant in which case $c = 0$ and q is constant) to

$$((\cosh(2x)W')^2)' = -(c^2e^{4W})', \tag{3.60}$$

which gives

$$\cosh^2(2x)W'^2 = -c^2e^{4W} + A^2, \tag{3.61}$$

for $A > 0$ an arbitrary constant. Taking the square root we get a separable variables ODE. After integration we get

$$W = -\frac{1}{2}\ln\frac{|\,c\,|}{A}\cosh(-2A\arctan e^{2x} + B), \tag{3.62}$$

for B and arbitrary constant. We need to find A and B that solves the end conditions for W i.e.

$$\frac{|\,c\,|}{A}\cosh B = e^{-2W_{-\infty}}, \tag{3.63}$$

$$\frac{|\,c\,|}{A}\cosh(-\pi A + B) = e^{-2W_\infty}. \tag{3.64}$$

Making the change of variables $A = \frac{B-D}{\pi}$ we get the equations

$$B = D + \pi\,|\,c\,|\,e^{2W_\infty}\cosh D, \tag{3.65}$$

$$D = B - \pi\,|\,c\,|\,e^{2W_{-\infty}}\cosh B. \tag{3.66}$$

Now the problem is to understand the solutions B and D. Graphing equations 3.65, 3.66 on a B, D coordinate axis, we see (observe the factor $|c|$ in front of $\cosh D$ and $\cosh B$) that there is some

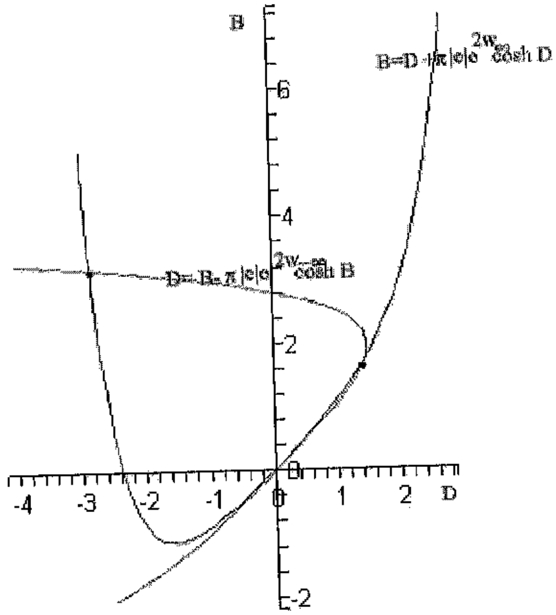

Figure 3.2: The graphs of equations 3.65 3.66

positive c_0 above which there are no solutions, at which there is only one and below which there are two solutions. See figure 3.2.

In the following we will analyze the solutions A and B as $c \to 0$. We will see that given a prescribed value $q_{-\infty}$ we get any asymptotic value for q_∞ by varying c from c_0 to zero. The equation

$$e^{2W_{-\infty}} \cosh B = e^{2W_\infty} \cosh D, \tag{3.67}$$

gives for the two branches the following behaviors:

1. For the first branch, either $W_\infty = W_{-\infty}$ for which we get, observing that $A = B - D > 0$

$$\begin{aligned} &B = -D \to 0, \\ &\frac{|c|}{A} \to e^{-2W_{-\infty}}, \end{aligned} \tag{3.68}$$

or $W_\infty \neq W_{-\infty}$ for which we get

$$\begin{aligned} &B \to \infty \; if \; W_\infty > W_{-\infty} \; (or \; -\infty \; if \; W_\infty < W_{-\infty}), \\ &B - D \to 2(W_\infty - W_{-\infty}) \; (or \; -2(W_\infty - W_{-\infty})), \\ &A \to \tfrac{2}{\pi}(W_\infty - W_{-\infty}) \; (or \; -\tfrac{2}{\pi}(W_\infty - W_{-\infty})). \end{aligned} \tag{3.69}$$

2. For the second branch, for any $W_\infty, W_{-\infty}$

$$\begin{aligned} &B \to \infty, D \to -\infty, \\ &B + D \to 2(W_\infty - W_{-\infty}), \\ &A \sim \tfrac{2B - 2(W_\infty - W_{-\infty})}{\pi}. \end{aligned} \tag{3.70}$$

With this behavior of A and B as $c \to 0$ we get:

1. In the first branch the formula for q'

$$q' = \frac{c}{(\cosh(2x))(\frac{|c|}{A}\cosh(-2A\arctan e^{2x} + B))^2}, \tag{3.71}$$

shows that starting at an arbitrary $q_{-\infty}$, q approaches uniformly to the constant function $q = q_{-\infty}$.

