Normal Two Dimensional Elliptic Singularities

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Abstract of the Dissertation

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Let p be a singularity of a normal two dimensional Stein space V with p as its only singularity. Let $\pi: M \to V$ be the minimal resolution of V with nonsingular A_i's and normal crossings, where the A_i's are the irreducible components of the exceptional set $A = \pi^{-1}(p)$. Suppose p is a weakly elliptic singularity. We introduce the concept of an elliptic sequence. This is defined purely topologically. Whenever the canonical divisor K' supported on A exists, we prove that -K' is actually equal to the summation of the elliptic sequence if π is the minimal resolution. Moreover $\dim H^1(M,\mathcal{O}) \leq$ the length of the elliptic sequence. A weakly elliptic singularity is called a maximally elliptic singularity if K' exists and $\dim H^1(M,\mathcal{O})$ = the length of the elliptic sequence. Maximally elliptic singularities may have $\dim H^1(M,\mathcal{O})$ arbitrary large. In case the length of the elliptic sequence is equal to one, then the singularity is minimally elliptic in the sense of Laufer. If

K' exists and the length of the elliptic sequence is equal to two, then p is called an almost minimally elliptic singularity. Minimally elliptic singularities and almost minimally elliptic singularities with ${}_V\mathcal{O}_p$ Gorenstein are maximally elliptic singularities Let m be the maximal ideal in ${}_V\mathcal{O}_p$. We prove that maximally elliptic singularities have ${}_V\mathcal{O}_p$ Gorenstein. For maximally elliptic singularities, if $Z_E \cdot Z_E \leq -2$ where Z_E is the fundamental cycle on $|E|$, then $m\mathcal{O} = \mathcal{O}(-Z)$. If $Z_E \cdot Z_E \leq -3$ and p is a hypersurface maximally elliptic singularity, then the Hilbert function for ${}_V\mathcal{O}_p$ is given by $-nZ \cdot Z$.

It is known that $\dim H^1(M,\mathcal{O})$ is independent of the resolution. We prove that if $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein, then p is a weakly elliptic singularity. Let $Z_{B_0}, ---, Z_{B_\ell}, Z_E$ be the elliptic sequence. We prove that $m\mathcal{O} \subseteq \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$. If $Z_E \cdot Z_E \leq -2$, then $m\mathcal{O} = \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$. In particular, the multiplicity of the singularity $\geq -\sum_{i=0}^{\ell} Z_{B_i}{}^2$ and the equality holds if $Z_E \cdot Z_E \leq -2$. If $Z_E \cdot Z_E \leq -3$, then the Hilbert function $\dim m^n/m^{n+1}$ for ${}_V\mathcal{O}_p$ is given by $-n(\sum_{i=0}^{\ell} Z_{B_i}{}^2)$. Examples show that these kinds of results are sharp. Laufer has an example which shows that $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ Gorenstein do not imply that p is an almost minimally elliptic singularity. However, a partial converse is shown for hypersurface singularities. We are able to list all possible weighted dual graphs for hypersurface singularities with $H^1(M,\mathcal{O}) = \mathbb{C}^2$. We prove that for hypersurface singularities, if $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and $H^1(|E|,\mathbb{Z}) = 0$,

then it is an almost minimally elliptic singularity. In fact, for hypersurface singularities, if $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and $Z_E \cdot Z_E \leq -2$, then it is an almost minimally elliptic singularity. For an almost minimally elliptic singularity p with ${}_V\mathcal{O}_p$ Gorenstein, p is absolutely isolated provided that $Z_E \cdot Z_E \leq -3$. In fact, after blowing up p at its maximal ideal, one obtains only rational double points and a minimally elliptic singularity. Examples also show that this result is sharp.

We are able to give a complete list of all weighted dual graphs for weakly elliptic double points by using the fact that -K' = the summation of an elliptic sequence. Moreover, each of these weighted dual graph, a typical defining equation is given. Later, we get a lower estimate on the dimension of Zariski tangent space of general two dimensional normal singularity in terms of the fundamental cycle Z,

$$\dim m/m^2 \geq \chi(Z) - Z \cdot Z + \dim H^1(M,\mathcal{O}(-Z)) - \dim H^1(M,\mathcal{O}(-2Z)).$$

This kind of estimate is sharp in the sense that equality holds for certain singularities. In case of maximally elliptic singularities, we show that $\dim H^1(M,\mathcal{O}(-Z)) = \dim H^1(M,\mathcal{O}(-2Z))$. In particular for maximally elliptic hypersurfaces, $Z \cdot Z \geq -3$.

