Relative Parametric Gromov-Witten Invariants and Symplectomorphisms

A Dissertation, Presented

by

Olguţa Buşe

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at Stony Brook

August 2002

State University of New York

at Stony Brook

The Graduate School

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Abstract of the Dissertation,

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Doctor of Philosophy

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Mathematics

State University of New York

at Stony Brook

2002

We study the symplectomorphism groups $G_\lambda = \mathrm{Symp}_0(M, \omega_\lambda)$ of an arbitrary closed manifold M equipped with a 1-parameter family of symplectic forms ω_λ with variable cohomology class. We show that the existence of nontrivial elements in $\pi_*(\mathcal{A}, \mathcal{A}')$, where $(\mathcal{A}, \mathcal{A}')$ is a suitable pair of spaces of almost complex structures, implies the existence of nontrivial elements in $\pi_{*-i} G_\lambda$, for $i = 1$ or 2. Suitable parametric Gromov Witten invariants detect nontrivial elements in $\pi_*(\mathcal{A}, \mathcal{A}')$. By looking at certain resolutions of quotient singularities we investigate the situation

$(M, \omega_\lambda) = (S^2 \times S^2 \times X, \sigma_F \oplus \lambda\sigma_B \oplus \omega_{st})$, with (X, ω_{st}) an arbitrary symplectic manifold. We find nontrivial elements in higher homotopy groups of G^X_λ, for various values of λ. In particular we show that the fragile elements w_ℓ previously found by Abreu-McDuff in $\pi_{4\ell}(G^{\mathrm{pt}}_{\ell+1})$ do not disappear when we consider them in $S^2 \times S^2 \times X$.

To Rodrigo and to my parents

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Acknowledgments

Very special thanks to my advisor Dusa McDuff. Her boundless energy and inspiring ideas, as well as her patience and enthusiasm were key in the making of this thesis. Further I would like to thank her for the generous guidance on how to give a talk and write a paper. Her careful editing of the thesis was crucial in bringing it up to the final form.

To Dennis Sullivan, many thanks for his inspiring spring classes in Stony Brook, for the opportunity of attending his CUNY seminars, and for the suggestions he gave me on many occasions.

I would also like to thank Siddhartha Gadgil for the several discussions he had with me that helped me clear up my ideas on the topological aspects of my thesis. To Sorin Popescu, thanks for talking with me about the section in Algebraic Geometry.

A great debt of gratitude I owe to my parents Luca and Cecilia Buşe. They invested countless efforts and sacrifices in supporting my education. They always had faith that in its worth. My warmest thanks to them, to my brother Lucian, and to Gina and Ştefan Buşe, for being there for me, at the other end of the phone, and making sure that I set the mark high.

Also, to my friends in the Mathematics or Physics Department who, at various times in Grad School, discussed with the thorny math and non-math problems I had to solve. Among them I would like to mention Rares Rasdeaconu, Ioana Suvaina, Ionut Chiose, Andrew McIntyre, Joe Coffey, Owen Dearricott, Dan Moraru, Lee-Peng Teo, Irina Mocioiu and Radu Roiban, and many many others, since this list cannot be either ordered or complete. Many thanks to all of them.

Last but not least, my deepest thanks to my husband Rodrigo Pérez whose love and continuous support makes my life beautiful. I want to thank him for believing in me even when I did not believe myself.

Chapter 1

Introduction

1.1 Aim

Consider (M^{2n}, ω) a $2n$ dimensional compact symplectic manifold. A symplectomorphism is a diffeomorphism that preserves the symplectic form. A basic invariant which distinguishes among different symplectic structures on M is the group of symplectomorphisms, $\mathrm{Symp}(M, \omega)$. This is an infinite dimensional group endowed with a natural C^∞ topology.

Two natural questions arise in relation with $\mathrm{Symp}(M, \omega)$ namely

(1) What can be said about the topological type of $\mathrm{Symp}(M, \omega)$?

(2) How does the topological type change as ω varies?

Any symplectic manifold admits a large set of almost complex structures. Among those, we are interested in those that are *tamed* by ω in the following sense:

Definition 1.1. *We say that an almost complex structure J is tamed by the symplectic form ω if $\omega_x(v, Jv) > 0$ for any vector field v on M, nonzero at*

$x \in M$.

We denote by $\mathcal{A}_{[\omega]}$ the space of almost complex structures that are tamed by some symplectic form that is isotopic [1] *to ω.*

Moser showed that the identity component $\mathrm{Diff}_0(M)$ of the group of diffeomorphisms acts transitively on the space $\mathcal{S}_{[\omega]}$ of symplectic forms isotopic to ω. As we will see in 4.1, this implies that the following fibration, first introduced by Kronheimer [9] and used in McDuff [13], exists:

$$\mathrm{Symp}_0(M,\omega) \longrightarrow \mathrm{Diff}_0(M) \longrightarrow \mathcal{A}_{[\omega]}. \tag{1.1}$$

where $\mathrm{Symp}_0(M,\omega) = \mathrm{Symp}(M,\omega) \cap \mathrm{Diff}_0(M)$.

Our strategy will be to define suitable pairs $(\mathcal{A},\mathcal{A}')$ of spaces of almost complex structures, such that information on nontrivial homotopy groups in $(\mathcal{A},\mathcal{A}')$ extends to information on $\mathrm{Symp}_0(M,\omega)$. We develop a version of relative GW invariants in family which detects such nontrivial elements in $\pi_*(\mathcal{A},\mathcal{A}')$.

Research regarding the symplectomorphisms groups has been by various authors [Abreu [2], Le-Ono [11], McDuff [12], Seidel [18]]. Their methods as well as ours use modern tools of symplectic geometry such as J-holomorphic curves. In the next section we will outline some relevant definitions and some earlier results that are relevant for our work.

[1] We say that two symplectic forms are isotopic if they can by joined by a path of cohomologous symplectic forms

1.2 Background and earlier results

Definition 1.2. *Consider a closed manifold (M, J) and a Riemann surface (Σ, j), both endowed with almost complex structures J and j respectively. We say that a smooth map $f : \Sigma \to M$ is J-holomorphic if its derivative $df_x : T_x\Sigma \to T_{f(x)}M$ satisfies*

$$df_x \circ j_x = J_{f(x)} \circ df_x$$

Roughly speaking, Gromov-Witten invariants are symplectic invariants of M that count J-holomorphic curves with given topological constraints.

We investigate the above mentioned questions by defining relative parametric GW invariants, which are sensitive to the topology of appropriate spaces of almost complex structures. Our method has been inspired by P. Kronheimer's work [9], as well as by the work of D. McDuff [12].

In [9], for situations with dim M=4, Kronheimer uses parametric Seiberg-Witten invariants, to exhibit some nontrivial families of symplectic structures. Roughly speaking, he discusses certain analytic families X_u, part of whose specific properties will be described in chapter 3, that arise as the fibers of a map $p : X \to U$ with U an open set in $\mathbb{C}^{m-1}$ and such that all fibers X_u are smooth and diffeomorphic, except for X_0 which has a special type of quotient singularity at a point x_0. This family admits an embedding in $\mathbb{C}P^N \times U$ through which the fibers X_u inherit symplectic forms from $\mathbb{C}P^N$. His result is the following:

Theorem 1.3. *(Kronheimer)*

Suppose p is as above, let $u \in U \setminus \{0\}$, and consider the fiber X_u as a symplectic manifold as above. Let Diff *and* Symp *be its group of diffeomorphisms and symplectomorphisms respectively. Then* $\pi_{2m-3}(\mathrm{Diff}/\mathrm{Symp})$ *is nonzero, provided that $b^+(X_u)$ is greater than $2m-1$.*

Other results regarding homotopy type of the symplectomorphism groups appear in the situations when M is a ruled surface or a product of $\mathbb{C}P^n$'s. We first set the notation

$$G_\lambda := \mathrm{Symp}_0(M, \omega_\lambda)$$

To serve our purposes, we will only exhibit those results that were found in [7], [3], [13] for the situation when $(M, \omega_\lambda) = (S^2 \times S^2, \sigma_F \oplus \lambda\sigma_B)$, where σ_F, σ_B are forms on the fiber $S^2 \times pt$ and base $pt \times S^2$ respectively, of total area 1, and $\lambda \geq 1$. For this particular manifold we will adopt the notation

$$G_\lambda^{pt} = \mathrm{Symp}_0(S^2 \times S^2, \sigma_F \oplus \lambda\sigma_B).$$

The first one to look at this groups was Gromov [7], who found that, in the case $\lambda = 1$, G_1^{pt} deformation retracts to the Lie group $SO(3) \times SO(3)$. He also pointed out a new element of infinite order that appear in $\pi_1 G_1$ when λ increases past 1.

Later Abreu-McDuff in [3] and [13] found natural maps $G_\lambda^{\mathrm{pt}} \to G_{\lambda+\epsilon}^{\mathrm{pt}}$, well defined up to homotopy, and proved:

Theorem 1.4. *(Abreu-McDuff)*

(i) The homotopy type of G_λ^{pt} *is constant on all the intervals* $(\ell-1,\ell]$ *with* $\ell \geq 2$ *a natural number. Moreover, as* λ *passes an integer* $\ell, \ell \geq 2$ *the groups* $\pi_i(G_\lambda^{\mathrm{pt}}), i \leq 4\ell-5$, *do not change.*

(ii) There is an element $w_\ell \in \pi_{4\ell-4}(G_\lambda^{\mathrm{pt}}) \times \mathbb{Q}$ *when* $\ell - 1 < \lambda \leq \ell$ *that vanishes for* $\lambda > \ell$.

As it will be explained later, we will extend their results and techniques to obtain information on the symplectomorphism groups of $S^2 \times S^2 \times X$.

1.3 Outline of the main results

In chapter **2** we will define the invariants as follows:

Consider a smooth family of symplectic forms $(\omega_\lambda)_{\lambda \in I}$, where the parameter λ varies in the interval I in $\mathbb{R}$ in such a manner that the cohomology classes $[\omega_\lambda]$ may also vary along a line L inside $H^2(M,\mathbb{R})$. For convenience we denote by $\mathcal{A}_\lambda := \mathcal{A}_{[\omega_\lambda]}$. Consider $D \in H_2(M,\mathbb{Z})$ and let $\mathcal{A}^c_{\lambda,\mathrm{D}} \subset \mathcal{A}_\lambda$ consist of those almost complex structures J which do not admit $J-$holomorphic stable maps in the class D. Further define

$$\mathcal{A}_I = \bigcup_{\lambda \in I} \mathcal{A}_\lambda,$$

and similarly let $\mathcal{A}^c_{I,D}$ be its subset given by

$$\mathcal{A}^c_{I,D} = \bigcup_{\lambda \in I} \mathcal{A}^c_{\lambda,D}$$

By Prop 4.2 below, (see [13]), $\mathcal{A}_I$ is homotopy equivalent with $\bigcup_{\lambda\in I}\mathcal{S}_{[\omega_\lambda]}$ and hence is connected. We will assume that there is a special almost complex structure $* = J_{basepoint}$ that belongs to all the spaces $\mathcal{A}^c_{\lambda,D}$. Consider a family of almost complex structures $(J_B, J_{\partial B}, *)$ that represent an element in $\pi_*(\mathcal{A}_I, \mathcal{A}^c_{I,D}, *)$. We will define a homomorphism

$$PGW^{M,(J_B,\partial J_B)}_{D,0,k} : \bigoplus_{i=1}^{k} H^{a_i}(M,\mathbb{Q})^k \to \mathbb{Q} \tag{1.2}$$

by counting J_b-holomorphic stable maps in class D, for all $b \in B$. This is well defined because the class D is never represented as a J_b-holomorphic stable maps if $b \in \partial B$. We have the following

Theorem 1.5. *(On relative parametric Gromov-Witten invariants)*

i) *The invariants $PGW^{M,(J_B,\partial J_B)}_{D,0,k}$ are symplectic deformation invariants and depend only on the relative homotopy class of the triple $(J_B, \partial J_B, *)$.*

ii) *For a fixed choice of k, D and α_i the map*
$\Theta_{0,k,\alpha_1,\ldots,\alpha_k} : \pi_*(\mathcal{A}_I, \mathcal{A}^c_{I,D}, *) \to \mathbb{Q}$*, given by*

$$\Theta_{k,\alpha_1,\ldots,\alpha_k}([(J_B, \partial J_B)] = PGW^{M,(J_B,\partial J_B)}_{D,0,k}(\alpha_1,\ldots,\alpha_k)$$

is a homomorphism.

One difficulty in counting Gromov-Witten invariants is to make sure that among the J-holomorphic curves that we can see we only count the ones that do not disappear under small perturbations of the almost complex structure. Those are called regular curves. Roughly speaking, this is equivalent to the fact that a certain linearized Cauchy-Riemann operator is surjective. We adapt these considerations to the family setting and call *parametric regular* those J_b-holomorphic curves that do not disappear under small perturbations of the whole family $(J_b)_{b\in B}$ of almost complex structures. In chapter 2 we introduce the following criterion of parametric regularity, which applies to families that appear in a fibered setting (in a sense that will be explained later):

Theorem 1.6. *Let $(J_z,\omega_z)_{z\in\mathbb{C}^m}$ be a family of pairs of almost complex structures and symplectic forms arising as the restrictions to the fibers $\pi^{-1}(z)\cong M$ of the pair $(\widetilde{J},\widetilde{\omega})$ of a symplectic fibration $(\widetilde{M},\widetilde{J},\widetilde{\omega})$*

$$\begin{tikzcd} M \arrow[r, "i"] & \widetilde{M} \arrow[d, "\pi"] \\ & \mathbb{C}^m \end{tikzcd} \tag{1.3}$$

Suppose that $f:\Sigma\longrightarrow M$ is a J_0 holomorphic map and consider the composite map

$$\widetilde{f}=i\circ f, \widetilde{f}:\Sigma\longrightarrow M\times 0\subset\widetilde{M}$$

which is $\widetilde{J}$-holomorphic. If $\widetilde{f}$ is regular then f is (J_z) parametric regular. Moreover, if $\Sigma=S^2$ then the reverse statement holds.

The proof of this theorem is contained in Appendix A.

In chapter **3** we will exhibit some examples of nontrivial PGW. There we consider the case when (M,ω) is $S^2\times S^2\times X$, where X is an arbitrary symplectic manifold, $\omega=\omega_\lambda\oplus\omega_{st}$, with ω_λ as before given by $\omega_\lambda=\sigma_F\oplus\lambda\sigma_B$ and with ω_{st} an arbitrary symplectic form on X. The families $(J_B,\partial J_B)$ of almost complex structures are provided for $S^2\times S^2$ in [9] and then further investigated in [3]. One has to look at a quotient singularity, $\mathbb{C}^2/C_{2\ell}$, where

$C_{2\ell}$ is the cyclic group of order 2ℓ acting diagonally by scalars on $\mathbb{C}^2$. The deformation space for the canonical resolution of this singularity provides a $4\ell-2$ family $(J_{B_\ell}, \partial J_{B_\ell}) \subset (\mathcal{A}_{[\ell,\ell+\epsilon]}, \mathcal{A}_\ell)$ for which suitable PGW are nontrivial.

The link between these examples and the corresponding groups of symplectomorphisms will be explained in chapter **4**. It will be there where we explain the extent to which the known homotopy properties (see [3]) of
$G_\lambda^{pt} = \mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda)$ are reflected in the higher homotopy groups of

$$G_\lambda^X := \mathrm{Symp}_0(S^2 \times S^2 \times X, \omega_\lambda \oplus \omega_{st}).$$

To be able to give any answers related to the two questions posed in the beginning, one has to establish first a more precise language in which they make sense. One of the difficulties is that when we deal with a general manifold M we no longer have either direct maps $G_\lambda \to G_{\lambda+\epsilon}$, or maps defined up to homotopy, as in the $M = S^2 \times S^2$ case.

To get around the fact that there is no map $G_\lambda \to G_{\lambda+\epsilon}$ we show that for any compact $K \subset G_\lambda$, the inclusion $0 \times K \subset G_\lambda$ extends to a map h that fits into the following commuting diagram:

$$\begin{tikzcd}
h : [-\epsilon, \epsilon] \times K \arrow[r] \arrow[d, "pr_1"] & \mathcal{G} := \bigcup (G_\lambda \times \lambda) \subset \mathrm{Diff}_0(M) \times \mathbb{R} \arrow[d, "pr_2"] \\
{[-\epsilon, \epsilon]} \arrow[r, "incl"] & (-\infty, \infty)
\end{tikzcd} \tag{1.4}$$

Moreover, for any two such maps h and h' which coincide on $0 \times K$, there is, for ϵ' small enough, a homotopy between them $H : [0,1] \times [-\epsilon', \epsilon'] \times K \to \mathcal{G}$ which also preserves the fibers of the natural projections. We therefore see that, for any cycle ρ in G_λ there are *extensions* ρ_ϵ in $G_{\lambda+\epsilon}$ which, for ϵ sufficiently small, are unique up to homotopy. Hence they give well defined elements in $\pi_* G_{\lambda+\epsilon}$.