2. In the second branch the formula for q approximates to

$$q' \sim \frac{\pm e^{-2W_{-\infty}}(2B - 2(W_\infty - W_{-\infty}))}{\pi \cosh B \cosh(2x)(e^{-2W_{-\infty}}(\cosh B)^{-1}\cosh(-2A\arctan e^{2x} + B))^2}, \tag{3.72}$$

and rearranged reads

$$q' \sim \frac{\pm e^{2W_{-\infty}}(2B - 2(W_\infty - W_{-\infty}))\cosh B}{\pi \cosh(2x)\cosh(-2A\arctan e^{2x} + B))^2}. \tag{3.73}$$

The factor in the denumerator

$$\cosh(-2A\arctan e^{2x} + B) = \cosh(B(-2\frac{A}{B}\arctan e^{2x} + 1)), \tag{3.74}$$

gets bounded above in the interval $-1 \leq x \leq 1$ by

$$\cosh 2Bx, \tag{3.75}$$

as $-2\frac{A}{B} \to \frac{-4}{\pi}$ (linearize and get a bound). The integral

$$\pm \int_{-1}^{1} \frac{e^{-2W_{-\infty}}(2B - 2(W_\infty - W_{-\infty}))\cosh B}{\cosh 2x(\cosh(-2A\arctan e^{2x} + B))^2} dx, \tag{3.76}$$

is equal after the change of variables $Bx = u$ to

$$\pm \int_{-B}^{B} \frac{e^{2W_{-\infty}}(2B - 2(W_{\infty} - W_{-\infty}))\cosh B}{B\cosh\frac{2u}{B}\cosh^2 2u}du, \tag{3.77}$$

that clearly goes to $\pm$ infinity as B goes to infinity.

`Solving for` a `and the analysis of the solution.`

To find out the expression for a we follow the same procedure as in the polarized case. We find $\dot{a}$ and a' from equations (3.54, 3.55) and then integrate in time (t) and space (x). The fact that W and q are time independent gives

$$4W_{\pm}^2 + q_{\pm}^2 e^{-W} = W'^2 + \frac{q'^2}{4}e^{-4W}. \tag{3.78}$$

Equation (3.61) gives for the right hand side

$$W'^2 + \frac{q'^2}{4}e^{-4W} = \frac{A^2}{\cosh^2 2x}. \tag{3.79}$$

which make equations (3.54)(3.55) to have the same form as equations (3.36) but with W_0^2 replaced by $\frac{A^2}{2}$. This gives for a

$$a(x,t) = a(0) - (\frac{1}{2} + \frac{A^2}{4})\frac{1}{2}\ln\cosh 2x + (\frac{3}{2} + \frac{A^2}{4})t. \tag{3.80}$$

The change of variables and the convergence-collapse follows exactly as in the polarized case.

3.2.5 The gluing.

`CMC states in a torus neck.`

For simplicity we will work with the polarized solution in a torus neck, but the computations carry over to the non polarized case as well. We will find a CMC slice, $t = s(x)$, of the solution

$$g = e^{2a}(-dt^2 + dx^2) + Re^{2W}d\theta_1^2 + Re^{-2W}d\theta_2^2, \tag{3.81}$$

with

$$a(x,t) = a(0) - (\frac{1}{2} + \frac{W_0^2}{2})\frac{1}{2}\ln\cosh 2x + (\frac{3}{2} + \frac{W_0^2}{2})t, \tag{3.82}$$

$$R(x,t) = R_0(e^{2(t+x)} + e^{2(t-x)}), \tag{3.83}$$

$$W(x,t) = W_1 + W_0\arctan e^{2x}, \tag{3.84}$$

that we found above, with $k = -3$ and asymptotically $t = s(x) \sim t_0 \pm \frac{(1+W_0^2)}{(3+W_0^2)}x$, so guaranteeing almost flat cone initial states at the ends compatible with the flat cone states at the cusps where the solution is going to be glued. The way to find such a CMC slice is by finding appropriate barriers. The first task is to get a general expression of the mean curvature of a general section $t = s(x)$. We keep the discussion brief. Given a slice $t = s(x)$ introduce a coordinate system

$$\begin{aligned} x &= \bar{x} + s'(\bar{x})\bar{t}, &(3.85)\\ t &= s(\bar{x}) + \bar{t}, &(3.86)\\ \theta_1 &= \bar{\theta}_1, &(3.87)\\ \theta_2 &= \bar{\theta}_2. &(3.88)\end{aligned}$$

The metric is written $((\bar{x},\bar{t})$ directions)

$$g = -\bar{N}^2 d\bar{t}^2 + \bar{g}(d\bar{x} + \bar{X}d\bar{t})(d\bar{x} + \bar{X}d\bar{t}), \qquad (3.89)$$

where

$$\begin{aligned} \bar{g} &= e^{2a}((1+s''\bar{t})^2 - s'^2), &(3.90)\\ \bar{N}^2 &= e^{2a}(1-s'^2). &(3.91)\end{aligned}$$