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LIST OF SYMBOLS

V = two dimensional analytic space

${}_V\mathcal{O}$ = the sheaf of germs of holomorphic functions on V.

${}_V\mathcal{O}_p$ = the stalk of the sheaf ${}_V\mathcal{O}$ over p.

$\mathcal{O}_V$ = the set of holomorphic functions on V.

E = minimally elliptic cycle.

Z = fundamental cycle.

Ω = canonical sheaf, i.e. the sheaf of germs of holomorphic 2-forms.

m = maximal ideal of ${}_V\mathcal{O}_p$.

$|D|$ = support of the divisor D.

Let $D = \Sigma\, d_iA_i$, $F = \Sigma f_iA_i$ be two cycles on complex two dimensional manifold M.

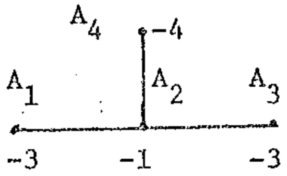
$\inf(D,F) = \sum_i \inf\,(d_i,f_i)A_i$.

Let F be a coherent sheaf on M.

$H^i_*(M,F)$ = cohomology with compact support.

Convention of weighted dual graphs: vertices without specifying genera are of genus zero. We record the multiplicity z_i of A_i in the fundamental cycle $Z = \Sigma z_iA_i$ by placing that integer in the corresponding position of the vertex

e.g.



$$Z = \begin{matrix} & 1 & \\ 1 & 3 & 1 \end{matrix} = A_1+3A_2+A_3+A_4$$

Let $D = \Sigma d_iA_i$ be a positive cycle. Let $B \subseteq |D|$. Then $D/B = \Sigma f_iA_i$ is a positive cycle where $f_i = d_i$ if $A_i \subseteq B$ and $f_i = 0$ if $A_i \nsubseteq B$.

ξ_p denotes the point bundle at p.

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INTRODUCTION

The classification of normal two dimensional singularities can be studied by the resolution of singularities. The resolution problem has been studied by Zariski [40], Hirzebruch [13], Hironaka [11], Brieskorn [5] and Abhyankar [1]. In resolving a two dimensional singularity p, one replaces p by a compact analytic space A. Because p is 2-dimensional, A is 1-dimensional. Let $A = \cup A_i$ be the decomposition of A into irreducible components. Thus, each A_i is a (possibly singular) Riemann surface. It is easy to reduce all considerations to the case where the A_i are nonsingular, intersect transversely, and no three meet at a point. There is a purely topological but very important criterion due to Grauert [7] and Mumford [26] which says that A comes from a resolution if and only if the intersection matrix $(A_i \cdot A_j)$ is negative definite.

The classification problem of isolated singularities of complex surfaces have been studied from various stand points. Taut singularities in the sense of Tyurina [35] have been studied by Grauert [7], Brieskorn [4], Laufer [22] and Wagreich [38]. The analytic structures of the taut singularities are determined by the topological information of their weighted dual graphs. The topological classification of normal two dimensional singularities has been studied by Mumford [26], Wagreich [37], [38] and Brieskorn [4].