It will therefore make sense to ask what will become of an element $\rho \in \pi_* G_\lambda$ inside $\pi_* G_{\lambda+\epsilon}$, for small ϵ. In this language we say that an element $\theta_\ell \in \pi_* G_\ell$ is *fragile* if any extension $\theta_{\ell+\epsilon}$ is null-homotopic in $\pi_*(G_{\ell+\epsilon})$ for $\epsilon > 0$. Also, we say that a family $\eta_{\ell+\epsilon} \in \pi_* G_{\ell+\epsilon}, 0 < \epsilon$ is *new* if there is no $\eta_\ell \in \pi_* G_\ell$ whose extension is $\eta_{\ell+\epsilon}$. We consider the space $\mathcal{A}_{\ell^+}$ roughly given by

$\mathcal{A}_{\ell^+} := (\bigcap_{0<\epsilon<\epsilon_0} \mathcal{A}_{\ell+\epsilon}) \bigcup \mathcal{A}_\ell$ (for the precise definition see (2.3)).

We say that an element $\alpha \in \pi_*(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell)$ is *persistent* if it has nonzero image under the map $\pi_*(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell) \to \pi_*(\mathcal{A}_{[\ell,\ell+\epsilon]}, \mathcal{A}_\ell)$. The content of our main theorem is the following:

Theorem 1.7. *(On Symplectomorphism groups) Assume that we have a persistent element $0 \neq \beta_\ell \in \pi_k(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell, *)$ Then we can construct an element $\theta_\ell \in \pi_{k-2} G_\ell$ such that either:*

A) *The element $\theta_\ell \in \pi_{k-2} G_\ell$ is a non-zero* **fragile** *element.*

or

B) *$\theta_\ell = 0$ and then there is an $\epsilon_\ell > 0$ such that we can construct a family of* **new** *elements $0 \neq \eta_{\ell+\epsilon} \in \pi_{k-1} G_{\ell+\epsilon}, 0 < \epsilon < \epsilon_\ell$.*

Any fragile element is null homotopic when viewed inside $\mathrm{Diff}_0(M)$. We should point out that our methods do not allow us to decide in general whether the image of $\eta_{\ell+\epsilon}$ in $\pi_{k-1}\mathrm{Diff}_0(M)$ is zero or not.

We show that the hypothesis of the theorem is verified when

$M = S^2 \times S^2 \times X$. We consider $D = A - \ell F$. Since $[\sigma_F \oplus \ell\sigma_B \oplus \omega_{st}](A - \ell F) = 0$ we get that $\mathcal{A}_\ell \subset \mathcal{A}^c_{[\ell,\ell+\epsilon],D}$. In this situation the $4\ell - 2$ dimensional elements

$(B_\ell, \partial B_\ell)$ obtained in section **3** are detected as nontrivial in $\pi_{4\ell-2}(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell)$ and are persistent. In fact in general PGW invariants detect persistent elements. By varying the value of the integer ℓ we obtain infinitely many values of λ for which higher order homotopy groups of G_λ^X are nontrivial and also make a more detailed discussion regarding the stability of the elements w_ℓ provided by theorem 1.4 inside $G_\lambda^X := \mathrm{Symp}_0(S^2 \times S^2 \times X, \omega_\lambda \oplus \omega_{st})$. This is the content of the following:

Corollary 1.8. *For any natural number $\ell \geq 1$, exactly one of the statements below holds.*

A) *We can construct a non-zero* **fragile** *element $w_\ell^X \in \pi_{4\ell-4} G_\ell^X$, which can be identified with $w_\ell \times id$.*

B) *There exists an $\epsilon_\ell > 0$ for which we can construct a family of* **new** *elements $0 \neq \eta_{\ell+\epsilon}^X \in \pi_{4\ell-3} G_{\ell+\epsilon}^X, 0 < \epsilon < \epsilon_\ell$.*

In particular this shows that the fragile elements obtained by Abreu-McDuff for $\ell > 1$ do not disappear when we consider them inside $S^2 \times S^2 \times X$. Either $0 \neq w_\ell \times id \in \pi_{4\ell-4}(G_\ell^X)$ as in **(A)** or, if $w_\ell \times id = 0$ then it yields the associated new $4\ell - 3$ dimensional elements $0 \neq \eta_{\ell+\epsilon}^X$ in $\pi_{4\ell-3} G_{\ell+\epsilon}^X$ f or small $\epsilon > 0$ (case **(B)**.

For general X and for $\ell = 1$ it is known by work of Le-Ono that **B** takes place and moreover $0 \neq i_*(\eta_{\ell+\epsilon}) \in \pi_1 \mathrm{Diff}_0(S^2 \times S^2 \times X)$. Also, for $X = \mathrm{pt}$ and $\ell > 1$ from the work of Abreu-McDuff we know that **A** takes place. We do not have examples when case **B** takes place and $i_*(\eta_{\ell+\epsilon}) = 0 \in \pi_* \mathrm{Diff}_0(M)$.

Similar work has been done in this direction by Le-Ono in [11]; by looking at related but slightly different parametric GW invariants they get results about $\pi_i(\mathrm{Symp}_0(S^2 \times S^2 \times X, \omega_1 \oplus \omega_{st}))$ when $i = 1, 3$. In chapter **3** we could consider $\mathbb{C}^2/C_{2\ell+1}$ instead and by carrying out similar arguments get the same type of results for $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \times X$.

Chapter 2

Relative parametric GW invariants

2.1 General setting

Consider a compact manifold B with boundary and a smooth map $i : (B, \partial B) \to (\mathcal{A}_I, \mathcal{A}^c_{I,D})$. Although the invariants can be defined in this generality, for the applications we have in mind we will consider B to be an n-ball such that i represents a relative homotopy class in $\pi_*(\mathcal{A}_I, \mathcal{A}^c_{I,D}, *)$. We will often write $J_b := i(b)$ and $J_B = \mathrm{im}(i)$, and refer to $\mathrm{im}B$ in $\mathcal{A}_I$ as J_B. Consider also a smooth family of symplectic forms $(\omega_b)_{b\in B}$ where ω_b tames J_b. We point out that the ω_b need not be cohomologous, as the taming condition is an open condition. Our goal here is to show how we can define parametric GW invariants relative to the boundary ∂J_B, which count J_b holomorphic maps for some $b \in B$. These will not depend either on deformations of the family ω_B or on the representative $(J_B, \partial J_B)$ of a relative homotopy class in $(\mathcal{A}_I, \mathcal{A}^c_{I,D})$.

Definition 2.1. *We say that $f : \Sigma \to M$ is simple if it is not the composite of a holomorphic branched covering map $(\Sigma, j) \to (\Sigma', j')$ of degree greater than 1 with a J-holomorphic map $\Sigma' \to M$.*

Consider $\widetilde{\mathcal{M}}^*_{0,k}(M, D, (J_B, \partial J_B))$ the space of tuples $(b, f, x_1, \ldots, x_k)$ where $f : S^2 \to M$ is a simple J_b-holomorphic map in class D, for some $b \in B$ and x_i are pairwise distinct points on S^2. We will consider

$$\mathcal{M}^*_{0,k}(M, D, (J_B, \partial J_B)) = \widetilde{\mathcal{M}}^*_{0,k}(M, D, (J_B, \partial J_B))/G$$

where $G = \mathrm{PSL}(2, \mathbb{C})$ acts on the moduli space by reparametrizations of the domain. Denote the elements of $\mathcal{M}^*_{0,k}(M, D, (J_B, \partial J_B))$ by $[b, f, x_1, \ldots, x_k]$.

In the best scenario, for a good choice of $(J_B, \partial J_B)$, the following hold:

(1) $\widetilde{\mathcal{M}}^*_{0,k}(M, D, (J_B, \partial J_B))$ is a manifold of dimension $2n+2c_1(D)+2k+\dim B$ and

(2) $\mathcal{M}^*_{0,k} := \mathcal{M}^*_{0,k}(M, D, (J_B, \partial J_B))$ is compact.

Then the image of the map

$$ev : \mathcal{M}^*_{0,k}(M, D, (J_B, \partial J_B)) \to M^k \tag{2.1}$$

with

$$ev([b, f, x_1, \ldots, x_k]) := (f(x_1), \ldots, f(x_k))$$

will provide a cycle $ev_*(\mathcal{M}^*_{0,k})$ in M^k which, by intersection with homology classes of complementary dimension in M^k, gives the parametric Gromov-Witten invariants.

2.2 Definition and properties of PGW

As we will see in the regularity discussion below, **(1)** can always be achieved by Sard-Smale theorem. However, even in the situations when **(1)** holds, **(2)** is seldom true. In order to compactify $\mathcal{M}^*_{0,k}(M, D, (J_B, \partial J_B))$ we need to add the following

Definition 2.2. *[10] A stable smooth rational map is given by a tuple* $(f, \Sigma, x_1, \ldots, x_k)$ *satisfying:*

1) $\Sigma = \bigcup_{i=1}^m \Sigma_i$ *is a connected rational curve with normal crossing singularities and* $x_1, \ldots, x_k$ *are distinct smooth points in* Σ

2) f *is continuous and each restriction* $f_{|\Sigma_i}$ *lifts to a smooth map from the normalization* $\overline{\Sigma_i}$ *to M;*

3) If $f_{|\Sigma_i}$ *is constant then* Σ_i *contains at least three special points. Here, a special point is either a singular point or a marked point.*

The compactification $\overline{\mathcal{M}}_{0,k}(M, D, (J_B, \partial J_B))$ of $\mathcal{M}^*_{0,k}(M, D, (J_B, \partial J_B))$ contains both *stable J- holomorphic maps* and *nonsimple curves* which we sometimes call multiple cover curves. These nonsimple curves could potentially produce boundary strata of high dimension in the compactification $\overline{\mathcal{M}}_{0,k}(M, D, (J_B, \partial J_B)$, and hence this space would not necessarily carry a fundamental class.

In the situation that $B = \mathrm{pt}$ there are various procedures [Li-Tian ([10]), Ruan ([17]), Fukaya-Ono ([5]) to build up a theory which would provide *a virtual moduli cycle*, that is, an object which carries a fundamental class required for the definition of the invariants.

Roughly speaking, locally one needs to consider here all the stable holomorphic maps as well as small perturbations of these. There are then various procedures to pass to a global object with the required properties. These go through without essential changes if one considers parameter spaces with no boundary [see Leung-Bryan([4]), Ruan([17])]

In our situation we need to make sure that the boundary causes no problem. In what follows denote by $[f, \Sigma, x_1, \ldots, x_k]$ the equivalence class of a stable map $(f, \Sigma, x_1, \ldots, x_k)$, where two maps are equivalent if they differ by an automorphism of the domain. Then the elements of $\overline{\mathcal{M}}_{0,k}(M, D, (J_B, \partial J_B))$ consist of such equivalence classes. The following lemma basically states that if we consider an appropriately small open neighborhood of $\overline{\mathcal{M}}_{0,k}(M, D, (J_B, \partial J_B))$ consisting of almost holomorphic stable maps, then its projection onto J_B stays away from ∂J_B.

Lemma 2.3. *For any compact set $J_B \in \mathcal{A}_I$ such that $\partial J_B \subset \mathcal{A}^c_{I,D}$ $\exists$ a $\delta > 0$ and $\epsilon(\delta) > 0$ for which there is no stable map $(f, \Sigma, x_1, \ldots, x_k)$ such that $\bar{\partial}_J f = \nu$, when $d(J, \partial J_B) < \delta$ and $\nu \in L^p(\Lambda^{0,1} \otimes_J f^*TM)$ with $|\nu| \leq \epsilon(\delta)$.*

Proof: We will prove this by assuming the opposite. Assume that we have a sequence J_i, ν_i and f_i such that $d(J_i, \partial J_B) \to 0$, $|\nu_i| = \epsilon_i \to 0$ and each f_i is a stable map in class D with the property that $\bar{\partial}_{J_i} f_i = \nu_i$. Since J_B is compact we find a convergent subsequence J_i, whose limit J_∞ is in ∂J_B. But this would lead to a contradiction because by the Gromov compactness theorem there is a subsequence of f_i which converges to a J_∞ stable holomorphic map in class D. This will contradict the fact that $J_\infty \in \partial J_B \subset \mathcal{A}^c_{I,D}$. □

With this lemma one shows as in [10], that every moduli space

$B \times \overline{\mathcal{M}}_{0,k}(M, D, (J_B, \partial J_B))$ carries a virtual fundamental cycle

$$[\mathcal{M}]^{vir} := [B \times \overline{\mathcal{M}}_{0,k}(M, D, (J_B, \partial J_B))]^{vir}$$

of degree $r = 2c_1(D) + 2k + 2n - 6 + \dim B$.

Moreover if we consider two homotopic maps $i : (B, \partial B, *) \to (\mathcal{A}_I, \mathcal{A}^c_{I,D}, *)$ and $i' : (B', \partial B', *) \to (\mathcal{A}_I, \mathcal{A}^c_{I,D}, *)$ that represent the same element in $\pi_*(\mathcal{A}, \mathcal{A}^c_D, *)$, then the corresponding fundamental cycles given by $[B \times \overline{\mathcal{M}}_{0,k}(M, D, (J_B, \partial J_B))]^{vir}$ and $[B' \times \overline{\mathcal{M}}_{0,k}(M, D, (J_{B'}, \partial J_{B'}))]^{vir}$ are oriented cobordant and hence the virtual fundamental class $[\mathcal{M}]^{vir}$ is independent of the choice of $(J_B, \omega_B, *)$ within the same class in $\pi_*(\mathcal{A}_I, \mathcal{A}^c_{I,D}, *)$. We denote by $\overline{\mathcal{F}}_D(M, 0, k)$ the space of all equivalences classes of stable maps $[f, \Sigma, x_1, \dots, x_k]$ with total homology D. To define relative parametric Gromov-Witten invariants we consider: $ev_i : B \times \overline{\mathcal{F}}_D(M, 0, k) \to M$ given by

$$ev_i(b, [f, \Sigma, x_1, \dots, x_k]) = f(x_i)$$

We then can define

$$PGW^{M,(J_B, \partial J_B)}_{D,0,k} : \bigoplus_{i=1}^{k} H^{a_i}(M, \mathbb{Q})^k \to \mathbb{Q}$$

by

$$PGW^{M,(J_B, \partial J_B)}_{D,0,k}(\alpha_1, \dots, \alpha_k) = ev_1^*(\alpha_1) \wedge \dots \wedge ev_k^*(\alpha_k)[\mathcal{M}]^{vir}$$

which are zero unless

$$\sum_{i=1}^{k} a_i = 2c_1(D) + 2k + 2n - 6 + \dim B \tag{2.2}$$

We should also point out that if one changes the orientation of B we obtain the same invariant but with a negative sign.

We have the following theorem:

Theorem 2.4.

(i) *The invariants $PGW_{D,0,k}^{M,(J_B,\partial J_B)}$ are symplectic deformation invariants and depend only on the relative homotopy class of $(J_B, \partial J_B)$.*

(ii) *For a fixed choice of k, D and α_i the map $\Theta_{0,k,\alpha_1,\ldots,\alpha_k} : \pi_*(\mathcal{A}_I, \mathcal{A}^c_{I,D}, *) \to \mathbb{Q}$, given by*

$$\Theta_{k,\alpha_1,\ldots,\alpha_k}([(J_B, \partial J_B)] = PGW_{D,0,k}^{M,(J_B,\partial J_B)}(\alpha_1, \ldots, \alpha_k)$$

is a homomorphism [1].

Proof: Point (**i**) follows from the properties of PGW listed above.

The fact that the morphism $\Theta_{0,k,\alpha_1,\ldots,\alpha_k}$ in **(ii)** is well defined also follows from properties of PGW listed above. To show that it is a homeomorphism we choose $(B_1, \partial B_1, *)$, and $(B_2, \partial B_2, *)$ representing two maps from the standard n-ball with boundary to $(\mathcal{A}_I, \mathcal{A}^c_{I,D}, *)$, that give 2 elements β_1 and β_2 inside $\pi_*(\mathcal{A}, \mathcal{A}^c_D *)$. We choose them such that by their concatenation we represent the element $\beta_1 + \beta_2$ by a map $j : (B, \partial B, *) \to (\mathcal{A}_I, \mathcal{A}^c_{I,D}, *)$ with $j(B \backslash \partial B) = (B_1 \backslash \partial B_1) \bigcup (B_2 \backslash \partial B_2)$, such that $j^{-1}(\mathcal{A}_I \backslash \mathcal{A}^c_{I,D})$ is included in the

[1] except the case of $\pi_1(\mathcal{A}_I, \mathcal{A}^c_{I,D} *)$, which is not a group.

disjoint union of two open subdiscs in B. We can therefore see that the new virtual cycle corresponding to the classes $\beta_1 + \beta_2$ will be a disjoint union of the virtual neighborhoods corresponding to β_1 and β_2. But this implies that the parametric invariants corresponding to the new class $\beta_1 + \beta_2$ are the sum of the PGW corresponding to β_1 and β_2. Therefore Θ is a homomorpism. $\square$

2.3 More on the relation between PGW and almost complex structures

In this subsection we will explain that PGW detect only certain kinds of relative homotopy classes of almost complex structures. As before, we refer to $\mathcal{A}_\lambda = \mathcal{A}_{\omega_\lambda}$. Denote by

$$\mathcal{A}_{\ell^+} = \{J \mid \text{ there is an } \epsilon_J > 0 \text{ s.t. } J \in \mathcal{A}_{\ell+\epsilon} \text{ for all } 0 < \epsilon < \epsilon_J\} \tag{2.3}$$

Then $\mathcal{A}_\ell \subset \mathcal{A}_{\ell^+}$ by lemma 4.6 below. Note that $\mathcal{A}_{\ell^+}$ may not be connected, but $\mathcal{A}_\ell$ is and we will consider our base point $* = J_{basepoint} \in \mathcal{A}_\ell$.