From them k is calculated as

$$k = -\frac{1}{e^a\sqrt{1-s'^2}}(\partial_{\bar{t}}a + \frac{s''}{1-s'^2} + \frac{\partial_{\bar{t}}R}{R}), \qquad (3.92)$$

where

$$\begin{aligned} \partial_{\bar{t}}a &= \partial_t a + s'\partial_x a = -(\frac{1}{2} + \frac{W_0^2}{2})s'\tanh 2x + (\frac{3}{2} + \frac{W_0^2}{2}), &(3.93)\\ \frac{\partial_{\bar{t}}R}{R} &= \frac{\partial_x R s' + \partial_t R}{2} = 2s'\tanh 2x + 2, &(3.94)\end{aligned}$$

which gives

$$k(x) = -\frac{1}{\sqrt{1-s'^2}}e^f(\frac{s''}{1-s'^2} - (\frac{1}{2} + \frac{W_0^2}{2})s'\tanh 2x + (\frac{3}{2} + \frac{W_0^2}{2}) + 2s'\tanh 2x + 2), \qquad (3.95)$$

with

$$f = -(a(0) - (\frac{1}{2} + \frac{W_0^2}{2})\frac{1}{2}\ln\cosh 2x + (\frac{3}{2} + \frac{W_0^2}{2})s). \qquad (3.96)$$

`Observation.` Note that $k(s(x)+\tau) = e^{-(\frac{3}{2}+\frac{W_0^2}{2})\tau}k(s(x))$, and so that once having a CMC slice with the desired properties a CMC foliation is obtained by shifting it in the (t) time direction.

Now, to construct the barriers, note that for the subsection $t = s(x) = t_0 + (\frac{1+W_0^2}{3+W_0^2})x$ has asymptotically (i.e. as $x \to +\infty$) constant $k = k_0$. Now, a direct calculation shows that for the pair of sections (on the right end)

$$t = s(x) = t_0 + \frac{1+W_0^2}{3+W_0^2}x \pm \frac{1}{x}, \tag{3.97}$$

the asymptotics to leading terms is

$$-k \sim -k_0 e^{\mp(\frac{3}{2}+\frac{W_0^2}{2})\frac{1}{x}}(1 + O(\frac{1}{x})). \tag{3.98}$$

The last formula shows that $-k(s_+) < -k_0 < -k(s_-)$ asymptotically. The extension of those sections to the center of the neck can be carried as follows. Take two sections symmetric with respect to the t-axis, that (say on the right) are i) any smooth section (s_+) from 0 to 10 with $s'' > 0$ and $s_+(10) + \frac{1+W_0^2}{3+W_0^2}(x-10) - \ln(x-9)$ thereafter ii) any smooth section (s_-) from 0 to 10 with $s'' > 0$ and $s_-(10) + \frac{1+W_0^2}{3+W_0^2}(x-10) + \ln(x-9)$ thereafter. It is easy to see using the observation above that by shifting the (-) section up at some shift the sections have disjoint range of their mean curvature (between the points of intersection) and that at the point of intersection their tangents are $\frac{1+W_0^2}{3+W_0^2}$ up to $\sim 1/x$. Due to that, it is easy to continue the sections as was said above, starting from an x sligtly less than the x where they intersect, having disjoint range of their mean curvatures and asymptotically approaching to $s(x) = t_0 + \frac{1+W_0^2}{3+W_0^2}$.

Note that given a CMC slice as described above, the same slice is CMC with the same mean curvature if on the metric we replace R_0 by $R_0e^{-2\delta}$. Also note that on the (x', t') coordinates, for large x' the metric is written approximately

$$g = e^{2t'}(-dt'^2 + dx^2) + R_0 e^{2(\frac{\pi}{2}W_0+W_1)}(e^{2(t'+x')} + e^{\frac{2}{2+W_0^2}(t'-x')})d\theta_1^2 + \dots \tag{3.99}$$

$$\dots + R_0 e^{-2(\frac{\pi}{2}W_0+W_1)}(e^{2(t'+x')} + e^{\frac{2}{2+W_0^2}(t'-x')})d\theta_2^2. \tag{3.100}$$

So by changing R_0 by $R_0e^{-2\delta}$ and changing the x' coordinate by $x'' = x' - \delta$ the metric approximates to any given desired order to the flat cone initial state,

$$g = e^{2t'}(-dt'^2 + dx''^2) + R_0 e^{2(\frac{\pi}{2}W_0+W_1)} e^{2(t'+x'')} d\theta_1^2 + \dots \tag{3.101}$$

$$\dots + R_0 e^{-2(\frac{\pi}{2}W_0+W_1)} e^{2(t'+x'')} d\theta_2^2. \tag{3.102}$$

However note that the distance between standard parts on the cusps get increased by $\sim 2\delta$. δ therefore parameterizes the family of CMC initial states displaying a convergence-collapse picture.