Let p be a singularity of a normal two dimensional analytic space V. In 1964 M. Artin introduced a definition for p to be rational. Rational singularities have also been studied by, for instance, DuVal [6], Tyurina [34], Lipman [25], and Laufer [20]. In 1970, Wagreich introduced a definition for p to be weakly elliptic. Let us recall the definition. Let $\pi: M \to V$ be the resolution of V and $A = \pi^{-1}(p)$ be the exceptional set. Let Z be the fundamental cycle [2 p.132] of A. Let $\mathcal{O}(-Z)$ be the sheaf of germs of holomorphic functions on M whose divisors are at least Z. Let $\mathcal{O}_Z = \mathcal{O}/\mathcal{O}(-Z)$. Then $\chi(Z) = \dim H^o(M,\mathcal{O}_Z) - \dim H^1(M,\mathcal{O}_Z)$ may be computed from the weighted dual graph Γ via the Riemann - Roch Theorem. Weak ellipticity is $\chi(Z) = 0$. The conditions for p to be weakly elliptic is in fact independent of the choice of the resolution [37 p.423]. In [24], Laufer defines a cycle $E > 0$ to be minimally elliptic if $\chi(E) = 0$ and $\chi(D) > 0$ for all cycles D such that $0 < D < E$. In the case of weakly elliptic singularities he proved that there exists a unique minimally elliptic cycle E. Weakly elliptic singularities have occurred naturally in papers by Grauert [7], Hirzebruch [12], Orlik and Wagreich [27], [28], Wagreich [38] and Laufer [22]. Karras and Saito have studied some of these particular weakly elliptic singularities. Recently, Laufer developed a theory for a general class of weakly elliptic singularities which satisfy a minimality condition. He proved that p is minimally elliptic if and only if $H^1(M,\mathcal{O}) = \mathbb{C}$ and ${}_V\mathcal{O}_p$ is Gorenstein. If $Z \cdot Z \leq -3$, then the Hilbert function for the ring ${}_V\mathcal{O}_p$ is given by $-nZ \cdot Z$. Also, the singularity p is absolutely isolated. After blowing up p at its maximal ideal, one obtains only rational double points as singularities. If $Z \cdot Z = -1$ or -2, then p is a double point.

Suppose p is a weakly elliptic singularity. When π is the minimal good resolution, we introduce the concept of an elliptic sequence. This is defined purely topologically. Whenever the canonical divisor K' supported on A exists, we prove that $-K'$ is actually equal to the summation of the elliptic sequence if π is the minimal resolution. Weakly elliptic singularities can be effectively studied by elliptic sequences. We prove that for weakly elliptic singularities, $\dim H^1(M,\mathcal{O}) \leq$ the length of the elliptic sequence. A weakly elliptic singularity is called a maximally elliptic singularity if K' exists and $\dim H^1(M,\mathcal{O}) =$ the length of the elliptic sequence. Maximally elliptic singularities may have $\dim H^1(M,\mathcal{O})$ arbitrarily large. In case the length of the elliptic sequence is equal to one, then the singularity is minimally elliptic in the sense of Laufer. If K' exists and the length of the elliptic sequence is equal to two, then p is called an almost minimally elliptic singularity. Minimally elliptic singularities and almost minimally elliptic singularities with ${}_V\mathcal{O}_p$ Gorenstein are maximally elliptic singularities. We prove that maximally elliptic singularities have ${}_V\mathcal{O}_p$ Gorenstein. For maximally elliptic singularities, if $Z_E \cdot Z_E \leq -2$ where Z_E is the fundamental cycle on $|E|$, then $m\mathcal{O} = \mathcal{O}(-Z)$. If $Z_E \cdot Z_E \leq -3$ and p is a hypersurface maximally elliptic singularity, then the Hilbert function of ${}_V\mathcal{O}_p$ is equal to $-nZ \cdot Z$.

Rational singularities have $H^1(M,\mathcal{O}) = 0$. The hypersurface singularities are actually double points. For $H^1(M,\mathcal{O}) = \mathbb{C}$, Laufer was able to list all weighted dual graphs of hypersurface singularities. It is a natural question to ask for a theory for those singularities with $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ Gorenstein. It should be mentioned that hypersurface singularities and complete intersections are Gorenstein. We can prove that if $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein, then p is a weakly elliptic singularity. Let $Z_{B_0}, \text{———}, Z_{B_\ell}, Z_E$ be the elliptic sequence. Let m be the maximal ideal in ${}_V\mathcal{O}_p$. We prove that $m\mathcal{O} \subseteq \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$. If $Z_E \cdot Z_E \leq -2$, then $m\mathcal{O} = \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$. In particular, the multiplicity of the singularity $\geq -\sum_{i=0}^{\ell} Z_{B_i}{}^2$ and equality holds if $Z_E \cdot Z_E \leq -2$. If $Z_E \cdot Z_E \leq -3$, then the Hilbert function $\dim m^n/m^{n+1}$ for ${}_V\mathcal{O}_p$ is given by $-n(\sum_{i=0}^{\ell} Z_{B_i}{}^2)$. Examples show that these kinds of results are sharp. Laufer has an example which shows that $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ Gorenstein do not imply that p is an almost minimally elliptic singularity. However, a partial converse is shown for hypersurface singularities. We are able to list all possible weighted dual graphs for hypersurface singularities with $H^1(M,\mathcal{O}) = \mathbb{C}^2$.