Definition 2.5. *Consider a nontrivial element $\beta_\ell \in \pi_*(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell)$. We say that β_ℓ is a* **persistent** *element if its image under the natural morphism*

$$i_*^\epsilon : \pi_*(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell, *) \to \pi_*(\mathcal{A}_{[\ell,\ell+\epsilon]}, \mathcal{A}_\ell, *) \tag{2.4}$$

is nonzero for any ϵ arbitrary small.

Assume that there is an ℓ such that no J in $\mathcal{A}_\ell$ has a J-holomorphic curve in the class D. Then we have the following proposition

Proposition 2.6. *Assume that no J in $\mathcal{A}_\ell$ admits J- holomorphic stable maps in class D. Consider an element $0 \neq \beta_\ell \in \pi_*(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell, *)$ obtained by counting nontrivial parametric Gromov-Witten invariants in class D. Then β_ℓ is a persistent element.*

The proof is quite evident from the manner in which the element β_ℓ arises. Namely, since $(\beta_\ell, \partial\beta_\ell) \subset (\mathcal{A}_{[\ell,\ell+\epsilon]}, \mathcal{A}_\ell)$, then by hypothesis we can apply Theorem 2.4. We obtain that

$$\Theta^{\epsilon}_{k,\alpha_1,\ldots,\alpha_k}([(\beta_\ell, \partial\beta_\ell)] = PGW^{M,(i^\epsilon_*\beta_\ell, \partial i^\epsilon_*\beta_\ell)}_{D,0,k}(\alpha_1, \ldots, \alpha_k) \neq 0. \tag{2.5}$$

Therefore $0 \neq i^\epsilon_*(\beta_\ell) \in \pi_*(\mathcal{A}_{[\ell,\ell+\epsilon]}, \mathcal{A}_\ell, *)$ and hence by the relation 2.4, β_ℓ is persistent. □

2.4 Computability of PGW

We will now get back to the two conditions we posed in the beginning of the section, sufficient to yield that the image of the map (2.1) is a cycle. In what will follow we will provide sufficient hypothesis on the parameter space $(J_B, \partial J_B, *)$ and on the class D such that **(1)** and **(2)** are satisfied, as well as a criterion how to check one of the hypothesis. It will then follow for such a family $(J_B, \partial J_B, *)$ the invariants PGW defined above are integer valued and can be obtained by intersecting the image of the cycle $ev_*(\overline{\mathcal{M}}^*_{0,k}(M, D, (J_B, \partial J_B)))$ with the classes $(PD(\alpha_1), \ldots, PD(\alpha_k))$ in $H_*(M)^k$. Moreover, they can be obtained by counting the number of J_b holomorphic maps in class D with k marked points which intersect generic cycles representing $(PD(\alpha_1), \ldots, PD(\alpha_k))$ in $f(z_i)$.

2.4.1 Parametric regularity

In this subsection we will show that the *D-parametric regular* families $(J_B, \partial J_B)$ are the ones for which **(1)** from 2.1 is satisfied. We begin by explaining what is *D-parametric regularity* and contrast it with the usual D-regularity for J (see [14]). For this we need to introduce the following facts.

Definition 2.7. *We say that a map $f : \Sigma \to M$ is somewhere injective if $df(z) \neq 0, f^{-1}(f(z)) = z$ for some $z \in \Sigma$.*

Observation: A simple J-holomorphic map is somewhere injective (see Proposition 2.3.1 page 18 in [14]).

Let $\mathcal{X} = \mathrm{Map}(\Sigma, M; D)$ be the space of *somewhere injective* C^∞ smooth maps $f : \Sigma \to M$ representing class D. This is an infinite dimensional manifold with $T_f\mathcal{X} = C^\infty(f^*TM)$. We will next consider the following generalized vector bundle $\mathcal{E} \longrightarrow B \times \mathcal{X}$, whose fiber at (b, f) is the space $\mathcal{E}_{b,f} = \Omega^{0,1}_{J_b}(f^*TM)$ of C^∞ smooth J_b antilinear forms with values in f^*TM. In this vector bundle we consider a section $\Phi : B \times \mathcal{X} \longrightarrow \mathcal{E}$, given by

$$\Phi(b, f) = \frac{1}{2}(df + J_b \circ df \circ j) \tag{2.6}$$

The zeros of Φ are precisely J_b holomorphic maps and thus the moduli space

$$\widetilde{\mathcal{M}}^*_{0,0}(M, D, (J_B, \partial J_B)) = \Phi^{-1}(0),$$

is the intersection of imΦ with the zero section of the bundle. Since we would like $\widetilde{\mathcal{M}}^*_{0,k}(M, D, (J_B, \partial J_B))$ to be a manifold we require that Φ is transversal to the zero section. This means that the image of $d\Phi(b, f)$ is complementary to the tangent space $T_bB \oplus T_f\mathcal{X}$ of the zero section. But for any f which is J_b holomorphic, $d\Phi$ is given by

$$d\Phi(b, f) : T_bB \oplus C^\infty(f^*TM) \longrightarrow T_bB \oplus T_f\mathcal{X} \oplus \mathcal{E}_{b,f}$$

If we consider now the projection onto the vertical space of the bundle:

$$proj_2 : T_bB \oplus T_f\mathcal{X} \oplus \mathcal{E}_{b,f} \longrightarrow \mathcal{E}_{b,f}$$

the above transversality translates into the fact that

$$d\Phi(b, f) \circ proj_2 : T_bB \oplus C^\infty(f^*TM) \longrightarrow \Omega^{0,1}_{J_b}(\Sigma, f^*TM) \tag{2.7}$$

is onto. We will make the notation $D\Phi(b, f) = d\Phi(b, f) \circ proj_2$. We then have:

Definition 2.8. *We say that a J_b holomorphic map f is J_B parametric regular if $D\Phi(b, f)$ is onto.*

Observation: The linearized operator is well defined if there is no pair (b, f) with f a J_b holomorphic and $b \in \partial B$. This is precisely the condition we imposed on $(J_B, \partial J_B)$ to give a relative cycle in $(\mathcal{A}_I, \mathcal{A}^c_{I,D})$.

Definition 2.9. *Consider (J_B, ω_B) as above. We say that $(J_B, J_{\partial B})$ is an D-parametric regular family of C^∞ smooth almost complex structures if any J_b holomorphic map in class D is parametric regular. We denote by $\mathcal{J}_{preg}(D)$ the* *set of all D-parametric regular families* $(J_B, \partial J_B) \subset (\mathcal{A}_I, \mathcal{A}^c_{I,D})$.

We have the following:

Theorem 2.10. *If* $(J_B, \partial J_B) \in \mathcal{J}_{preg}(D)$, *then the moduli space* $\widetilde{\mathcal{M}}^*_{0,0}(M, D, (J_B, \partial J_B))$ *is a smooth open manifold of dimension* $2n + 2c_1(D) + \dim B$, *with a natural orientation.*

Moreover, if one considers $\widetilde{\mathcal{M}}^*_{0,0}(M, D, (J_B, \partial J_B)) \times (S^2)^k$ and takes away all the diagonals of the type $\widetilde{\mathcal{M}}^*_{0,0}(M, D, (J_B, \partial J_B)) \times \mathrm{diag}_{i,j}$, what we obtain is precisely $\widetilde{\mathcal{M}}^*_{0,k}(M, D, (J_B, \partial J_B))$. This will therefore be a manifold of dimension $2n + 2c_1(D) + \dim B + 2k$, when $(J_B, \partial J_B) \in \mathcal{J}_{preg}(D)$.

The proof of theorem 2.10 is based on the following characterization of parametric regularity.

We write $\widetilde{\mathcal{M}}^*_{0,0}(M, D, \mathcal{A}_I)$ for the *universal* moduli space consisting of pairs (f, J) where $J \in \mathcal{A}_I$ is a C^∞ smooth almost complex structure and f is a J -holomorphic map.

Proposition 2.11. *Consider the diagram*

$$\begin{tikzcd} & \widetilde{\mathcal{M}}^*_{0,0}(M, D, \mathcal{A}_I) \arrow[d, "\Pi"] \\ (B, \partial B) \arrow[r, "i"] & (\mathcal{A}_I, \mathcal{A}^c_{I,D}) \end{tikzcd} \tag{2.8}$$

Then $J_B \in \mathcal{J}_{preg}(A)$ *iff* $i \pitchfork \Pi$.

Proof: For simplicity we will denote by $D_{f,b} = D\Phi(b, f)_{|C^\infty(f^*(TM))}$. By (2.7) the surjectivity of $D\Phi(b, f)$ is then equivalent with the surjectivity of the following linear operator

$$D\phi_{|T_bB} : T_bB \to \mathrm{coker} D_{b,f}$$

We will denote $i(b) = J$. The tangent space $T_J\mathcal{A}_I$ to $\mathcal{A}_I$ consists of all sections Y of the bundle $\mathrm{End}(TM, J)$ whose fiber at $p \in M$ is the space of linear maps $Y : T_pM \to T_pM$ such that $YJ + JY = 0$; we will consider the map

$$R : T_J\mathcal{A}_I \to \Omega_J^{0,1}(\Sigma, f^*TM)$$

given by $R(Y) = \frac{1}{2}Y \circ df \circ j$. The map

$$d\Pi : T_{f,J}\widetilde{\mathcal{M}}^*_{0,0}(M, D, \mathcal{A}_I) \to T_J\mathcal{A}_I$$

is given by $d\Pi(\xi, Y) = Y$, where the pair (ξ, Y) is in $T_{f,J}\widetilde{\mathcal{M}}^*_{0,0}(M, D, \mathcal{A}_I)$ if and only if

$$D_{f,b}(\xi) + R(Y) = 0 \tag{2.9}$$

From this one can see that $\mathrm{im} D_{f,b} = R(\mathrm{im}(d\Pi))$. Since $D_{b,f}$ is elliptic and $\ker R \subset \mathrm{im} d\Pi$, it follows that $\mathrm{coker} d\Pi$ has finite dimension. If we consider the map

$$\mathcal{F} : \mathcal{X} \times \mathcal{A}_I \to \mathcal{E} \quad , \mathcal{F}(f, J_b) = \bar{\partial}_J(f) \tag{2.10}$$

then (see Prop 3.4.1 in [14]) the linearization at a zero (f, J) with f simple is onto. That is

$$D\mathcal{F}(f, J_b)(\xi, Y) = D_{f,b}\xi + R(Y) \tag{2.11}$$

is onto.This implies that coker$D_{f,b}$ is covered by R. Therefore there is an induced map

$$\widetilde{R} : \mathrm{coker} d\Pi \to \mathrm{coker} D_{f,b}$$

which is isomorphism. The proof of the proposition then follows easily.

$D\Phi_{|T_bB}(Y) = R \circ di$, so we have $i \pitchfork \Pi \Leftrightarrow di \to \mathrm{coker} d\Pi$ onto $\Leftrightarrow$

$\widetilde{R} \circ di \to \mathrm{coker} D_{b,f}$ onto. $\square$

Sketch of proof for theorem 2.10

Due to the similarity of the proof to the one of Theorem 3.1.2 in [14], we will only sketch the proof. We begin by making the following remarks:

i) As we will explain below, proofs of various statements involving regularity and transversality requires one to use results regarding elliptic (hence Fredholm) operators whose domains and targets are Banach manifolds. Therefore, rather than working with Fréchét manifolds consisting on C^∞ objects, we must consider their completions under suitable Sobolev norms.

More precisely, one should to work with spaces consisting of almost complex structures of class C^ℓ,and also with $\mathcal{X}^{k,p}$, where $kp > 2$, the space of maps whose k-th derivatives are of class L^p. One should also use the completion

$$\mathcal{E}^p_f = W^{k-1,p}(\Lambda^{0,1}T^*\Sigma \otimes_J f^*TM))$$

rather than $\Omega^{0,1}_J(\Sigma, f^*TM)$. The tangent space to the space $\mathcal{X}^{k,p}$ at a point f will now be the space $W^{k,p}(f^*TM)$ of $W^{k,p}$ sections of the bundle f^*TM. This will replace the space $C^\infty(f^*TM)$ of C^∞ sections in f^*TM.

ii) We should point out that the kernel and the cokernel of a smooth elliptic operator are independent of the smoothness of the functions in the domain and in the range. This is a consequence of elliptic regularity (see Prop 3.2.2 [14]). This is the reason why we can state Prop 2.11 for the C^∞ case.

For instance, if we consider elements in J_B of class ℓ, and the operator

$$D\Phi(b,f) : T_bB \oplus W^{k,p}(f^*TM) \longrightarrow W^{k-1,p}(\Lambda^{0,1}T^*\Sigma \otimes_{J_b} f^*TM) \tag{2.12}$$

then its kernel and cokernel do not depend on the choice of k and p as long as $k \leq \ell + 1$.

iii) A proof of theorem 2.10 involves the use of the infinite dimensional version of the implicit function theorem. ([14]). More precisely, the surjectivity of the operator $D\Phi(b,f)$ defined on space of C^∞ sections implies the surjectivity of the operator $D\Phi$ defined as in 2.12 on the Banach spaces completions. Denote by $(\mathcal{A}_I^\ell, \mathcal{A}_{I,D}^{c,\ell})$ paired spaces of almost complex structures of class ℓ, and by

$$\mathcal{M}_{0,0}^{*,\ell}(M, D, \mathcal{A}_I) = \{(f, J) \in \mathcal{X}^{k,p} \times \mathcal{A}_I^\ell | \partial_J(f) = 0\}$$

with $p > 2$ and $1 \leq k \leq \ell$. Due to elliptic regularity this is independent of k and p. We will now apply proposition 2.11 for the operator Π^ℓ now defined on Banach manifolds , $\Pi^\ell : \widetilde{\mathcal{M}}_{0,0}^{*,\ell}(M, D, \mathcal{A}_I) \longrightarrow (\mathcal{A}_I^\ell, \mathcal{A}_{I,D}^{c,\ell})$ and obtain transversality for the following diagram:

$$\begin{tikzcd} & \widetilde{\mathcal{M}}^{*,\ell}_{0,0}(M,D,\mathcal{A}_I) \arrow[d, "\Pi^\ell"] \\ (B,\partial B) \arrow[r, "i"] & (\mathcal{A}^\ell_I, \mathcal{A}^{c,\ell}_{I,D}) \end{tikzcd} \tag{2.13}$$

Then one applies the infinite dimensional implicit function theorem, which says that, given a C^ℓ Fredholm operator $\mathcal{P} : X \longrightarrow Y$, transversal to a C' imbedding $g : A \longrightarrow \mathcal{P}$, $dimA < \infty$, then the preimage $\mathcal{P}^{-1}(g(A))$ is $(index\mathcal{P} + dimA)$- dimensional manifold provided that $\ell > index\mathcal{P} + dimA$.

Elliptic regularity again gives the result for the C^∞ category. $\square$

Remark 2.12. In the proof of proposition 2.11 we use the fact that the linearization $D\mathcal{F}(f, J_b)(\xi, Y) = D_{f,b}\xi + R(Y)$ of the operator $\mathcal{F}$ defined in 2.10 is surjective for any zero (f, J_b). To see this we complete its image and range under suitable Sobolev norms; we obtain the following operator defined on

$$D\mathcal{F}(f,J_b) : W^{k,p}(f^*TM) \times C^\ell(End(TM,J_b)) \to W^{k-1,p}(\Lambda^{0,1}T^*\Sigma \otimes_{J_b} f^*TM)), \tag{2.14}$$

as in [14]. Afterwords one uses Hahn-Banach theorem to show that its range, which is closed because it is the range of a Fredholm operator, is also dense. Elliptic regularity then provides the surjectivity result for the C^∞ operator $D\mathcal{F}(f, J_b)$.

Definition 2.13. *We will say that* $(J_B, \partial J_B)$ **satisfies hypothesis** H_2 *if it is a D-parametric regular family of almost complex structures.*

There are few key points to be noticed here. Notice that parametric regularity is a generalization of the usual regularity. Indeed, if we consider $J_b = J$ to be constant for b in a neighborhood around b_0 then the regularity of an almost complex structure J simply says, following the diagram above, that $d\Pi$ is surjective. If we now regard J within an arbitrary family J_B, this no longer needs to be the case. It will then suffice that the cokernel of $d\Pi$ is covered by the variation of J in the direction of B.

In fact, when we count rational maps, the criterion of parametric regularity described below reduces the problem to the usual regularity in some suitable ambient space.