`A traceless transverse tensor.`

The gluing of the metric described above (depending on δ) and the flat cone on H_1 and H_2 is going to take place on an interval of length one on one end of the neck and on an interval of length one on the cusp, so the metric on the rest of neck and on the rest of the hyperbolic manifold remains unchanged.

We are going to find a transverse traceless $\hat{K}$ by keeping the values on the neck and the hyperbolic manifold except on the gluing region where it is going to be defined.

So we want to solve for $\nabla_i \hat{K}^{ij} = 0$. The θ_i components are,

$$\nabla_i \hat{K}^{i\theta_i} = \Gamma^i_{ik} \hat{K}^{k\theta_i} + \Gamma^{\theta_i}_{ik} \hat{K}^{ik}, \tag{3.103}$$

which are automatically zero, because for a T^2 symmetric metric, $\Gamma^{\theta_i}_{\theta_j \theta_j} = 0, i, j = 1, 2$ and $\Gamma^{\theta_i}_{xx} = 0,\ i = 1, 2$. The x-component gives

$$\partial_x \hat{K}^{xx} + (2\Gamma^x_{xx} + \Gamma^{\theta_i}_{\theta_i x} - \Gamma^x_{\theta_2 \theta_2}) \hat{K}^{xx} + (\Gamma^x_{\theta_1 \theta_1} - \Gamma^x_{\theta_2 \theta_2}) \hat{K}^{\theta_1 \theta_1} = 0. \tag{3.104}$$

On the neck and right before the gluing $\hat{K} \sim 0$ (in C^3). We need to find a solution to equation (3.104), being, after an interval of length one, exactly zero. To do that we choose the glued metric in such a way that $\Gamma^x_{\theta_1 \theta_1} \neq \Gamma^x_{\theta_2 \theta_2}$ slightly on an interval of length one half inside the gluing interval. Then choose $\hat{K}^{\theta_1 \theta_1}$ such that the solution to (3.104) is exactly zero right after the gluing region. Such a thing can always be done due to the explicit expression of the solution to a first order ODE.

Estimates.

Once having (g, K) with $divK = 0$ and $tr_g K = k$ we invoke a theorem of Isenberg [I] (see the background section), which guarantees the existence of a solution to the Lichnerowicz equation in any manifold if $\hat{K} \neq 0$ and $k \neq 0$ as is our case. To estimate the solution to the Lichnerowicz equation

$$\Delta \phi = \frac{1}{8} R_g \phi - \frac{1}{8} |\hat{K}|^2_g \phi^{-7} + \frac{k^2}{12} \phi^5, \tag{3.105}$$

we use the maximum principle and the standard local elliptic regularity. The maximum principle tells that

$$R_g \phi(x_{max}) - |\hat{K}|^2 \phi(x_{max})^{-7} + \frac{k^2}{12} \phi(x_{max})^5 \leq 0, \tag{3.106}$$

Now note that $R_g = |\hat{K}|^2 - \frac{2}{3} k^2 + \epsilon(x)$ where $\epsilon(x)$ is nonzero only on the gluing region. Using that

$$|\hat{K}|^2 (\phi(x_{max}) - \phi^{-7}(x_{max})) + \frac{2}{3} k^2 (\phi(x_{max})^5 - \phi(x_{max})) + \epsilon(x_{max}) \phi(x_{max}) \leq 0. \tag{3.107}$$

From it we see that with a bound in $\|\hat{K}\|_{L^\infty}$, $\|\epsilon\|_{L^\infty}$ controls $\|\phi - 1\|_{L^\infty}$. Standard elliptic regularity gives the control in $C^{2,\alpha}$ at zero.

3.3 Stability of pure H ground states.

The proof is based on the following ideas. First we prove a lemma, which guarantees that where the local geometry doesn't degenerate, then the reduced energy minus its infimum together with the zero and first order Bel-Robinson energies control at zero the $W^{2,2}_{loc}$ (local) norm of $\hat{K}$ and the $W^{1,2}_{loc}$ (local) norm of $\hat{Ricc}$ and thus $\mathcal{E}$ and $H - H_{inf}$ (assuming Comment 2) control (up to diffeomorphism) the states around $(g_H, -g_H)$ in $W^{3,2}_0 \times W^{2,2}_0$. Then we recall a formula in [AM] showing that as long as the CMC flow remains (with any shift) inside a ball $B_{(g_H,-g_H)}(\epsilon)$ with ϵ sufficiently small in $W^{3,2}_0 \times W^{2,2}_0$, then the energy of the normalized states decreases at least exponentially in logarithmic time $t = -\ln -k$. Due to the existence of a uniform (around $k=-3$) CMC foliation for solutions with initial states in $B_{(g_H,-g_H)}(\epsilon)$ with ϵ sufficiently small, we deduce long time existence in logarithmic time. Finally we prove a proposition showing that if the volume stays bounded and the volume radius stays away from zero, then $\|\hat{K}\|_{L^2}$ and $\mathcal{E}$ control $H - H_{Hyperbolic}$ at zero. We use that and the formula for the derivative of the reduced hamiltonian to show that in the long logarithmic time the normalized CMC states converge (after a suitable choice of the shift vector) to $(g_H, -g_H)$ in $W^{3,2}_0 \times W^{2,2}_0$.