We prove that for hypersurface singularities, if $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and $H^1(|E|, Z) = 0$, then it is an almost minimally elliptic singularity. In fact, for hypersurface singularities, if $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and $Z_E \cdot Z_E \leq -2$, then it is an almost minimally elliptic singularity. For an almost minimally elliptic singularity p with ${}_V\mathcal{O}_p$ Gorenstein, p is absolutely isolated provided that $Z_E \cdot Z_E \leq -3$. In fact, after blowing up p at its maximal ideal, one obtains only rational double points and a minimally elliptic singularity. Examples also show that this result is sharp.

One of the important questions in normal two dimensional singularities is "the classification of all weighted dual graphs for hypersurface singularities". Double points are hypersurface singularities. In 1970, Wagreich proved that for double points, $Z \cdot Z \geq -2$. Using this fact, he listed most of the possible weighted dual graphs of weakly elliptic double points. Using the fact that $-K'$ = the summation of an elliptic sequence and a combinatorial argument, we give a complete list of all weighted dual graphs for weakly elliptic double points. Moreover, each of these weighted dual graphs, a typical defining equation is given. Later, we get a lower estimate on the dimension of Zariski tangent space of general two dimensional normal singularity in terms of the fundamental cycle Z,

$$\dim m/m^2 \geq \chi(Z) - Z \cdot Z + \dim H^1(M,\mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-2Z))$$

This kind of estimate is sharp in the sense that equality holds for certain singularities. In case of maximally elliptic singularities, we show that $\dim H^1(M,\mathcal{O}(-Z)) = \dim H^1(M,\mathcal{O}(-2Z))$. In particular for maximally elliptic singularity, $Z \cdot Z \geq -3$. This enables us to list all the possible maximally elliptic hypersurface singularities. However, the list is too long to be included.

In Chapter II, we introduce the concept called maximal ideal cycle Y. Whenever the canonical divisor K' supported on A exists, we prove that Y cannot be greater than -K'. In Chapter VI, we give a necessary and sufficient conditions for a weakly elliptic singularity to be Gorenstein or maximally elliptic. A weakly elliptic singularity is said to be quasi-simple elliptic if the minimally elliptic cycle consists of elliptic curve. It is known that if ${}_V\mathcal{O}_p$ is Gorenstein, then the canonical divisor supported on A exists. Conversely if Γ is a weighted dual graphs of an almost minimally quasi-simple elliptic singularity and the canonical divisor K' exists, then there exists a Gorenstein structure for an associated singularity.

CHAPTER I

PRELIMINARIES

For the sake of convenience to readers, we include the basic knowledge for reading this paper. Most of these can be found in [24].

Let V be a complex analytic subvarity of a domain in $\mathbb{C}^m$ given by $V = \{z = (z_1,z_2,\ldots,z_m) : f_i(z) = 0,\ i = 1,2,3,\ldots,r\}$ Let $V = \bigcup_{i=1}^{k} V_i$ be the decomposition of V into irreducible components.

Definition 1.1 $\dim V = \max_{1\leq i\leq k} \dim V_i$

Definition 1.2 A point $p \in V$ is a regular point of V if the jacobian $(\frac{\partial f_i}{\partial z_j})(p)$, $1 \leq j \leq m$, $i \in I$ where I is a subset of $(1,2,\ldots,r)$ and $\{f_i\}_{i\in I}$ is a minimal set of defining equations for V at p, has maximal rank. If p is not a regular point of V, p is called a singular point of V. A singular point p of V is called a two dimensional singularity of V if V is two dimensional near p.