More precisely, note that the regularity of a holomorphic map is a local statement within B and it is directly related to the almost complex structure data rather than to the symplectic structure data. Therefore, for each $b \in IntB$ we can restrict our attention to a neighborhood of b, and without loss of generality the following discussion can be made for smoothly trivial fibrations. We say that the family (J_B, ω_B) descends from a fibration $M \to \widetilde{M} \to B$ if $\widetilde{M}$ comes with an almost complex structure $\widetilde{J}$ such that the restriction to each fiber $M \times b$ is an almost complex structure J_b. Moreover $\widetilde{M}$ admits a closed two form $\widetilde{\omega}$ which restricts on each fiber $M \times b$ to a symplectic form ω_b that tames J_b. Here we chose a trivialization of the fibration such that smoothly $\widetilde{M} = B \times M$ and π is just the projection on the first factor. In fact, every family $(J_b)_{b \in B}$, locally around a point b_0, descends from a fibration. In the following theorem we consider the family of parameters B to be a subspace of $\mathbb{C}^m$ and we denote by z the parameters in $\mathbb{C}^m$.

Theorem 2.14. *Let $(J_z, \omega_z)_{z\in\mathbb{C}^m}$ be a family on M descending from the symplectic fibration $(\widetilde{M}, \widetilde{J}, \widetilde{\omega})$*

$$\begin{tikzcd} M \arrow[r, "i"] & \widetilde{M} \arrow[d, "\pi"] \\ & \mathbb{C}^m \end{tikzcd} \tag{2.15}$$

Suppose that $f : \Sigma \longrightarrow M$ is a J_0 holomorphic map and consider the composite map

$$\widetilde{f} = i \circ f, \widetilde{f} : \Sigma \longrightarrow M \times 0 \subset \widetilde{M}$$

which is $\widetilde{J}$-holomorphic. If $\widetilde{f}$ is $\widetilde{J}$-regular, then f is (J_z) parametric regular. Moreover, if $\Sigma = S^2$ then the reverse statement holds.

For the proof of the theorem see Appendix A.

2.4.2 Compactness

Even in those situations when **(1)** from section 2.1, when is easily achieved using Sard-Smale, **(2)** is seldom true. However, **(2)** is true when k is either 0 or 1, and the class D is J_b *indecomposable* for any $b \in B$. This means that no J_b holomorphic map in class D can decompose into a connected union of J_b holomorphic spheres $C = C^1 \bigcup C^2 \bigcup \ldots \bigcup C^N$, with $N > 1$ such that each C^i represents the class $D_i \neq 0$ and $D = D_1 + \ldots + D_N$. Then as a consequence of Gromov's compactness theorem it follows that $\mathcal{M}^*_{0,k} := \mathcal{M}^*_{0,k}(M, D, (J_B, \partial J_B))$ is compact and hence in this situation the image of $ev : \mathcal{M}^*_{0,k}(M, D, (J_B, \partial J_B)) \to M^k$ is a cycle.

Definition 2.15. *We will say that the* **hypothesis H_2 is satisfied** *by $(J_B, \partial J_B)$ and D if the class D is J_b indecomposable for every $b \in B$.*

Note that if D is J_b indecomposable and $k \geq 2$ then in order to compactify the image of the evaluation map, one only needs to include the limits of sequences of J-holomorphic maps for which two distinct marked points converge to each other. Hence $ev(\mathcal{M}^*_{0,k})$ will have boundary of codimension 2 or more and hence it will carry a fundamental class.

Chapter 3

Resolutions of singularities and relative PGW

3.1 Quotient singularities; the local picture

In this subsection we will give an overview of work of Kronheimer [9] and Abreu-McDuff [3] on how to construct special families of almost complex structures arising from the study of the total spaces of deformations for some quotient singularities. In the end of the section we will explain how these families serve our purpose of counting nontrivial PGW. The local picture is as follows (see Kronheimer [9]):

We consider the particular type of Hirzebruch-Jung singularity $Y_0 = \mathbb{C}^2/C_{2\ell}$, given by the diagonal action by scalars of $C_{2\ell}$ on $\mathbb{C}^2$, where $C_{2\ell}$ is the cyclic group of order 2ℓ. This admits a resolution $\sigma_0 : \widetilde{Y_0} \to Y_0$ where $\widetilde{Y_0}$ is the total space of the line bundle $\mathcal{O}(-2\ell)$ of degree -2ℓ over $\mathbb{C}P^1$. The exceptional curve of the resolution, we will call it E, is a curve of self-intersection -2ℓ and is the zero section of $\widetilde{Y_0}$. This resolution admits a $2\ell-1$ complex dimensional parameter family of deformations ,$\widetilde{Y}_t, t \in \mathbb{C}^{2\ell-1}$. The total space $\widetilde{Y} = \bigcup \widetilde{Y}_t$ of this family of deformations is the total space of the vector bundle $\mathcal{O}(-1)^{2\ell}$.

With the exception of the case $\ell = 2$, this coincides with the versal space of deformations of the cone over the rational curve of degree 2ℓ in $\mathbb{C}P^{2\ell}$. When $\ell = 2$, the versal space of deformations has two components, one in complex dimension 3 and another one of dimension 1, which intersect transversally (see Pinkham [16]). The total space of the bundle $\mathcal{O}(-1)^{2\ell}$ coincides with the 3-dimensional component.

More precisely, we consider the exact sequence of bundles

$$\mathcal{O}(-2\ell) \ \rightarrow \ \mathcal{O}(-1)^{2\ell} \ \xrightarrow{r} \ \mathcal{O}^{2\ell-1} \tag{3.1}$$

where r is given by evaluating at $2\ell - 1$ generic sections of the dual of $\widetilde{Y}$, $\widetilde{Y}^* = \mathcal{O}(1)^{2\ell}$. Since holomorphically $\mathcal{O}^{2\ell-1}$ is trivial, we can project it to its fiber $\mathbb{C}^{2\ell-1}$ and hence we obtain a submersion $\widetilde{q} : \mathcal{O}(-1)^{2\ell} \rightarrow \mathbb{C}^{2\ell-1}$ with $\widetilde{Y}_t = \widetilde{q}^{-1}(t)$. Also it can be seen that $\widetilde{Y}$ is diffeomorphic with $\widetilde{Y}_0 \times \mathbb{C}^{2\ell-1}$ and a choice of trivialization provides a fiberwise diffeomorphism

$$\theta : \widetilde{Y} \ \stackrel{C^\infty}{\cong} \ \widetilde{Y}_0 \times \mathbb{C}^{2\ell-1} \tag{3.2}$$

where $\widetilde{Y}_0$ is the total space of the bundle $\mathcal{O}(-2\ell)$.

One way of seeing $\tilde{Y}$ is to identify the base $\mathbb{C}P^1$ of the bundle $\mathcal{O}(-1)^{2\ell} \longrightarrow \mathbb{C}P^1$ with the set of all directions in $\mathbb{C}^2$. Then any element in the total space $\widetilde{Y}$ of the bundle, which is not on the zero section, will be a 2ℓ-tuple of vectors in $\mathbb{C}^2$ that have the same direction given by the base point $z \in \mathbb{C}P^1$. This viewpoint can be formalized as follows:

Consider 4ℓ sections in the dual $\widetilde{Y}^*$. Here the space of holomorphic sections is given by $H^0(CP^1, \mathcal{O}(1))^{2\ell}$. Denote by Y the subspace of $(\mathbb{C}^2)^{2\ell}$ consisting of 2ℓ -tuples of vector in $\mathbb{C}^2$ which span either zero or a line. By evaluating all the 4ℓ section we obtain a map

$$\sigma : \widetilde{Y} \to Y \subset \mathbb{C}^{4\ell}$$

which contracts E to a point $\gamma_0 = \sigma(E)$. Moreover, γ_0 is the only singular point of Y. and the morphism is one to one outside E. We also define as in 3.1 the map $q : Y \to \mathbb{C}^{2\ell-1}$ by evaluating at the original $2\ell - 1$ generic sections. The following diagram commutes

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\sigma"] \arrow[d, "\widetilde{q}"] & Y \arrow[d, "q"] \\ \mathbb{C}^{2\ell-1} \arrow[r, "id"] & \mathbb{C}^{2\ell-1} \end{tikzcd} \tag{3.3}$$

Remark 3.1. Consider τ_{st} the standard Kähler form on $\mathbb{C}^{4\ell}$. By restriction, this gives a Kähler form on Y. If we denote by $(v_1, \dots, v_{2\ell})$ an element in $Y \subset (\mathbb{C}^2)^{2\ell}$, we can consider the S^1 action $S^1 \times Y \longrightarrow Y$ given by

$$e^{2\pi i\gamma} \cdot (v_1, \dots, v_{2\ell}) = (e^{2\pi i\gamma} v_1, \dots, e^{2\pi i\gamma} v_{2\ell}).$$

It follows that this is a hamiltonian action with respect to the form τ_{st} on Y. In particular the form τ_{st} is S^1 invariant.

Similarly, we can give a fiberwise S^1 action on $\widetilde{Y}$ by multiplying with $e^{2\pi i\gamma}$ the 2ℓ dimensional vectors in each fiber. It is then immediate that the holomorphic map σ commutes with the S^1 action. Via σ^* we pullback the form τ_{st} and obtain a closed S^1 invariant form $\tau_{\widetilde{Y}} = \sigma^*\tau_{st}$ on $\widetilde{Y}$ which restricts to a Kähler form τ_t on each fiber $\widetilde{Y}_t$ if $t \neq 0$ but degenerates along E when $t = 0$. E is contained in the fixed point set for the action on $\widetilde{Y}$. If we further push forward through θ defined in 3.2, these forms can be seen as a family of forms on the manifold $\widetilde{Y}_0$.

3.2 Quotient singularities; the compactified global picture

In this section we will compactify the local picture, as follows:

We consider the following associated exact sequence

$$\mathcal{O}(-2\ell) \xrightarrow{i} \mathcal{O}(-1)^{2\ell} \oplus \mathcal{O} \xrightarrow{r+id} \mathcal{O}^{2\ell-1} \oplus \mathcal{O} \tag{3.4}$$

By taking the projectivization of the second and third term in the sequence, we obtain a map

$$P(\mathcal{O}(-1)^{2\ell} \oplus \mathcal{O}) \overset{\bar{r}}{- - - >} P(\mathcal{O}^{2\ell-1} \oplus \mathcal{O}) \tag{3.5}$$

which is defined everywhere except at those points in each fiber $P(\mathcal{O}(-1)^{2\ell} \oplus \mathcal{O})_z$ that belong to the kernel of $\bar{r}$. These points describe a section $E_\infty := [i(\mathcal{O}(-2\ell))_z : 0]_{z\in\mathbb{C}P^1}$, to which we will refer from now on as the section at infinity. Let us consider the image of the zero section E in the projectivizations of the bundle $\mathcal{O}(-1)^{2\ell}$, $[0_z : \lambda]_{z\in\mathbb{C}P^1}$. We will denote it by E_0 and refer to it as to the zero section of $P(\mathcal{O}(-1)^{2\ell}\oplus\mathcal{O})_z$. We now blow up each fiber $P(\mathcal{O}(-1)^{2\ell} \oplus \mathcal{O})_z$ at the point $[i(\mathcal{O}(-2\ell))_z : 0]$, and in this manner we obtain a fibration $\widetilde{P}(\mathcal{O}(-1)^{2\ell} \oplus \mathcal{O}) \to \mathbb{C}P^1$ with fibers $\mathbb{C}P^{2\ell}\#\overline{\mathbb{C}P^{2\ell}}$. We will denote by D_z the exceptional divisor of the blow up. This will be a $\mathbb{C}P^{2\ell-1}$. We hence produce a morphism $\tilde{r}$ that extends the morphism $\bar{r}$; We have:

$$\widetilde{P}(\mathcal{O}(-1)^{2\ell} \oplus \mathcal{O}) \quad \overset{\tilde{r}}{\to} \quad P(\mathcal{O}^{2\ell-1} \oplus \mathcal{O}) \tag{3.6}$$

Basically, on each fiber, the map $\tilde{r}_z$ is equal to the map $\bar{r}_z$ outside the exceptional divisor D_z. We will denote by $[\lambda_1 : \ldots : \lambda_{2\ell-1} : \lambda_{2\ell}]_z$ a point in the fiber of $P(\mathcal{O}^{2\ell-1} \oplus \mathcal{O})$ over z. With this notation we have that

$\bar{r}_z^{-1}([\lambda_1 : \ldots : \lambda_{2\ell-1:\lambda_{2\ell}}]_z)$ is a line $L_{[\lambda_1:\ldots:\lambda_{2\ell-1}:\lambda_{2\ell}]} \setminus \infty$ in $P(\mathcal{O}(-1)^{2\ell} \oplus \mathcal{O})_z$. Each point p on the exceptional divisor D_z is uniquely determined by its property of being the point at infinity for precisely one of the lines $L_{[\lambda_1:\ldots:\lambda_{2\ell-1}:\lambda_{2\ell}]}$. We will define $\tilde{r}_z(p) = [\lambda_1 : \ldots : \lambda_{2\ell-1} : \lambda_{2\ell}]_z$.

As in the local picture, we can project onto a fiber $\mathbb{C}P^{2\ell-1}$ of the trivial bundle $P(\mathcal{O}^{2\ell-1} \oplus \mathcal{O}) \tilde{=} \mathbb{C}P^{2\ell-1} \times \mathbb{C}P^1$ and obtain a submersion

$$\widetilde{q} : \tilde{P}(\mathcal{O}(-1)^{2\ell} \oplus \mathcal{O}) \to \mathbb{C}P^{2\ell-1}$$

with the fibers are diffeomorphic to $S^2 \times S^2$. Moreover, each fiber naturally inherits a holomorphic structure. It is a classical result that the only complex structures on $S^2 \times S^2$ are the ones which will give the manifolds the structure of a Hirzebruch surface. We remind the reader that a Hirzebruch surface H_n

is a complex manifold given by :

$$H_n = P(\mathcal{O}(-n) \oplus \mathcal{O})$$

As a particular note, the Hirzebruch surfaces H_{2k} of even order are diffeomorphic to $S^2 \times S^2$, and all the Hirzebruch surfaces of odd order H_{2k-1} are diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ (see [6]). The next result shows that the fibers of $\tilde{q}^{-1}(t)$ are Hirzebruch surfaces H_{2n}, with $n < \ell$ if $t \neq 0$.

Proposition 3.2. *For $\tilde{q}$ as above, the following hold:*

(i) $\overline{Y_0} := \widetilde{q}^{-1}([0 : \ldots : 0 : 1]) = P(\mathcal{O}(-2\ell) \oplus \mathcal{O}) = H_{2\ell}$

(ii) *If $t = [\lambda_1 : \ldots : \lambda_{2\ell-1} : 1] \neq [0 : \ldots : 0 : 1]$, we have that*
$\overline{Y}_t := \widetilde{q}^{-1}([\lambda_1 : \ldots : \lambda_{2\ell-1} : 1]) = H_{2n}$ *for some $n < \ell$.*

Proof: **(i)** Since $\tilde{r}([i(\mathcal{O}(-2\ell))_z : \lambda]) = \bar{r}([i(\mathcal{O}(-2\ell))_z : \lambda]$ for $\lambda \neq 0$, we can easily conclude that $\widetilde{q}^{-1}([0 : \ldots : 0 : 1])$ contains all the points of the type $[i(\mathcal{O}(-2\ell))_z : \lambda]_{z \in \mathbb{C}P^1}$ for nontrivial λ. The section at infinity described as $[i(\mathcal{O}(-2\ell))_z : 0]_{z \in \mathbb{C}P^1}$ appears by adding all the points on D_z, with $z \in \mathbb{C}P^1$, whose image through $\tilde{r}$ is precisely $[0 : \ldots : 0 : 1]$. The conclusion follows, that $\widetilde{q}^{-1}([0 : \ldots : 0 : 1]) = P(\mathcal{O}(-2\ell) \oplus \mathcal{O})$.

(ii) Throughout this proof, we will denote by $t = [\lambda_1 : \ldots : \lambda_{2\ell-1} : 1]$. From the construction of $\widetilde{q}$ we can see that $\widetilde{q}^{-1}(t)$ is a holomorphic $\mathbb{C}P^1$ bundle over $\mathbb{C}P^1$. The associated integrable almost complex structure J_t is obtained as a restriction from the complex structure on $\widetilde{P}(\mathcal{O}(-1)^{2\ell} \oplus \mathcal{O})$ using the holomorphic map $\widetilde{q}$. Moreover, as we have already mentioned, all the fibers of $\widetilde{q}$ are diffeomorphic to $S^2 \times S^2$, and hence J_t gives the fiber the structure of a

Hirzebruch surface H_{2n}.

In order to show that $n < \ell$, we will show that $\widetilde{q}^{-1}(t)$ does not contain any J_t holomorphic curves of self-intersection $-2p \leq -2\ell$.