Lemma 4 *Let (g, K) be a sequence of states satisfying conditions* 1 *and* 2 *and $\mathcal{E} \to 0$. Let Ω be the sequence of ϵ-thick (ϵ fixed) domains according to Theorem 1. Then on Ω, $\|\hat{K}\|_{W^{2,2}_{loc}}$ and $\|\hat{Ricc}\|_{W^{1,2}_{loc}}$ are controlled at zero by $\mathcal{E}$ and $\|\hat{K}\|_{L^2}$.*

Proof: The proof is based on studying the equation (2.7). We observe that $div\hat{K} = 0$ and since $d^\nabla(\hat{K})_{ijk} = \nabla_i \hat{K}_{jm} - \nabla_j \hat{K}_{im}$

$$d^\nabla(\hat{K}) = -W^T = \epsilon_{ij}{}^{l} B_{lm}. \tag{3.108}$$

Also

$$d^{\nabla *}(-W^T) = -\epsilon_j{}^{li}\nabla_i B_{lm} - \epsilon_m{}^{li}\nabla_i B_{lj} = -2curl(B). \tag{3.109}$$

According to equation (2.37) we can express $curl(B)$ as

$$curl(B) = E(\nabla_T W) + \frac{3}{2}(E \times K) - \frac{1}{2}kE, \tag{3.110}$$

Equations (3.108), (3.109), (3.110) give the elliptic equation

$$2\nabla^*\nabla\hat{K} = -\mathcal{R}(\hat{K}) - 2(E(\nabla_T W) + \frac{3}{2}(E \times K) - \frac{1}{2}kE). \tag{3.111}$$

The coefficients of $\mathcal{R}$ involve only $Ricc$, and therefore are bounded in L^2. On the other hand E and $E(\nabla_T W)$ are controlled at zero in L^2 by $\mathcal{E}$. Let's restate the equation as the elliptic equation

$$\nabla^*\nabla\hat{K} = a\hat{K} + b, \tag{3.112}$$

where a is bounded in L^2 and b is controlled at zero by $\mathcal{E}$.

To prove the lemma, first we observe that

1. if $\hat{K}$ is in $L^\alpha(D)$ with $\alpha > 2$ then

$$\|a\hat{K}\|_{L^\beta(D)} \leq \|a\|_{L^2(D)}\|\hat{K}\|_{L^\alpha(D)}, \tag{3.113}$$

with $\beta = \frac{2\alpha}{(2+\alpha)}$.

2. if $\hat{K}$ is in $L^\alpha(D)$ then $\|\hat{K}\|_{W^{2,\beta}(D')} \leq C(\|a\hat{K}\|_{L^\beta(D)} + \|\hat{K}\|_{L^\beta(D)} + \|b\|_{L^\beta(D)})$ for $\beta = \frac{2\alpha}{(2+\alpha)}$ and $D' \subset\subset D$.

3. Sobolev embeddings give:

   (a) if $2\beta < 3$ then $\|\hat{K}\|_{L^{\frac{3\beta}{3-2\beta}}(D)} \leq C\|\hat{K}\|_{W^{2,\beta}(D)}$, $q = \frac{3\beta}{3-2\beta}$.

   (b) if $3+\beta > 2\beta > 3$ then $\|\hat{K}\|_{C^{0,2-\frac{3}{\beta}}(D)} \leq C\|\hat{K}\|_{W^{2,\beta}(D)}$.

To prove that $\|\hat{K}\|_{W^{2,2}_{loc}}$ is controlled at zero by $\mathcal{E}$ and $\|\hat{K}\|_{L^2}$, we note first that $\|\hat{K}\|_{L^4_{loc}}$ and $\|\hat{K}\|_{W^{1,2}_{loc}}$ are controlled at zero by $\mathcal{E}$ and $\|\hat{K}\|_{L^2}$ by Lemma 1. Now iterate the control following observations 1, 2, 3 in that order, starting with $\alpha = 4$. With this we get $\beta = \frac{4}{3}$ and $q = 12$. In the second iteration of control $\alpha = q = 12$, $\beta = \frac{12}{7}$ and $2\beta = \frac{24}{7} > 3$. Then $\|\hat{K}\|_{C^0_{loc}}$ is controlled at zero too. After we guarantee that $\hat{K}$ is controlled in $C^{0,\frac{1}{4}}_{loc}$, elliptic regularity on equation (3.112) gives the desired control on $\|\hat{K}\|_{W^{2,2}_{loc}}$.