In order to do so, let us first notice that we can extend the S^1 action described in 3.1 on $\widetilde{Y}$ to a S^1 action on the total space of the bundle $\mathcal{O}(-1)^{-2\ell} \oplus \mathcal{O}$ by acting on the trivial bundle $\mathcal{O}$ also by multiplication in the fiber by $e^{2\pi i\gamma}$. Moreover, if we denote the coordinate on the fiber $\mathcal{O}_z$ by λ, we can extend $\tau_{\widetilde{Y}}$, the S^1 invariant form on $\widetilde{Y}$ (see Remark 3.1) to $\mathcal{O}(-1)^{-2\ell} \oplus \mathcal{O}$. We do so by taking

$$\tau = \tau_{\widetilde{Y}} + d\lambda \wedge d\overline{\lambda}.$$

It follows that τ is also a closed S^1 invariant form, such that τ is 0 when evaluated on vector fields tangent to sections of the type $(0_z, \lambda)_{z \in \mathbb{C}P^1}$ and nondegenerate everywhere else.

Finally, let us point out that associated to the hamiltonian S^1 action on $Y \subset \mathbb{C}^{4\ell}$ there is a moment map given by $H : Y \to \mathbb{R}$,

$$H(v_1 \ldots, v_{2\ell}) = -\pi|v|^2$$

Consequently, we obtain a map $\widetilde{H} : \mathcal{O}(-2\ell) \oplus \mathcal{O} \longrightarrow \mathbb{R}$, given by

$$\widetilde{H} = \sigma^*(H) - \pi|\lambda|^2.$$

If we consider the level set $\widetilde{H}^{-1}(\pi)$ then this is invariant under the S^1 action.

If we consider the quotient $\widetilde{H}^{-1}(\pi)/S^1$, then this will be diffeomorphic with $P(\mathcal{O}(-1)^{-2\ell} \oplus \mathcal{O})$. Moreover, the effect of taking the quotient is to provide a simultaneous symplectic reduction in each fiber $(\mathcal{O}(-2\ell)\oplus\mathcal{O})_z$ which will now be a symplectic $\mathbb{C}P^{2\ell}$. In fact the closed S^1 invariant 2 form τ restricts to the level set $\widetilde{H}^{-1}(\pi)$ and can be pushed forward to the quotient $\widetilde{H}^{-1}(\pi)/S^1$. We will denote by $\overline{\tau}$ the 2-form obtained on the quotient space.

Exactly as in the symplectic reduction (see McDuff-Salamon [15] Lemma 5.2), for any point $p \in \widetilde{H}^{-1}(\pi)$ which is not on the submanifold $((0_z,\lambda)_{z\in\mathbb{C}P^1}, |\lambda| = 1)$ we obtain nondegeneracy of the form $\overline{\tau}_p$. Moreover, the closed 2 form $\overline{\tau}$ degenerates along E_0, the "zero section" of the projectivized bundle.

In conclusion, this construction provides us with a closed 2-form on $P(\mathcal{O}(-2\ell)\oplus\mathcal{O})$ that only degenerates along the zero section E_0. We have constructed $\widetilde{P}(\mathcal{O}(-2\ell)\oplus\mathcal{O})$ by blowing up $P(\mathcal{O}(-2\ell)\oplus\mathcal{O})$ along the section at infinity E_∞. It easily follows that the form $\overline{\tau}$ pulls back to $\widetilde{P}(\mathcal{O}(-2\ell)\oplus\mathcal{O})$, to a closed 2-form which also degenerates only along the zero section E_0. For simplicity we will denote it by $\overline{\tau}$ as well.

Let us also notice that the forms $\overline{\tau}_t$ are compatible with the complex structure J_t since they are obtained by pulling back via a holomorphic map. Since the fiber $\widetilde{q}^{-1}(t), t \neq 0$, is disjoint from $E_0 \subset \widetilde{q}^{-1}(0)$, it follows that it is in fact a Kähler manifold $(\overline{Y}_t, J_t, \overline{\tau}_t)$ diffeomorphic to $S^2 \times S^2$.

Since the forms $\overline{\tau}_t$ are obtained by restricting the closed form $\overline{\tau}$ to fibers of $\widetilde{q}$ it is immediate that they are all in the same cohomology class. Moreover, since $[\overline{\tau}_0]_{|A-\ell F} = (\overline{\tau}_0)_{|E_0} = 0$ we obtain that $\forall t \neq 0, [\overline{\tau}_t] = [\omega_\ell] = [\sigma_F \oplus \ell\sigma_B]$.

Suppose now that we have a J_t holomorphic curve f of self-intersection $-2p \leq 2\ell$ in $\overline{Y}_t \cong S^2 \times S^2$. This would be then in the class $A - pF$, and since is

J_t- holomorphic, we would have that $\overline{\tau}_t([imf] > 0$. But this is contradicted by the fact that $\overline{\tau}_t([imf] = [\overline{\tau}_t]([A-pF] = [\sigma_F \oplus \ell\sigma_B]([A-pF]) = \ell - p \leq 0$.

$\square$

Remark 3.3. What we shall use from what we proved about the structure of the fibers of $\widetilde{q}$ is that for $t = [\lambda_1 : \ldots : \lambda_{2\ell-1} : 1] \neq [0 : \ldots : 0 : 1]$, the space $\overline{Y}_t := \widetilde{q}^{-1}([\lambda_1 : \ldots : \lambda_{2\ell-1} : 1])$ is a Kähler manifold $(\overline{Y}_t, J_t, \overline{\tau_t})$ diffeomorphic to $S^2 \times S^2$, that doesn't admit any J_t holomorphic curves of self-intersection $-2p \leq -2\ell$. This implies that there are no J_t- holomorphic stable maps in the class $A - \ell F$. Indeed, if there was any, then its irreducible components would be J_t - holomorphic curves $f_i : P\Sigma_i = S^2 \longrightarrow \overline{Y}_t$,with the homology classes given by $[f_1] = A - pF$ and $[f_i]_{i \neq 1} = m_i F$, such that $p > \ell$ and $m_i > 0$. But this contradicts the fact that the self-intersection of f_1, equal to $-2p$, has to be greater than -2ℓ.

In what follows we will give a direct proof of the fact that the fibers $\widetilde{q}^{-1}(t)$ are Hirzebruch surfaces, for the case $\ell = 1$. The reader may skip and go directly to section 3.3.

Proposition 3.4. *If we consider the sequence*

$$\mathcal{O}(-2) \;\rightarrow\; \mathcal{O}(-1) \oplus \mathcal{O}(-1) \;\stackrel{r}{\rightarrow}\; \mathcal{O} \tag{3.7}$$

and follow the compactification procedure described above, then the following hold:

(i) $\widetilde{q}^{-1}([0:1]) = P(\mathcal{O}(-2) \oplus \mathcal{O}) = H_2$

(ii) $\widetilde{q}^{-1}([a:1]) = P(\mathcal{O} \oplus \mathcal{O}) = H_0$

Proof: As before, we identify Y, the total space of the bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ with the the zero section collapsed to a point, with the subset of $\mathbb{C}^2 \times \mathbb{C}^2$, consisting from pairs $(\overline{v_1}, \overline{v_2})$ of colinear vectors in $\mathbb{C}^2$. Consider $\overline{e_1}$ and $\overline{e_2}$ to be the standard vectors $(1,0)$ and $(0,1)$ in $\mathbb{C}^2$.

We will define the map r on $Y \subset \widetilde{Y}$ by

$$r(\overline{v_1}, \overline{v_2}) = <\overline{e_1}, \overline{v_1}> + <\overline{e_2}, \overline{v_2}>, \tag{3.8}$$

where $< .,. >$ is the standard hermitian product on $\mathbb{C}^2$. This map naturally extends to $\widetilde{Y} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ by taking the value 0 for points on the zero section. This extended map provides a morphism of bundles which satisfies the properties of 3.7.

Consider now (U, z) and (U', z') the two coordinates charts on the base $\mathbb{C}P^1$, where $0 \in U$, $\infty \in U'$ and $zz' = 1$. We can now describe the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in coordinate charts as follows:

The coordinates over the open set containing zero are given by $(z, \xi_1, \xi_2)_0 \in U \times \mathbb{C} \times \mathbb{C}$, and the coordinates over the open set containing infinity are $(z', \xi_1', \xi_2')_\infty \in U' \times \mathbb{C} \times \mathbb{C}$, with the change of coordinates satisfying the relations:

$$\begin{aligned} z &= \tfrac{1}{z'} \\ \xi_1 &= z'\xi_1' \\ \xi_2 &= z'\xi_2' \end{aligned} \tag{3.9}$$

With this coordinates charts, the vectorial description of any point on Y given in the coordinates is:

$$\begin{aligned} (z,\xi_1,\xi_2)_0 &= (\overline{(\xi_1 z,\xi_1)},\overline{(\xi_2 z,\xi_2)}) \\ (z',\xi_1',\xi_2')_\infty &= (\overline{(\xi_1',z'\xi_1')},\overline{(\xi_2',z'\xi_2)'}) \end{aligned} \tag{3.10}$$

Combining 3.8 and 3.10 we obtain:

$$\begin{aligned} r(z,\xi_1,\xi_2)_0 &= \xi_1 z+\xi_2 \\ r(z',\xi_1',\xi_2')_\infty &= \xi_1'+\xi_2' z' \end{aligned} \tag{3.11}$$

It is obvious from 3.9 that the two definitions agree on the intersection $U\bigcap U'$. These definitions extend immediatly to define the map

$$\overline{r}: P(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}) ----> P(\mathcal{O}\oplus\mathcal{O}) \tag{3.12}$$

given by:

$$\begin{aligned} \overline{r}(z,[\xi_1:\xi_2:\lambda])_0 &= (z,[\xi_1 z+\xi_2:\lambda]) \\ \overline{r}(z',[\xi_1':\xi_2':\lambda])_\infty &= (z',[\xi_1'+\xi_2' z':\lambda]) \end{aligned} \tag{3.13}$$

These formulas yield the following descriptions for

$\widetilde{q}^{-1}([a:1])=\widetilde{q}^{-1}([\lambda a:\lambda])$, for any $\lambda\neq 0, (z,\xi)\in U\times\mathbb{C}, (z',\xi')\in U'\times\mathbb{C}$:

$$\begin{aligned} \widetilde{q}^{-1}([a:1]) &= (z,[\xi_1:a\lambda-xi_1 z:\lambda])_0 \\ &= (z',[a\lambda-\xi_2' z':\xi_2':\lambda])_\infty), \end{aligned} \tag{3.14}$$

where now we will allow $\lambda=0$ as a reflection of the fact that the preimages of $\widetilde{q}$ lie in $\widetilde{P}(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O})$ rather that in $P(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O})$. The proof of the proposition is now quite obvious, once we recall the following result from Kodaira [8]:

Lemma 3.5. *(Kodaira)*

Consider a holomorphic fibration P over $\mathbb{C}P^1$ with the fiber $\mathbb{C}P^1$, given by the change of coordinates:

$$\begin{aligned} z &= \tfrac{1}{z'} \\ \eta &= (z')^m \eta' + a(z')^k \end{aligned} \tag{3.15}$$

where $k \leq m/2$ and $\eta, \eta' \subset \mathbb{C} \bigcap \infty = \mathbb{C}P^1$. Then E is biholomorphic to H_m if $a = 0$ and to H_{m-2k} otherwise.

To conclude our proof, we take $m = 2$ and $k = 1$ in the lemma above and we get a biholomorphism between P and $\widetilde{q}^{-1}([a:1])$ by taking $(z, \eta)_0$ into (z, ξ_1) and $(z', \eta')_\infty$ into $(z', -\xi_2')$, and then plug in the formulas 3.14. One checks that this morphism commutes with the change of coordinates in the two manifolds.

□

3.3 A parametric regular family of almost complex structures on $S^2 \times S^2 \times X$

3.3.1 The description of the family

Let $B^{4\ell-2}$ be the unit ball in $\mathbb{C}^{2\ell-1}$. With the proposition 3.2, we have provided a family $(\overline{Y}_t, J_t^\ell, \overline{\tau}_t)_{t \in B^{4\ell-2}}$, where each $(\overline{Y}_t, J_t^\ell, \overline{\tau}_t^\ell), t \neq 0$ is a Kähler manifold diffeomorphic with $S^2 \times S^2$, and, $(\overline{Y}_0, J_0^\ell)$ is a complex manifold, also diffeomorphic with $S^2 \times S^2$ and $\overline{\tau}_0$ degenerates along $A - \ell F$. The total space of the family has the following properties:

a) The space $\overline{Y} = \cup_{t \in B^{4\ell-2}} \overline{Y}_t$ is smoothly diffeomorphic with $S^2 \times S^2 \times B^{4\ell-2}$. Moreover $\overline{Y}$ is a complex manifold with a complex structure $\widetilde{J}^\ell$ which restricts to each fiber $\overline{Y}_t$ to the complex structure J_t^ℓ. Also, $\overline{Y}$ has a closed $(1,1)$ form $\overline{\tau}$ which is satisfies all the properties of a Kähler form outside the zero fiber and restricts at each fiber to the forms, $\overline{\tau}_t$.

b) The form $\overline{\tau}$ restricted to $\overline{Y_0}$ degenerates along the exceptional curve $A - \ell F$

Moreover, $\forall t \in B^{4\ell-2}, [\overline{\tau}_t^\ell] = [\omega_\ell]$.

From **(a)** we see that there is a holomorphic projection $\pi : \overline{Y} \to S^2 \times B^{4\ell-2}$. This is because every $\overline{Y}_t$ is a ruled surface therefore it fibers over S^2. If we denote by α the area form on S^2 we can construct a two form

$$\overline{\tau}^\lambda = \overline{\tau} + (\lambda - \ell)\pi^*(\alpha)$$

For $\lambda > \ell$ these forms are Kähler forms and moreover they restrict to each $\overline{Y}_t$ to symplectic forms in the class $[\omega_\lambda]$. This proves that any J_t^ℓ (including J_0^ℓ) is tamed by a form isotopic with ω_λ as long as $\lambda > \ell$. We now follow a similar procedure to construct a family of symplectic forms $\omega_t, t \in B^{4\ell-2}$ such that each ω_t tames J_t^ℓ. We will now change the forms $\overline{\tau}_t$ by perturbing with a a positive factor of $\pi^*(\alpha)$ only around $t = 0$ and smooth with a cut-off function. By this procedure we obtain symplectic forms ω_t with variable cohomology classes.

In conclusion, we have pairs $(S^2 \times S^2, J^\ell_t, \omega_t)_{t \in B^{4\ell-2}}$ where ω_t is a *symplectic* structure on $S^2 \times S^2$ that tames J^ℓ_t. Moreover $[\omega_t]_{t \in S^{4\ell-3}} = [\omega_\ell]$. This gives a family of almost complex structures which we denote by abuse of notation B_ℓ such that $(B_\ell, \partial B_\ell) \in (\mathcal{A}_{[\ell,\ell+\epsilon]}, A_\ell)$ for any $\epsilon > 0$. More importantly, only J^ℓ_0 admits the exceptional curve in the class $A - \ell F$.

We will then obtain a family of almost complex structures on $(S^2 \times S^2 \times X)$ by taking $(J^\ell_t \times J_{st})$, and by abuse of notation, we will call this family also B_ℓ. Therefore we just produced on $(S^2 \times S^2 \times X)$ pairs $(B_\ell, \partial B_\ell) \subset (\mathcal{A}_{[\ell,\ell+\epsilon]}, A_\ell)$, with $\epsilon > 0$ that represents an element β_ℓ in $\pi_*(\mathcal{A}_{[\ell,\ell+\epsilon]}, A_\ell)$. Moreover each $B_\ell \subset \mathcal{A}_{\ell+\epsilon}$ for any small $\epsilon > 0$.

From the choice of the J's we know that the only structure which admits $A - \ell F$ curves is $J_0 \times J_{st}$.

3.3.2 A computation of PGW

Here we prove that (H_1) and (H_2) are satisfied for the family $(B_\ell, \partial B_\ell)$, and therefore the invariant is integer valued and can be obtained by counting holomorphic maps intersecting generic cycles of appropriate dimension.

Claim 1. The family $(B_\ell, \partial B_\ell)$ satisfies H_1 .

Proof of claim 1: For the proof, we need the following

Lemma 3.6. *(McDuff-Salamon)*

Assume that J is integrable and let $f : \mathbb{C}P^1 \to M$ be a J-holomorphic curve. Suppose that every summand of f^*TM has Chern number $c_1 \geq -1$. Then D_f is onto.

From the sequence (3.1) we have that the map E, which is $\widetilde{J}^\ell$-holomorphic has the normal bundle $\mathcal{O}(-1)^{2\ell}$ and therefore we can apply the above lemma for the integrable almost complex structure $\widetilde{J}$. If follows that E is $\widetilde{J}^{\ell}$ regular inside $\overline{Y}$. If we consider now $\overline{Y} \times X$ and $\widetilde{J}^{\ell} \times J_{st}$, the curve E lies entirely inside $\overline{Y}$ and therefore the normal bundle inside $\overline{Y} \times X$ is $\mathcal{O}(-1)^{2\ell} + trivial$, and therefore the curve is is $\widetilde{J}^{\ell} \times J_{st}$ regular. This splitting and therefore regularity use the fact that the map E is of genus zero. Theorem 2.14 implies parametric regularity and therefore (H_1) holds. $\square$

Claim 2. The family $(B_\ell, \partial B_\ell)$ satisfies (H_2) .

Proof of claim 2: This is proved by inspection. Only $J_0^{\ell} \times J_{st}$ admits $A - \ell F$ stable maps, and the only maps in this class are copies of the imbedded map E in any fiber $S^2 \times S^2 \times \mathrm{pt}$. Hence there are no decomposable J_b holomorphic maps. We should point out that for other almost complex structures J on $S^2 \times S^2 \times X$ one could have decomposable J-holomorphic maps in the class $A - \ell F$.

$\square$

Observation: For an example of a situation when one could have decomposable J-holomorphic maps, we can consider $M = S^2 \times S^2 \times \mathbb{C}P^n, \omega = \omega_{1+\epsilon} \oplus \omega_{st}$. If we denote by H the hyperplane class in $\mathbb{C}P^n$, we can take ω_{st} such that $\omega(A - F - H) > 0$ and one can get a symplectic embeding of S^2 into M, in the class $A - F - H$. We can choose an ω tamed almost complex structure $\tilde{J}$ on M which fibers over the base $S^2 \times S^2$ and such that the class H has a $\tilde{J}$ holomorphic representative. Then the class $A - F$ is $\tilde{J}$ decomposable where the decomposition is given by a C with $C = C_1 \bigcup C_2$ with $[C_1] = A - F - H$ and $[C_2] = H$. $\square$

We can therefore conclude that the invariants

$$PGW^{S^2\times S^2\times X,(B_\ell,\partial B_\ell)}_{A-\ell F,0,k} : \bigoplus_{i=1}^{k} H^{a_i}(S^2\times S^2\times X,\mathbb{Q})^k \to \mathbb{Z}$$

are integer valued. We have two situations. First, if $X = \mathrm{pt}$ then the moduli space of unparametrized curves has dimension 0 so we would count isolated curves. This follows immediately from the fact that $c_1(A-\ell F) = -4\ell + 2$ (adjunction formula) and therefore

$$\begin{aligned}\dim \mathcal{M}^*_{0,0}(S^2\times S^2, A-\ell F,(B_\ell,\partial B_\ell)) &= 2\times 2 + 2c_1(A-\ell F) + \dim B^\ell - 6\\ &= 4 - 4\ell + 4 + 4\ell - 2 - 6 = 0.\end{aligned}$$

Moreover, the invariant $PGW^{S^2\times S^2\times X,(B_\ell,\partial B_\ell)}_{A-\ell F,0,0}([\mathrm{pt}]) = 1$ because it counts E, the only $(J_b)_{b\in B^\ell}$ parametric holomorphic map in the class $A-\ell F$.

In the situation that $\dim X = 2n > 0$, we will count maps with one marked point. $c_1(A-\ell F)$ will be the same since the holomorphic maps in class $A-\ell F$ will be copies of the curve E and hence will have the image entirely in the fibers $S^2\times S^2\times \mathrm{pt} \subset S^2\times S^2\times X$. We therefore have

$$\begin{aligned}\dim \mathcal{M}^*_{0,1}(S^2\times S^2\times X, A-\ell F,(B_\ell,\partial B_\ell)) &= 2\times(2+n) + 2c_1(A-\ell F) + \dim B^\ell\\ &\quad - 6 + 2 = 2n+2\end{aligned}$$

We will consider a cycle in the homology class F which will lie in a fiber $S^2\times S^2\times \mathrm{pt}$ inside $S^2\times S^2\times X$. It easily follows that the only J_{b_ℓ} holomorphic map with one marked point which intersect this cycle transversely is a copy of the map E inside the fiber $S^2 \times S^2 \times \mathrm{pt}$. We obtain that

$$PGW^{S^2\times S^2\times X,(B_\ell,\partial B_\ell)}_{A-\ell F,0,1}(PD([F]) = \pm 1$$

where the sign depends on the orientation of the parameter space B_ℓ. Applying theorem 2.4 we obtain that the morphism Θ in both situations is nontrivial and therefore there is a nonzero element

$$\beta_\ell \in \pi_{4\ell-2}((\mathcal{A}_{[\ell,\ell+\epsilon]}, A_\ell) \text{ for all } \epsilon > 0 \tag{3.16}$$

that is represented by the cycle $(B_\ell, \partial B_\ell) \subset (\mathcal{A}_{\ell+\epsilon}, \mathcal{A}^c_{\ell+\epsilon,D})$.

Chapter 4

Almost complex structures and symplectomorphism groups

In the next two sections we explain the fibration (1.1), and discuss some known facts on the symplectomorphisms groups and the spaces of almost complex structures for the situation when $M = S^2 \times S^2$. The results here are either standard facts in symplectic topology or they come from work of Abreu, McDuff,[13] and Kronheimer [9]. Then we introduce the notions of fragile and new elements of the symplectomorphism groups as ω varies, and discuss some of their properties.

4.1 The main fibration

Consider as before $\mathcal{S}_{[\omega]}$ to be the space of symplectic forms ω' that can be joined to ω through a path of symplectic cohomologous forms $(\omega_t)_{t\in[0,1]}$. First recall:

Theorem 4.1. *Moser's stability theorem*

Let M be a closed manifold, and suppose that ω_t is a smooth family of symplectic cohomologous forms on M. Then there is a family of diffeomorphism ϕ_t of M such that $\phi_0 = id$, $\phi_t^*\omega_t = \omega_0$.

This implies that the group $\mathrm{Diff}_0 M$ acts transitively on $\mathcal{S}_{[\omega]}$ via the action

$$\phi \cdot \omega = \phi_*(\omega) = (\phi^{-1})^*\omega_0.$$

Hence, we have the following fibration (see [9]):

$$\mathrm{Symp}_0(M,\omega) \longrightarrow \mathrm{Diff}_0(M) \xrightarrow{\psi\to(\psi^{-1})^*\omega} S_{[\omega]} \tag{4.1}$$

Later, McDuff observed the following

Proposition 4.2. *$\mathcal{S}_{[\omega]}$ is homotopy equivalent to $\mathcal{A}_{[\omega]}$.*

In order to prove this she considers the space of pairs

$$\mathcal{X}_{[\omega]} = \{(\omega', J) \in \mathcal{S}_{[\omega]} \times \mathcal{A}_{[\omega]} | \omega' \text{ tames } J\}$$

The maps $\mathcal{X}_{[\omega]} \to \mathcal{S}_{[\omega]}$ and $\mathcal{X}_{[\omega]} \to \mathcal{A}_{[\omega]}$ are fibrations with contractible fibers, and hence homotopy equivalences and the conclusion follows.

This proposition allows one to conclude that there is the following fibration in homotopy (see [13])

$$\mathrm{Symp}_0(M,\omega) \longrightarrow \mathrm{Diff}_0(M) \longrightarrow \mathcal{A}_{[\omega]}. \tag{4.2}$$

4.2 Structural and stability results for the case $M = S^2 \times S^2$

If one considers the situation when $(M, \omega_\lambda) = (S^2 \times S^2, \sigma_F \oplus \lambda\sigma_B)$, Abreu-McDuff [3], and McDuff [13] provide extensive information regarding the structure of the spaces $\mathcal{A}_\lambda$ and G_λ as well as about the variation of their homotopy type as λ varies.

More precisely, we have:

Proposition 4.3. *Consider, as above* $(M, \omega_\lambda) = (S^2 \times S^2, \sigma_F \oplus \lambda\sigma_B)$, *and* $\lambda > 1$. *Then* $\mathcal{A}_\lambda \subset \mathcal{A}_{\lambda+\epsilon}$ *for any* $\epsilon > 0$.

As a corollary of the above results, one gets

Proposition 4.4. *For* $\lambda > 1, \epsilon > 0$, *there are maps* $\mathcal{S}_\lambda \to \mathcal{S}_{\lambda+\epsilon}$ *and* $G_\lambda^{pt} \to G_{\lambda+\epsilon}^{pt}$ *that are well defined up to homotopy and make the following diagrams commute*

$$\begin{tikzcd}
G_\lambda^{pt} \arrow[r] \arrow[d] & \mathrm{Diff}_0 S^2 \times S^2 \arrow[r] \arrow[d, "="] & \mathcal{S}_\lambda \arrow[d] \\
G_{\lambda+\epsilon}^{pt} \arrow[r] & \mathrm{Diff}_0 S^2 \times S^2 \arrow[r] & \mathcal{S}_{\lambda+\epsilon}
\end{tikzcd} \tag{4.3}$$

and

$$\begin{tikzcd}
G_\lambda^{pt} \arrow[r] \arrow[dr] & G_{\lambda+\epsilon}^{pt} \arrow[d] \\
& G_{\lambda+\epsilon+\epsilon'}^{pt}
\end{tikzcd} \tag{4.4}$$

Moreover, the investigation in [13] is based on a detailed description of the spaces $\mathcal{A}_\lambda$. Some of the results are collected in the following

Proposition 4.5. *The spaces $\mathcal{A}_\lambda$ are stratified spaces where the strata $\mathcal{A}_{\lambda,\ell}$, $0 < \ell < \lambda$ consist of almost complex structures tamed by ω_λ and that admit almost holomorphic curves in the class $A - \ell F$. Each stratum is a Fréchét suborbifold of $\mathcal{A}_\lambda$ of codimension $4\ell - 2$.*

We should point out that in particular the homotopy type of $\mathcal{A}_\lambda$ only changes as λ passes an integer.

4.3 Almost complex structures and symplectomorphisms; deformations along compact subsets

In this section we will get back to the situation when M is an arbitrary closed manifold. The aim here is to describe what can be said about the behavior of spaces of almost complex structures and about the symplectomorphism groups as the symplectic form varies along the line L.

If L happens to be a ray $\lambda\omega, \lambda > 0$ then G_λ is independent of λ. It will therefore make sense to consider $L \neq ray$.

For M an arbitrary symplectic manifold we no longer have either inclusion of $\mathcal{A}_\lambda$ in $\mathcal{A}_{\lambda+\epsilon}$ or the maps $G_\lambda \to G_{\lambda+\epsilon}$, or the stratification exhibited in the previous section. Nevertheless, as a consequence of the fact that taming is an open condition, we are able to establish the following proposition, which we use in the proof of the theorem 2.14.

Proposition 4.6. i) *Let K' to be an arbitrary compact subset of $\mathcal{A}_\lambda$. For $\mathcal{G}$ as in (1.4), there is an $\epsilon_{K'} > 0$ such that K' is contained in $\mathcal{A}_{\lambda+\epsilon}$, for $|\epsilon| < \epsilon_{K'}$.*

ii) *Consider K an arbitrary compact set in G_λ. Then there is an $\epsilon_K > 0$ and a map $h : [-\epsilon_K, \epsilon_K] \times K \to \mathcal{G}_{|L}$ such that the following diagram commutes*

$$\begin{tikzcd}
h : [-\epsilon_K, \epsilon_K] \times K \arrow[r] \arrow[d, "pr_1"] & \mathcal{G}_{|L} \arrow[d, "pr_2"] \\
{[-\epsilon_K, \epsilon_K]} \arrow[r, "incl"] & (-\infty, \infty).
\end{tikzcd} \tag{4.5}$$

Moreover, for any two such maps h and h' which coincide on $0 \times K$, there exists, for an $0 < \epsilon'$ small enough, a homotopy $H : [0,1] \times [-\epsilon', \epsilon'] \times K \to \mathcal{G}_{|L}$ between them which satisfies

$$\begin{tikzcd}
H : [0,1] \times [-\epsilon', \epsilon'] \times K \arrow[r] \arrow[d, "pr_1"] & \mathcal{G}_{|L} \arrow[d, "pr_2"] \\
{[-\epsilon', \epsilon']} \arrow[r, "incl"] & (-\infty, \infty).
\end{tikzcd} \tag{4.6}$$

Proof: Subpoint **(i)** is an immediate consequence of the openness of the taming condition.

For the proof of **(ii)**, let's first notice that, since the symplectic condition is an open condition, there is a convex open neighborhood U of ω_λ inside the space of 2-forms such that any closed ω' in U is still symplectic.

Moreover, there is an $\epsilon(K) > 0$ such that for any $g_k \in K$,

$g_k^* \omega_{\lambda+\epsilon} \in \mathrm{U}$, for all $0 \le \epsilon < \epsilon(K)$.

This claim can be proved by contrapositive as follows. Assume that there is a sequence $\{\epsilon(k_i)\} \to 0$ and a sequence $\{g_{k_i}\} \subset K$ for which $g_{k_i}^* \omega_{\epsilon(k_i)}$ is in the complement of U. Since K is compact, we can assume, if necessary by passing to a subsequence, that $\{g_{k_i}\}$ is convergent.

Then $\{g_{k_i}^*\omega_{\epsilon(k_i)}\} \to g_{limit}^*\omega_\lambda = \omega_\lambda \in U$, which is impossible, since U is open.

We will construct the elements $h(\epsilon, k)$ as follows. For $t \in [0,1]$ the forms

$$\omega_{k,\lambda+\epsilon}^t := tg_k^*\omega_{\lambda+\epsilon} + (1-t)\omega_{\lambda+\epsilon}$$

are symplectic since both $g_k^*\omega_{\lambda+\epsilon}$ and $\omega_{\lambda+\epsilon}$ are inside the convex set U. Moreover, since $K \subset G_\lambda \subset \mathrm{Diff}_0(M)$, any g_k is smoothly isotopic to the identity and hence $[g_k^*\omega_{\lambda+\epsilon}] = [\omega_{\lambda+\epsilon}]$. Therefore the forms $\omega_{k,\lambda+\epsilon}^t$ are cohomologous as we vary t. We now apply Moser's argument for the one parameter family of symplectic forms $\omega_{k,\lambda+\epsilon}^t$ and obtain a family of diffeomorphisms $\xi_{k,\lambda+\epsilon,t}$ with the property that $\xi_{k,\lambda+\epsilon,t}^*\omega_{k,\lambda+\epsilon}^t = \omega_{\lambda+\epsilon}$. We will now define $h(\epsilon,k) := g_k \circ \xi_{k,\lambda+\epsilon,1}$. Then h has the required properties.

For an arbitrary $h : [-\epsilon, \epsilon] \times K$ satisfying (4.5) we take the homotopy

$$F : [0,1] \times [-\epsilon,\epsilon] \times K \to \mathbb{R} \times \mathrm{Diff}_0 M$$

given by $F(t,\epsilon,k) := (\epsilon, h(t\epsilon, k))$.

This gives a homotopy between h and $h_0 : [-\epsilon,\epsilon] \times K \to \mathbb{R} \times \mathrm{Diff}_0 M$, where $h_0(\epsilon', k) = h(0,k)$. We similarly obtain a homotopy F' between h' and h_0, where h' also satisfies (4.5). By concatenating one homotopy with the opposite of the other we obtain a homotopy between h and h' which we call

$G : [0,1] \times [-\epsilon_1, \epsilon_1] \times K \to \mathbb{R} \times \mathrm{Diff} M$. Denote by $g_{s,\epsilon,k} := G(s,\epsilon,k)$. We will now follow the same procedure as before. Namely, we restrict to a short interval $[-\epsilon', \epsilon']$ such that, if we call

$$\omega^t_{s,k,\lambda+\epsilon} := t g^*_{s,\epsilon,k}\omega_{\lambda+\epsilon} + (1-t)\omega_{\lambda+\epsilon}$$

then these are symplectic, $\forall 0 \leq |\epsilon| < \epsilon'$ and $\forall t, s \in [0,1]$. This is possible because $\omega^t_{s,k,\lambda} = \omega_\lambda$. Again, the diffeomorphisms $g_{s,\epsilon,k}$ are smoothly isotopic with the identity and hence, as above, we apply Moser's argument to the isotopic forms $\omega^t_{s,k,\lambda+\epsilon}$ and we obtain diffeomorphisms $\xi_{s,k,\lambda+\epsilon,t}$ with the property that $\xi^*_{s,k,\lambda+\epsilon,t}\omega^t_{s,k,\lambda+\epsilon,} = \omega_{\lambda+\epsilon}$. We will now define $H(s,\epsilon,k) := g_{s,\epsilon,k} \circ \xi_{s,k,\lambda+\epsilon,1}$. Then H has the required properties. □

Definition 4.7. *Let $\rho : B \to G_\lambda$ be a cycle in G_λ. An* **extension** *ρ^ϵ of ρ is a smooth family of cycles $\rho^\epsilon : B \to G_{\lambda+\epsilon}$ defined for $|\epsilon| \leq \epsilon_0$ such that $\rho^0 = \rho$ and satisfying (4.6). Using 4.6* **(i)** *every cycle ρ has an extension.*

Observation : Consider two extensions $\rho^\epsilon_1, 0 \leq |\epsilon| < \epsilon_1$ and $\rho^\epsilon_2, 0 \leq |\epsilon| < \epsilon_2$. By (4.6) there is an $\epsilon' > 0$ and a homotopy between ρ^ϵ_1 and ρ^ϵ_2 defined for all $0 \leq \epsilon \leq \epsilon'$. Hence any extension provides well defined elements in $\pi_* G_{\lambda+\epsilon}$ for small values of ϵ. Therefore each $[\rho] \in \pi_*(G^X_\lambda)$ has an extension $[\rho^\epsilon] \in \pi_*(G^X_{\lambda+\epsilon})$ whose **germ** at $\epsilon = 0$ is independent of the choices of ρ.