To prove that $\|\hat{Ricc}\|_{W^{1,2}_{loc}}$ is controlled at zero by $\|\hat{K}\|_{L^2_{loc}}$ and $\mathcal{E}$ observe that by the proved control on $\|\hat{K}\|_{W^{2,2}_{loc}}$, it is enough by taking the covariant derivative of the defining equation of E

$$\hat{Ricc} - \hat{K} \circ \hat{K} + \frac{1}{3}k\hat{K} + \frac{1}{3}|\hat{K}|^2 g = E, \tag{3.114}$$

to prove that E is controlled at zero in $W^{1,2}_{loc}$. For this observe that the system (d^∇, div) is uniformly elliptic and that

$$d^\nabla(E)_{ijk} = (curl(E)_{il} + \frac{1}{2}div(E)_m \epsilon^m{}_{il})\epsilon^l{}_{jk}, \tag{3.115}$$

$$div(E)_m = (K \wedge B)_m. \tag{3.116}$$

On the other hand

$$curl(E) = -B(\nabla_T W) - \frac{3}{2}(B \times K) + \frac{1}{2}kB. \tag{3.117}$$

Using the C^0_{loc} control of K, standard elliptic regularity gives the result. □

Given the state (g, K) (with any k) in a CMC foliation, denote the normalized states as $(\tilde{g}, \tilde{K}) = (\frac{k^2}{9}g, \frac{-k}{3}K)$.

PROPOSITION 4 *(Inequality for the evolution of the normalized energy,* [AM] **Lemma 5.6***). For ϵ sufficiently small, while the solution remains on $B_{(g_H,-g_H)}(\epsilon)$ in $W_0^{3,2} \times W_0^{2,2}$ one has*

$$\partial_t \tilde{\mathcal{E}} \leq -(2 - C\tilde{\mathcal{E}}^{\frac{1}{2}})\tilde{\mathcal{E}}, \tag{3.118}$$

where $t = -\ln -k$ is the logarithmic time.

PROPOSITION 5 *For normalized states with $\nu(x) > \delta$ and bounded volume, $\|\hat{K}\|_{L^2}$ and Q_0 control $H - H_{Hyperbolic}$ at zero.*

Proof: It is enough to prove that any such sequence with $\|\hat{K}\|_{L^2} \to 0$ and $Q_0 \to 0$ has a subsequence converging in $W_0^{2,2}$ (up to diffeomorphisms) to g_H. By Lemma 1, $\|\hat{K}\|_{L^2}$ and Q_0 control $\|\hat{K}\|_{L^4}$ at zero. Using that, the defining equation for E (equation (2.31)) and the energy constraint give $\|\hat{Ricc}\|_{L^2} \to 0$ and $\|R+6\|_{L^2} \to 0$. As the volume is bounded and the volume radius bounded from below, there is a subsequence converging after a suitable choice of diffeomorphism to g_H in $W_0^{2,2}$.□

Proof of the stability of pure H ground states.

Proof: Let S be a local transversal subsection through $(g_H, -g_H)$ to the orbits of the action of the diffeomorphism group in $W_0^{3,2} \times W_0^{2,2}$. Assume ϵ is small enough such that S intersects $\partial(B_{(g_H,-g_H)}(\epsilon))$ transversely and Proposition 4 holds on $B_{(g_H,-g_H)}(\epsilon)$. Assuming Comment 2, Theorem 1 and Lemma 3 imply that for ϵ' sufficiently small the set of normalized states for which $H - H_{Hyperbolic} \leq \epsilon'$ and $\mathcal{E} \leq \epsilon'$ intersects $B_{g_H,-g_H)}(\epsilon) \cap S$ inside $B_{(g_H,-g_H)}(\frac{\epsilon}{2}) \cap S$. This fact together with the fact that H is monotonically decreasing makes it possible to find (differentiably along the flow) spatial diffeomorphisms (or a trivialization) that keep (while the flow is defined) the normalized states of a solution that starts at $B_{(g_H,-g_H)}(\epsilon)$ inside $B_{(g_H,-g_H)}(\frac{\epsilon}{2}) \cap S$. In fact, to achieve that trivialization, take any trivialization and project that flow along the orbits of the diffeomorphism group into $B_{(g_H,-g_H)}(\frac{\epsilon}{2}) \cap S$ (see Figure 3.3). This proves long time existence in logarithmic time for the solutions with initial states inside a ball of radius small enough around $(g_h, -g_H)$ in $W_0^{3,2} \times W_0^{2,2}$.