Definition 4.8. *We say that a smooth family of elements $[\rho^\epsilon] \in \pi_* G_{\lambda+\epsilon}$, $0 < \epsilon < \epsilon_\rho$ is* **new** *if it is not the extension for $\epsilon > 0$ of any element $[\rho] \in \pi_* G_\lambda$.*

Definition 4.9. *We say that an element $[\rho] \in \pi_* G_\lambda$ is* **fragile** *if it admits a null homotopic extension to the right $0 = [\rho^\epsilon] \in \pi_* G_{\lambda+\epsilon}$, for $\epsilon > 0$.*

In the next section we will use the same letter ρ to refer both to cycles as well as to the homotopy class they represent.

4.4 The relation between almost complex structures and symplectomorphism groups

In this section we use the fibration (4.2) and standard methods in algebraic topology, to study how persistent elements in the relative homotopy groups of almost complex structures affect homotopy groups of symplectomorphisms. Our result is the following:

Theorem 4.10. *(On Symplectomorphism groups) Assume that we have a persistent element $0 \neq \beta_\ell \in \pi_k(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell, *)$ Then we can construct an element $\theta_\ell \in \pi_{k-2}G_\ell$ such that either:*

A) *The element $\theta_\ell \in \pi_{k-2}G_\ell$ is a non-zero* **fragile** *element.*

or

B) *$\theta_\ell = 0$ and then there is an $\epsilon_\ell > 0$ such that we can construct a family of* **new** *elements $0 \neq \eta_{\ell+\epsilon} \in \pi_{k-1}G_{\ell+\epsilon}, 0 < \epsilon < \epsilon_\ell$.*

Proof: We will consider the long exact sequence of relative homotopy groups of the pair $(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell)$

$$\cdots \longrightarrow \pi_k\mathcal{A}_{\ell^+} \longrightarrow \pi_k(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell) \longrightarrow \pi_{k-1}\mathcal{A}_\ell \longrightarrow \pi_{k-1}\mathcal{A}_{\ell^+} \longrightarrow \cdots$$

Since by construction $\beta_\ell \in \pi_k(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell)$ is nontrivial, then one of the two following cases can happen:

Case 1 $\beta_\ell \mapsto \gamma_\ell \neq 0 \in \pi_{k-1}\mathcal{A}_\ell$

Case 2 $\beta_\ell \mapsto 0 \in \pi_{k-1}\mathcal{A}_\ell$. In this situation, there is an element $0 \neq \alpha_\ell \in \pi_k\mathcal{A}_{\ell^+}$ where $\alpha_\ell \mapsto \beta_\ell$.

We will do the analysis case by case for our situation:

Case 1 If we are in this case then we consider the fibration (4.2), that yields

$$G_\ell \longrightarrow \mathrm{Diff}_0(M) \longrightarrow \mathcal{A}_\ell$$

We consider the long exact sequence in homotopy

$$\cdots \longrightarrow \pi_{k-1}(G_\ell) \longrightarrow \pi_{k-1}\mathrm{Diff}_0(M) \longrightarrow$$

$$\longrightarrow \pi_{k-1}\mathcal{A}_\ell \longrightarrow \pi_{k-2}G_\ell \longrightarrow \pi_{k-2}\mathrm{Diff}_0(M) \longrightarrow \cdots$$

Again, there are two possibilities:

i) $\gamma_\ell \to \theta_\ell \neq 0 \in \pi_{k-2}G_\ell$. In this situation, we have a nontrivial element $\theta_\ell \in \pi_{k-2}G_\ell$, such that $\theta_\ell \mapsto 0 \in \pi_{k-2}\mathrm{Diff}_0(M)$. Then we are in case **A**. This element is *fragile*. This can be proved by contrapositive. Assume that θ_ℓ can be extended by $\theta_{\ell+\epsilon}$ which yields nontrivial classes in $\pi_{k-2}G_{\ell+\epsilon}$. Then $\theta_{\ell+\epsilon} \mapsto 0 \in \pi_{k-2}\mathrm{Diff}_0(M)$ as well. Therefore it appears as a boundary of an element $\gamma_{\ell+\epsilon} \in \pi_{k-1}\mathcal{A}_{\ell+\epsilon}$ which is homotopic with γ_ℓ. But by construction and lemma (4.6), we know that γ_ℓ is a contractible cycle inside $\mathcal{A}_{\ell+\epsilon}$. This contradicts the existence of $\gamma_{\ell+\epsilon}$.

ii) $\gamma_\ell \mapsto 0 \in \pi_{k-2}G_\ell$. Then γ_ℓ is in the image of the morphism $\pi_{k-1}\mathrm{Diff}_0(M) \to \pi_{k-1}\mathcal{A}_\ell$, and therefore there is an element, $\gamma'_\ell \in \pi_{k-1}\mathrm{Diff}_0(M)$ such that $0 \neq \gamma'_\ell \mapsto \gamma_\ell$.

In this situation, we can choose a cycle $S \subset \mathcal{A}_\ell$ representing $\gamma_\ell \in \pi_{k-1}(\mathcal{A}_\ell)$, and, using lemma (4.6), there is an $\epsilon_S > 0$ such that for any ϵ such that $0 < \epsilon < \epsilon_S, S \subset \mathcal{A}_{\ell+\epsilon}$. We make the following

Claim (1) $0 = [S] \in \pi_{k-1}\mathcal{A}_{\ell+\epsilon}$.

By hypothesis S is the boundary of a cycle B_ℓ such that $B_\ell \subset \mathcal{A}_{\ell+\epsilon}$ for all small $\epsilon > 0$. Therefore we have a k dimensional ball inside $\mathcal{A}_{\ell+\epsilon}$ whose boundary is S, which proves the claim. We therefore have:

$$\gamma'_\ell \longrightarrow \quad [S] = 0 \in \pi_{k-1}\mathcal{A}_{\ell+\epsilon}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{k-1}(G_{\ell+\epsilon}) \arrow[r] & \pi_{k-1}\mathrm{Diff}_0(M) \arrow[r] \arrow[d, equal] & \pi_{k-1}\mathcal{A}_{\ell+\epsilon} \arrow[r] & \pi_{k-2}(G_{\ell+\epsilon}) \arrow[r] & \cdots \\
\cdots \arrow[r] & \pi_{k-1}(G_\ell) \arrow[r] & \pi_{k-1}\mathrm{Diff}_0(M) \arrow[r] & \pi_{k-1}\mathcal{A}_\ell \arrow[r] \arrow[u, dotted, "i_{|k}"'] & \pi_{k-2}(G_\ell) \arrow[r] & \cdots
\end{tikzcd}$$

$$\gamma'_\ell \longrightarrow \quad \gamma_\ell \in \pi_{k-1}\mathcal{A}_{\ell+\epsilon}$$

Here, from the first row, since γ'_ℓ is in the kernel of the map $\pi_{k-1}\mathrm{Diff}_0(M) \to \pi_{k-1}\mathcal{A}_{\ell+\epsilon}$, it has to be in the image of the map $\pi_{k-1}(G_{\ell+\epsilon}) \to \pi_{k-1}\mathrm{Diff}_0(M)$, and therefore we are able to produce an element $0 \neq \eta_{\ell+\epsilon} \in \pi_{k-1}(G_{\ell+\epsilon})$ such that $\eta_{\ell+\epsilon}$ persists in the topology of the group of diffeomorphisms. Thus we are in case **B**.

The elements we obtain here are *new*. This follows easily by assuming the opposite. That is, if we consider that there is an element $0 \neq \eta_\ell \in \pi_{k-1}G_\ell$ whose germ is given by $\eta_{\ell+\epsilon}$, then the image of η_ℓ in $\mathrm{Diff}_0(M)$ has to be γ'_ℓ. But this contradicts the fact that $\gamma'_\ell \mapsto \gamma_\ell \neq 0$.

Case 2. In this situation we have a nontrivial element $\alpha_\ell \in \pi_k\mathcal{A}_{\ell^+}$. We then have the following :

Claim (2) There is an ϵ such that for $0 < \delta < \epsilon$, the element α_ℓ has a representative C inside $\mathcal{A}_{\ell+\delta}$, $0 \neq [C] \in \pi_k\mathcal{A}_{\ell+\delta}$. The proof of this statement follows from the construction of α_ℓ from a persistent element and from the definition of $\mathcal{A}_{\ell^+}$.

Namely, since $\beta_\ell \mapsto 0 \in \pi_{k-1}\mathcal{A}_\ell$ we conclude that there exist a k-dimensional disk D inside $\mathcal{A}_\ell$ whose boundary is ∂B_ℓ; by lemma (4.6) **(i)** this can be viewed inside $\mathcal{A}_{\ell+\delta}$ for small δ. Moreover $B_\ell \subset \mathcal{A}_{\ell+\delta}$ for small δ, which follows from 2.3. We can now glue B_ℓ and D along their boundary ∂B_ℓ. In this manner we get a cycle $C \subset \mathcal{A}_{\ell+\delta}$ which represents the class α_ℓ Moreover, C does not contract inside $\mathcal{A}_{\ell+\delta}$, for small δ. This follows basically from the fact that β_ℓ is persistent. Namely, since β_ℓ maps to a trivial element in $\pi_{k-1}\mathcal{A}_\ell$, and since the relation 2.4 implies that the $i_*^\delta(\beta_\ell)$ is nontrivial then $i_*^\delta(\beta_\ell)$ has to lift also to a nontrivial element $i_*^\delta(\alpha_\ell)$ in $\pi_k\mathcal{A}_{[\ell,\ell+\delta]}$. Moreover, from its construction, the cycle C we have built also represents the element $i_*^\delta\alpha_\ell$. If C were to contract inside $\mathcal{A}_{\ell+\delta}$ then that would imply that it contracts inside $\mathcal{A}_{[\ell,\ell+\delta]}$ and it would mean that $i_*^\delta\alpha_\ell$ is trivial which is false.

We can therefore consider again the sequence

$$\cdots \longrightarrow \pi_k(G_{\ell+\delta}) \longrightarrow \pi_k\mathrm{Diff}_0(M) \longrightarrow$$

$$\longrightarrow \pi_k\mathcal{A}_{\ell+\delta} \longrightarrow \pi_{k-1}G_{\ell+\delta} \longrightarrow \pi_{k-1}\mathrm{Diff}_0(M) \longrightarrow \cdots$$

Claim (3) $[C]$ doesn't lift to $\pi_k\mathrm{Diff}_0(M)$.

Proof of claim (3): We should first make the observation that there is a map

$$\pi_k\mathrm{Diff}_0(M) \to \pi_k\mathcal{A}_\lambda \tag{4.7}$$

for any λ and moreover as λ varies this maps are homotopic in $\mathcal{A}_I$. If C did lift, the map $\pi_k\mathrm{Diff}_0(M) \to \pi_k\mathcal{A}_\ell$ would produce a cycle $[B] \in \mathcal{A}_\ell$, which by means of lemma (4.6) can be viewed inside all $\mathcal{A}_{\ell+\epsilon}$ for small ϵ and which moreover is homotopic with C inside $\mathcal{A}_{[\ell,\ell+\epsilon]}$. Therefore $[C]$ would map to $0 \in \pi_k(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell)$, which would contradict its definition. $\square$

Since $[C]$ cannot be in the image of the map $\pi_k \mathrm{Diff}_0(M) \to \pi_k \mathcal{A}_{\ell+\delta}$, we know that [C] must have nonzero image $[C] \mapsto \eta_{\ell+\delta} \neq 0$ in $\pi_{k-1} G_{\ell+\delta}$. Moreover from the obvious properties of exact sequences again, $\eta_{\ell+\delta} \to 0$ through the natural inclusion map $\pi_{k-1} G_{\ell+\delta} \to \pi_{k-1} \mathrm{Diff}_0(M)$. The fact that this elements are *new* follows again by assuming the opposite. If they were the **extension** of an element η_ℓ in $\pi_{k-1} G_\ell$, then η_ℓ would also be null homotopic inside $\mathrm{Diff}_0(M)$ so it would therefore come from a class $[C']$ in $\pi_k \mathcal{A}_\ell$. Moreover, C' would be homotopic with C inside $\mathcal{A}_{[\ell,\ell+\delta]}$ therefore also in $(\mathcal{A}_{[\ell,\ell+\delta]}, \mathcal{A}_\ell)$ which is false given that C has to yield a nontrivial element in $\pi_k(\mathcal{A}_{[\ell,\ell+\delta]}, \mathcal{A}_\ell)$. Thus we are in the case **B** of the theorem.

With this, we have exhausted all the possible cases given by the nontrivial PGW. $\square$

Now consider the manifold $(S^2 \times S^2 \times X, \omega_\lambda \oplus \omega_{st})$. As explained in (3.16) the cycles $(B^\ell, \partial B^\ell)$ satisfy the definition (2.3), so they give by Prop (2.6) **persistent** elements in $\pi_{4\ell-2}(\mathcal{A}_{\ell^+}, \mathcal{A}_\ell)$.

Therefore theorem (4.10) applies and so the following corollary holds:

Corollary 4.11. *For any natural number $\ell \geq 1$, exactly one of the statements below holds.*

A) *We can construct a non-zero* **fragile** *element $w_\ell^X \in \pi_{4\ell-4} G_\ell^X$, which can be identified with $w_\ell \times id$.*

B) *There exists an $\epsilon_\ell > 0$ for which we can construct a family of* **new** *elements $0 \neq \eta_{\ell+\epsilon}^X \in \pi_{4\ell-3} G_{\ell+\epsilon}^X, 0 < \epsilon < \epsilon_\ell$.*

Appendix A: A criterion of parametric regularity

In this appendix we give a proof of the regularity criterion stated in theorem 2.14, namely:

Theorem 4.12. *Let $(J_z, \omega_z)_{z\in\mathbb{C}^m}$ be a family on M descending from the symplectic fibration $(\widetilde{M}, \widetilde{J}, \widetilde{\omega})$*

$$\begin{tikzcd} M \arrow[r, "i"] & \widetilde{M} \arrow[d, "\pi"] \\ & \mathbb{C}^m \end{tikzcd} \tag{4.8}$$

Suppose that $f : \Sigma \longrightarrow M$ is a J_0 holomorphic map and consider the composite map

$$\widetilde{f} = i \circ f, \widetilde{f} : \Sigma \longrightarrow M \times 0 \subset \widetilde{M}$$

which is $\widetilde{J}$-holomorphic. If $\widetilde{f}$ is regular then f is (J_z) parametric regular. Moreover, if $\Sigma = S^2$ then the reverse statement holds.

Let $T_{|_{\pi^{-1}(0)}}\widetilde{M}$ be the tangent space along the preimage of $0 \in \mathbb{C}^m$. We will denote by H the subbundle of $T_{|_{\pi^{-1}(0)}}\widetilde{M}$ which is $\widetilde{\omega}$ orthogonal to the fiber $\{0\} \times M$. We would like H to coincide with the horizontal space of $T\widetilde{M}$ with respect to the trivialization π and to be $\widetilde{J}$ invariant. This can be arranged by deforming the form $\widetilde{\omega}$ so that near the zero fiber $\{0\} \times M$ it is given by

$$\widetilde{\omega} = \omega_0 + \pi^*(\sigma_{base}),$$

where σ_{base} is a standard symplectic two form on the holomorphic base B. Throughout this deformation process $\widetilde{J}$ is still $\widetilde{\omega}$ tamed.

Let g_0 be a metric on M_0 and ∇ be the Levi-Civita connection on M associated with it. ∇^{st} will be the standard Levi-Civita connection on $\mathbb{C}^m$. We will denote from now on $\widetilde{\nabla} = \nabla \times \nabla^{st}$, the product connection on $\widetilde{M} \simeq \mathbb{C}^m \times M$. The regularity of $\widetilde{f} : \Sigma \longrightarrow \widetilde{M}$ is by definition, equivalent to the fact that $D_{\widetilde{f}}$ is surjective, where $D_{\widetilde{f}}$ is the linearization of $\bar{\partial}$,

$$D_{\widetilde{f}} : C^\infty(\widetilde{f}^* T\widetilde{M}) \longrightarrow \Omega^{0,1}_{\widetilde{J}}(\Sigma, \widetilde{f}^* T\widetilde{M}).$$

Using the connection $\widetilde{\nabla}$ we will derive formulas for $D_{\widetilde{f}}$ and express them in terms of the linearization $D\Phi$.