It is clear too from equation (3.118), that $\tilde{\mathcal{E}} \to 0$ as the logarithmic time diverges. It remains to prove that $H - H_{Hyperbolic} \to 0$. By Proposition 2 if $H - H_{inf} \geq \gamma > 0$ (observe that H is monotonically decreasing) it must be $\|\tilde{\hat{K}}\|_{L^2} \geq \delta > 0$ necessarily after some time.

For ϵ small enough $\|\tilde{N} - \frac{1}{3}\|_{L^\infty} < \frac{1}{6}$, thus equation 2.24

$$\frac{dH}{dt} = -3^4 \int_M \tilde{N} |\tilde{\hat{K}}|^2 dv_{\tilde{g}}, \tag{3.119}$$

shows that $H - H_{\inf}$ must go below zero at some time, giving a contradiction.

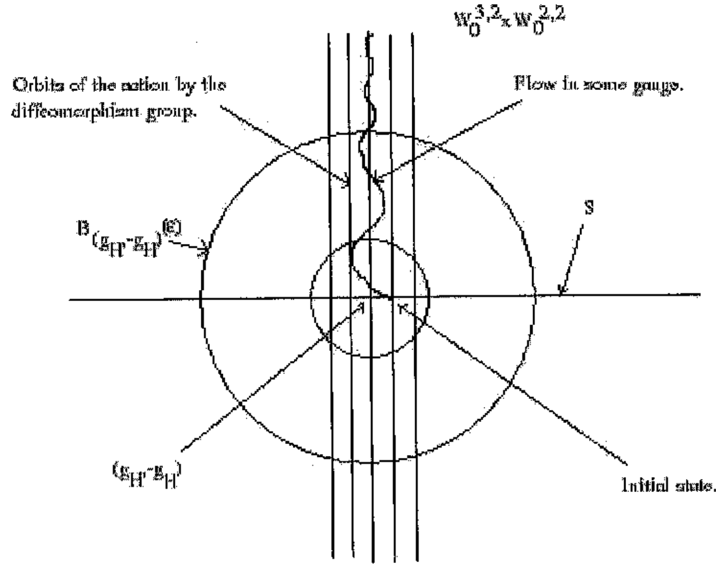

Figure 3.3: .

Schematic view of the construction in Theorem 2. The set of vertical lines represents the set $H - H_{inf} \leq \epsilon'$ and $\mathcal{E} \leq \epsilon'$.

3.4 Stability of the flat cone.

As explained in the background section, H has a strict local minimum at $(g_H, -g_H)$ on a local slice of normalized states S through $(g_H, -g_H)$ and transversal to the orbits of the diffeomorphism group. Take then a ball $B_{(g_H,-g_H)}(\epsilon)$ in $W_0^{3,2} \times W_0^{2,2}$ with ϵ sufficiently small where: i) H takes the value $H_{Hyperbolic}$ only at $(g_H, -g_H)$ on $B_{(g_H,-g_H)} \cap S$, ii) $H - H_{Hyperbolic} \geq \delta > 0$ on $\partial(B_{(g_H,-g_H)} \cap S)$, iii) Comment 4 holds. Finally, set $U = B_{(g_H,-g_H)}(\epsilon) \cap \{(g,K) \in W_0^{3,2} \times W_0^{2,2} / H - H_{Hyperbolic} < \frac{\delta}{2}, k = -3\} \cap S$.

Theorem 8 (Stability of the flat cone.) (under a suitable choice of the shift vector) U is stable under the normalized Einstein flow and every normalized trajectory converges in $W_0^{3,2} \times W_0^{2,2}$ to the flat cone state $(g_H, -g_H)$.

Proof: Suppose the initial state (for $k = -3$) is at U. As in the proof of Theorem 2, take any trivialization (starting with $\phi_{-3} = Id : M \to M$) and project the resulting flow through the orbits of the action of the diffeomorphism group into $B_{(g_H,-g_H)}(\epsilon) \cap S$. Observe that because H is monotonically decreasing the projection can never escape U (at the boundary of U (which doesn't intersect $\partial(B_{(g_H,-g_H)} \cap S)$) it is $H - H_{Hyperbolic} = \frac{\delta}{2}$), thus the projection extends to the full orbit, and realizes a trivialization (or a choice of spatial gauge) for which U is invariant under the flow. The proof finishes exactly along the same lines as in Theorem 2. □

Comment 5. It is proved in [AM] (**Theorem 6.7**) that these space times are future geodesically complete, therefore coincide with the maximally globally hyperbolic solutions (to the future).