Since $\widetilde{M} \simeq \mathbb{C}^m \times M$ and $\mathrm{im}\widetilde{f} \subset \{0\} \times M$, we have the following relations:

$$\widetilde{f}^* \left(T\widetilde{M} \right) = \widetilde{f}^* \left(T\widetilde{M}_{\pi^{-1}(0)} \right) = \widetilde{f}^*(H \oplus TM) = triv \oplus f^*(TM)$$

where by $triv$ we denote the trivial m-dimensional complex bundle over Σ. This gives

$$C^\infty(\widetilde{f}^* T\widetilde{M}) \simeq C^\infty(triv) \oplus C^\infty(f^* TM) \tag{4.9}$$

Given that each fiber is $\widetilde{J}$ invariant, and that H is $\widetilde{J}$ invariant along $\pi^{-1}(0)$, we obtain

$$\Omega^{0,1}_{\widetilde{J}}(\Sigma, \widetilde{f}^*T\widetilde{M}) \simeq \Omega^{0,1}_{J}(\Sigma, f^*TM) \oplus \Omega^{0,1}_{\widetilde{J}}(\Sigma, H) \tag{4.10}$$

From (4.9) and (4.10) we obtain

$$D_{\widetilde{f}} : C^{\infty}(triv) \oplus C^{\infty}(f^*TM) \longrightarrow \Omega^{0,1}_{J}(\Sigma, f^*TM) \oplus \Omega^{0,1}_{\widetilde{J}}(\Sigma, H)$$

and by considering the appropriate restrictions we obtain the following operators

$$\begin{array}{rccc} D_{1,vert} : & C^{\infty}(triv) & \longrightarrow & \Omega^{0,1}_{J}(\Sigma, f^*TM) \\ D_{1,hor} : & C^{\infty}(triv) & \longrightarrow & \Omega^{0,1}_{\widetilde{J}}(\Sigma, H) \\ D_{2,vert} : & C^{\infty}(f^*TM) & \longrightarrow & \Omega^{0,1}_{J}(\Sigma, f^*TM) \\ D_{2,hor} : & C^{\infty}(f^*TM) & \longrightarrow & \Omega^{0,1}_{\widetilde{J}}(\Sigma, H) \end{array}$$

We will sometimes use $D_k = (D_{k,vert}, D_{k,hor}),\ k = 1, 2.$

To compute the formulas for these operators we will use the following general method (see [1]).

Consider $\xi \in C^{\infty}(\Sigma, \widetilde{f}^*T\widetilde{M})$ and $\widetilde{F}_{\xi} : [0,1] \times \Sigma \longrightarrow \widetilde{M}$ given by

$$\widetilde{F}_{\xi}(t, x) = \exp^{\widetilde{\nabla}}_{\widetilde{f}(x)}(t\xi(x)),$$

for ξ sufficiently small. Let $s : \Sigma \longrightarrow T\Sigma$ be a section and $\widetilde{s}$ its lift to $T([0,1] \times \Sigma)$. We denote $\frac{\partial}{\partial t}$ the vector field in $T([0,1] \times \Sigma)$ corresponding to the parameter in $[0,1]$. Define $\widetilde{f}_t(x) := \widetilde{F}_{\xi}(t, x)$. For any $x \in \Sigma$, define the path $\widetilde{\gamma}^{\xi}_x : [0,1] \longrightarrow \widetilde{M}$ given by

$$\widetilde{\gamma}^{\xi}_x(t) = \widetilde{F}_{\xi}(t, x),$$

the image under $\widetilde{F}_\xi$ of $[0,1]\times x$ in $\widetilde{M}$. By the definition of $\widetilde{F}_\xi$, $\widetilde{\gamma}^\xi_x$ is a geodesic path in $\widetilde{M}$ relative to the connection $\widetilde{\nabla}$.

Denote by $\tau^\xi_{t,x} : T_{\gamma_x(t)}\widetilde{M} \longrightarrow T_{\gamma_x(0)}\widetilde{M}$ the parallel transport in $\widetilde{M}$ along the curve $\gamma_x := \widetilde{\gamma}^\xi_x$. To compute $D_{\widetilde{f}}(\xi)(s)$ in general, one needs to consider the expression $\frac{1}{2}\tau^\xi_{t,x}(d\widetilde{f}_t(s) + \widetilde{J}d\widetilde{f}_t(js))$ and take its derivative with respect to t at $t = 0$ i.e.

$$D_{\widetilde{f}}(\xi)(s) = \frac{1}{2}\frac{\partial}{\partial t}\left(\tau^\xi_{t,x}(d\widetilde{f}_t(s) + \widetilde{J}d\widetilde{f}_t(js))\right)_{|t=0} \tag{4.11}$$

We define $Const$ to be the subspace of $C^\infty(triv)$ made out of constant sections. For the proof of the theorem, we are particularly interested in computing $D_{1,hor}$ and the restriction of $D_{1,vert}$ to $Const$.

In order to simplify the notation, we denote by x the coordinate on Σ and write the points in $\mathbb{C}^m \times M$ as $(z_1,\ldots,z_m,y)$ where $z_1 = w_1 + iv_1$ and so on. For simplicity we denote the vector field in $Const$ by $\frac{\partial}{\partial w_k} = \partial_{w_k}$ and so on. Since we are going to work with an arbitrary choice of w_k and v_k we will refer to them simply as ∂_w, unless we need to be more specific.

Lemma 4.13. *The following relations hold:*

i) $D_{2,hor} = 0$

ii) $D_{2,vert} = D_f$

iii) $D_{1,hor}(\xi) = \bar{\partial}_{\mathbb{C}^m}(\xi), \forall\xi \in C^\infty(triv)$*, where* $\bar{\partial}_{\mathbb{C}^m}$ *is the delbar operator in* $\mathbb{C}^m$.

iv) $(D_{1,vert})(\partial_z)(s) = \frac{1}{2}\frac{\partial}{\partial z}(J(z))_{|z=0}(df(js)$ *for* ∂_z *a typical vector field in* $Const \subset C^\infty(triv)$*.*

Proof: Since $\widetilde{f} = f\circ i \subset \{0\}\times M$ we can naturally view any $\xi \in C^\infty(f^*TM)$ as an element in $C^\infty(\widetilde{f}^*T\widetilde{M})$ with values in the vertical direction tangent to

$\{0\} \times M$. We have that

$$\widetilde{F}_\xi(t,x) = \exp^{\widetilde{\nabla}}_{\widetilde{f}(x)}(t\xi) = \exp^{\nabla}_{f(x)}(t\xi),$$

with $\mathrm{im}\widetilde{F} \subset \{0\} \times M$. This implies that the $d\widetilde{f}_t(s)$ are also vertical vector fields supported in $\{0\} \times M$ and, since $\widetilde{J}$ keeps $T(\{0\} \times M)$ invariant, we have as well that the $\widetilde{J}d\widetilde{f}_t(js)$ are vertical vector fields in $\{0\} \times M$. Similarly, $\widetilde{F}_\xi^* \frac{\partial}{\partial t}$ is a vertical section in $T\widetilde{M}$ supported in $\{0\} \times M$ and parallel transport along $\widetilde{f}(x)$ with respect to $\widetilde{\nabla}$ is the same as parallel transport with respect to ∇.

A direct application of (4.11) is that

$$\left(D_{\widetilde{f}}\xi\right)(s) = \frac{1}{2}\frac{\partial}{\partial t}\left(\tau^\xi_{t,x} d\widetilde{f}_t(s) + \tau^\xi_{t,x}\widetilde{J}df_t(js)\right)_{|t=0} = \left(D_f\xi\right)(s),$$

which proves (**i**). Relation (**ii**) follows immediately from the formula above, taking into account that $D_{\widetilde{f}}\xi = D_{2,vert}(\xi)$, and that $\mathrm{im}D_{\widetilde{f}}|_{C^\infty f^*TM} \subset \Omega^{0,1}_J(\Sigma, f^*TM)$.

For the proofs of (**iii**) and (**v**) we now consider $\xi \in C^\infty(triv)$. We can assume $\xi = \phi(x)\partial_w$ where $\phi : \Sigma \leftarrow \mathbb{C}^m$. In this situation,

$$\widetilde{F}_\xi(t,x) = \exp^{\widetilde{\nabla}}_{\widetilde{f}(x)}(t\partial_w) = (\phi(x)t, 0, \ldots, 0, f(x))$$

It then follows that the paths γ_x are straight lines in $\mathbb{C}^n \times f(x) \subset \widetilde{M}$ and therefore the parallel transport along γ_x, $\tau_{t,x} : T_{(t,f(x))}\widetilde{M} \longrightarrow T_{0,f(x))}\widetilde{M}$ is the identity. We are also going to consider the coordinates $x \in \Sigma$ of the type $x = x_1 + ix_2$, and do our computations for $s = \partial_{x_1}$.

If $\widetilde{J}(t)$ is the almost complex structure at $\widetilde{\gamma}_x^\xi(t)$ then

$$\widetilde{J}(t) = \begin{pmatrix} A_t & 0 \\ B_t & J_t \end{pmatrix}$$

with respect to the product structure $\mathbb{C}^m \times M$. Moreover along $\pi^{-1}(0)$ we have

$$\widetilde{J}(0) = \begin{pmatrix} J_{\mathbb{C}^m} & 0 \\ 0 & J_t \end{pmatrix}$$

Therefore $\frac{\partial}{\partial t}\widetilde{J}(t)$ preserves the fibers, the same as $\widetilde{J}(t)$ does. Moreover, along $\{0\}\times M$, $\widetilde{J}(0)$ preserves the splitting into TM and H. As we have seen parallel transport along $\widetilde{\gamma}_x^\xi(t)$ is just the identity.

Considering local coordinates on Σ given by $x = x_1 + ix_2$ and taking $s = \partial_{x_1}$, we have:

$$\begin{aligned}
D_{1,hor}(\phi\partial_w)(\partial_{x_1}) &= \tfrac{1}{2} proj_H \tfrac{\partial}{\partial t}\left(\tau_{t,x}^\xi d\widetilde{f}_t(\partial_{x_1}) + \tfrac{1}{2}\tau_{t,x}^\xi \widetilde{J} d\widetilde{f}_t(j\partial_{x_1})\right)_{|t=0} \\
&= \tfrac{1}{2} proj_H \tfrac{\partial}{\partial t}\left(d\widetilde{f}_t(\partial_{x_1}) + \tfrac{1}{2}\widetilde{J} d\widetilde{f}_t(\partial_{x_2})\right)_{|t=0} \\
&= \tfrac{1}{2}\tfrac{\partial}{\partial t}\left(\partial_{x_1}(\phi(x))t, 0, \ldots, 0\right)_{|t=0} + \tfrac{1}{2} proj_H \tfrac{\partial}{\partial t}\left(\widetilde{J}_t\right)_{|t=0} df(\partial_{x_2}) + \\
&\quad \tfrac{1}{2} proj_H \widetilde{J}_0 \tfrac{\partial}{\partial t}\left(\partial_{x_2}(\phi(x))t, 0, \ldots, 0, df(x)\right)_{|t=0}
\end{aligned}$$

where, as mentioned before, $\phi : \Sigma \to \mathbb{C}^m$. But here the middle term vanishes because $df(\partial_{x_2})$ is a vertical vector and $\frac{\partial}{\partial t}\widetilde{J}$ preserves fibers so we get that $\frac{\partial}{\partial t}\left(\widetilde{J}_t\right)_{|t=0} df(\partial_{x_2})$ is also a vertical vector. Then

$$D_{1,hor}(\phi\partial_w)(\partial_{x_1}) = \frac{1}{2}\partial_{x_1}\phi(x) + \frac{1}{2}J_{\mathbb{C}^m}(\partial_{x_2})\phi(x) \tag{4.12}$$

For the last expression we have to use that along $\pi^{-1}(0)$, $\widetilde{J}_0$ preserves the horizontal space H, so $proj_H \circ \widetilde{J}_0 = \widetilde{J}_{\mathbb{C}^m} \circ proj_H$. Therefore, the conclusion follows that

$$D_{1,hor} = \bar{\partial}_{\mathbb{C}^m}$$

To prove point **(v)** of the theorem we need to consider now $\xi = \partial_w$, that is $\xi \in Const$. Under this assumption we have $\tau_{t,x}^{\partial_w} d\widetilde{f}_t = df_0$. Thus

$$\frac{\partial}{\partial t}\tau_{t,x}^{\partial_w} d\widetilde{f}_t(s) = 0.$$

As before, s is a just a section in $T\Sigma$. We then have

$$\begin{aligned} D_{1,vert}(\partial_w)(s) &= \tfrac{1}{2} proj_V \tfrac{\partial}{\partial t}\left(\tau_{t,x}^{\partial_w} d\widetilde{f}_t(s) + \tfrac{1}{2}\tau_{t,x}^{\partial w}\widetilde{J} d\widetilde{f}_t(js)\right)_{|t=0} \\ &= \tfrac{1}{2} proj_V \tfrac{\partial}{\partial t}\left(\tau_{t,x}^{\partial_w} d\widetilde{f}_t(s)\right)_{|t=0} + \tfrac{1}{2} proj_V \tfrac{\partial}{\partial t}\left(\tau_{t,x}^{\partial_w}\widetilde{J}(\tau_{t,x}^{\partial_w})^{-1}\right)_{|t=0} df(js) \\ &\quad + \tfrac{1}{2} proj_V \widetilde{J}_0 \left(\tfrac{\partial}{\partial t}\tau_{t,x}^{\partial_w} d\widetilde{f}_t(js)\right)_{|t=0} = \tfrac{1}{2} proj_V \left(\widetilde{\nabla}_{\partial_w}\widetilde{J}\right) df(js) \end{aligned}$$

where we denote by $proj_V$ the projection onto the fibers. Recall that $\frac{\partial}{\partial t}\widetilde{J}$ takes vertical vector fields into vertical vector fields. Therefore

$$\frac{1}{2} proj_V \widetilde{\nabla}_{\partial_w}\widetilde{J} df(js) = \frac{1}{2}\frac{\partial}{\partial w}(J(z))(df(js))$$

precisely because $df(js)$ is a vertical vector field and because the covariant derivative along horizontal vector fields was chosen to be the standard connection in $\mathbb{C}^m$. Applying the same reasoning for $i\partial_v$ we see that

$$(D_{1,vert})(\partial_z)(s) = \frac{1}{2}\frac{\partial}{\partial z}(J(z))_{|z=0}(df(js))$$

It is worth to point out that $\frac{\partial}{\partial z}(J(z)_{|z=0} = d\psi_0^*(\frac{\partial}{\partial z})$.

$\square$

Proof of the theorem:

Implication "⇒"

Using lemma 4.13, point **(v)** we get the commutativity of the following diagram

$$\begin{tikzcd} T_0\mathbb{C}^m \arrow[r, "d\psi"] \arrow[d, "i"] & T_J\mathcal{A}_I \arrow[d, "R"] \\ Const \arrow[r, "D_{1,vert}"] & \Omega_J^{0,1}(\Sigma, f^*TM), \end{tikzcd} \tag{4.13}$$

where $i : T_0\mathbb{C}^n \to Const \subset C^\infty(triv)$ is the natural identification map and ψ is the morphism from the parameter space to the space of almost complex structures. R is, as mentioned before, given by $R(Y) = \frac{1}{2}Y \circ df \circ j$.

Since $D_{\tilde{f}}$ is surjective by the hypothesis of this implication, this means that $D_1 \oplus D_2$ is surjective. We therefore have, by lemma (4.13) **(i)**,**(ii)**,

$$D_1 = (D_{1,vert}, D_{1,hor}) : C^\infty(triv) \longrightarrow \mathrm{coker} D_f \oplus \Omega_{\tilde{J}}^{0,1}(\Sigma, H) \tag{4.14}$$

is surjective. Since the kernel of the $\bar{\partial}_{\mathbb{C}^m}$ operator on $\mathbb{C}^m$ consists precisely of constant sections, lemma 4.13 **(iii)** implies that $D_{1,hor}^{-1}(0) = Const$. Therefore we have that the operator

$(D_{1,vert})_{|Const} : Const \longrightarrow \mathrm{coker} D_f$ is surjective. But this will imply that $D_{1,vert}{}_{|Const} \circ i : T_0\mathbb{C}^m \twoheadrightarrow \mathrm{coker} D_f$. But as we saw in the proof of 2.11, R induces an isomorphism $\widetilde{R} : \widetilde{\mathrm{coker} d\Pi} \longrightarrow \mathrm{coker} D_2$ and moreover the diagram 4.13 will be still commutative if we restrict $d\psi$ and $D_{1,vert}$ to $\mathrm{coker} d\Pi$ and cokerD_2 respectively. Therefore $d\psi : T_0\mathbb{C}^n \longrightarrow \mathrm{coker} d\Pi.$ is surjective. By proposition 2.11, this yields exactly the parametric regularity.

For the inverse implication, we notice that since $D_{1,hor}$ is $\bar{\partial}_{\mathbb{C}^m}$, it will cover the space $\Omega^{0,1}_{\tilde{J}}(\Sigma, H)$ when $\Sigma = S^2$. By hypothesis we have that $d\psi : T_0\mathbb{C}^n \longrightarrow \mathrm{coker} d\Pi.$ is surjective and the above observation implies that

$$D_1 = (D_{1,vert}, D_{1,hor}) : C^\infty(triv) \longrightarrow \mathrm{coker} D_f \oplus \Omega^{0,1}_{\tilde{J}}(\Sigma, H) \tag{4.15}$$

is also surjective. Therefore $D_{\tilde{f}}$ is a surjective operator. $\square$

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