Bibliography

[A1] Anderson, Michael T. On long-time evolution in general relativity and geometrization of 3-manifolds. *Comm. Math. Phys. 222 (2001), no. 3, 533–567.*

[A2] Anderson, Michael T. Remarks on Perelman's Papers. *Preprint.*

[A3] Anderson, Michael T. Scalar curvature and the existence of geometric structures on 3-manifolds. I. *J. Reine Angew. Math. 553 (2002), 125–182.*

[A4] Anderson, Michael T. Extrema of curvature functionals on the space of metrics on 3-manifolds. *Calc. Var. Partial Differential Equations 5 (1997), no. 3, 199–269.*

[A5] Anderson, Michael T. Scalar curvature and geometrization conjectures for 3-manifolds. *Comparison geometry (Berkeley, CA, 1993–94), 49–82, Math. Sci. Res. Inst. Publ., 30, Cambridge Univ. Press, Cambridge, 1997.*

[A6] Anderson, M. T. Scalar curvature, metric degenerations, and the static vacuum Einstein equations on 3-manifolds. II. *Geom. Funct. Anal. 11 (2001), no. 2, 273–381.*

[AM] Andersson, Lars; Moncrief, Vincent. Future complete vacuum space times. *The Einstein equations and the large scale behavior of gravitational fields, 299–330, Birkhuser, Basel, 2004.*

[AM1] Andersson, Lars; Moncrief, Vincent. Elliptic-hyperbolic systems and the Einstein equations. *Ann. Henri Poincar 4 (2003), no. 1, 1–34.*

[Ba1] Bartnik, Robert. The existence of maximal surfaces in asymptotically flat spacetimes. *Asymptotic behavior of mass and spacetime geometry (Corvallis, Ore., 1983), 57–60, Lecture Notes in Phys., 202, Springer, Berlin, 1984.*

[Ba2] Bartnik, Robert. Remarks on cosmological space times and constant mean curvature surfaces. *Comm. Math. Phys. 117 (1988), no. 4, 615–624.*

[CIP] Chruściel, Piotr T; Isenberg, James; Pollack, Daniel Initial data engineering. *Comm. Math. Phys. 257 (2005), no. 1, 29–42.*

[CB] Choquet-Bruhat, Yvonne Future complete U(1) symmetric Einsteinian space times, the unpolarized case. *The Einstein equations and the large scale behavior of gravitational fields, 251–298, Birkhuser, Basel, 2004.*

[CBY] Choquet-Bruhat, Yvonne; York, James W., Jr. The Cauchy problem. it General relativity and gravitation, Vol. 1, pp. 99–172, Plenum, New York-London, 1980.

[CK] Christodoulou, Demetrios; Klainerman, Sergiu The global nonlinear stability of the Minkowski space. *Princeton Mathematical Series, 41. Princeton University Press, Princeton, NJ, 1993.*

[FM1] Fischer, Arthur E.; Moncrief, Vincent The reduced hamiltonian of general relativity and the σ-constant of conformal geometry. *Mathematical and quantum aspects of relativity and cosmology (Pythagoreon, 1998), 70–101, Lecture Notes in Phys., 537, Springer, Berlin, 2000.*

[FM2] Fischer, Arthur E. Moncrief, Vincent The reduced Einstein equations and the conformal volume collapse of 3-manifolds. *Classical Quantum Gravity 18 (2001), no. 21, 4493–4515.*

[GT] Gilbarg, David; Trudinger, Neil S. Elliptic partial differential equations of second order. *Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.*

[Ge] Gerhardt, Claus H-surfaces in Lorentzian manifolds. *Comm. Math. Phys. 89 (1983), no. 4, 523–553.*

[H] Hamilton, Richard SNon-singular solutions of the Ricci flow on three-manifolds. *Comm. Anal. Geom. 7 (1999), no.4, 695–729.*

[I] Isenberg, James, Constant mean curvature solutions of the Einstein constraint equations on closed manifolds. *Classical Quantum Gravity 12 (1995), no. 9, 2249–2274.*

[MT] Marsden, Jerrold E.; Tipler, Frank J. Maximal hypersurfaces and foliations of constant mean curvature in general relativity. *Phys. Rep. 66 (1980), no. 3, 109–139.*

[Re] Rendall, Alan D. Constant mean curvature foliations in cosmological space-times. *Journes Relativistes 96, Part II (Ascona, 1996). Helv. Phys. Acta 69 (1996), no. 4, 490–500.*

[R1] Ringström, Hans On a wave map equation arising in general relativity. *Comm. Pure Appl. Math. 57 (2004), no. 5, 657–703.*

[R2] Ringström, Hans. The future asymptotics of Bianchi VIII vacuum solutions. *Classical Quantum Gravity 18 (2001), no. 18, 3791–3823.*

[R3] Ringström, Hans Future asymptotic expansions of Bianchi VIII vacuum metrics. *Classical Quantum Gravity 20 (2003), no. 11, 1943–1989.*

[R4] Ringström, Hans On curvature decay in expanding cosmological models. Preprint.