Definition 1.3 A germ h of a function defined on the regular points of V near p is said to be weakly holomorphic at p if h is holomorphic on the regular points near p and locally bounded near p, Let $\tilde{\mathcal{O}}$ and $\mathcal{O}$ be respectively the sheaf of germs of weakly holomorphic functions and sheaf of germs of holomorphic functions on V. There is a natural inclusion $\mathcal{O} \subset \tilde{\mathcal{O}}$. V is normal at p if

$\mathcal{O}_p \subset \tilde{\mathcal{O}}_p$ is an isomorphism. V is normal if $\mathcal{O} \cong \tilde{\mathcal{O}}$, i.e. if V is normal at each of its points.

Definition 1.4 If V is an analytic space, a resolution of the singularities of V consists of a manifold M and a proper holomorphic map $\pi: M \to V$ such that π is biholomorphic on the inverse image of R, the regular points of V, and such that $\pi^{-1}(R)$ is dense in M.

Definition 1.5 A nowhere discrete compact analytic subset A of an analytic space G is called exceptional (in G) if there exists an analytic space Y and a proper holomorphic map $\Phi: G \to Y$ such that $\Phi(A)$ is discrete, $\Phi: G - A \to Y - \Phi(A)$ is biholomorphic and such that for any open set $U \subset Y$, with $V = \Phi^{-1}(U)$, $\Phi^*: \Gamma(U,\mathcal{O}) \to \Gamma(V,\mathcal{O})$ is an isomorphism.

If A is exceptional in G, we shall sometimes say that A can be "blown down" or Φ blows down A.

Definition 1.6 A resolution $\pi: M \to V$ of the singularities of V (with nonsingular A_i's and normal crossings) is a minimal (good) resolution if for any other resolution (with nonsingular A_i's and normal crossings) $\pi': M' \to V$, there is a unique holomorphic map $\rho: M' \to M$ such that $\pi' = \pi \circ \rho$.

A minimal good resolution for isolated two dimensional singularities always exists and is unique [19].

Let $\pi: M \to V$ be a resolution of normal two dimensional Stein space V. We assume that p is the only singularity of V. Let $\pi^{-1}(p) = A = \bigcup_i A_i$, $1 \leq i \leq n$, be the decomposition of the exceptional set A into irreducible components. Suppose π is the minimal good resolution. The topological nature of the embedding of A in M is described by the weighted dual graph Γ [14], [19]. The vertices of Γ correspond to the A_i. The edge of Γ connecting the vertices corresponding to A_i and A_j, $i \neq j$, correspond to the points of $A_i \cap A_j$. Finally, associated to each A_i is its genus, g_i, as a Riemann surface, and its weight, $A_i \cdot A_i$, the topological self-intersection number. Γ will denote the graph, along with the genera and the weights.

Definition 1.7 $\quad \deg A_i = \Sigma A_j \cdot A_i$, $j \neq i$

A cycle (or divisorial cycle) D on A is an integral combination of the A_i. $D = \Sigma d_i A_i$, $1 \leq i \leq n$ with d_i an integer. In this paper, "cycle" will always mean a cycle on A. There is a natural partial ordering, denoted by <, between cycles defined by comparing the coefficients. We shall only be considering cycles $D \geq 0$. We let $\text{supp}D = |D| = \bigcup A_i$, $d_i \neq 0$, denote the support of D.

Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on M. Let $\mathcal{O}(-D)$ be the sheaf of germs of holomorphic functions on M which vanish to order d_i on A_i. Let $\mathcal{O}_D$ denote $\mathcal{O}/\mathcal{O}(-D)$. We use "dim" to denote dimension over $\mathbb{C}$.

$$(1.1) \qquad \chi(D) = \dim H^0(M,\mathcal{O}_D) - \dim H^1(M,\mathcal{O}_D)$$

Some authors work instead with the arithmetic genus $P_a(D) = 1 - \chi(D)$. The Riemann - Roch Theorem [31 p.75] says

$$(1.2) \qquad \chi(D) = -\frac{1}{2}(D\cdot D + D\cdot K).$$

In (1.2), K is the canonical divisor on M. $D\cdot K$ may be defined as follows. Let ω be a meromorphic 2-form on M, i.e. a meromorphic section of K. Let (ω) be the divisor of ω. Then $D\cdot K = D\cdot(\omega)$ and this number is independent of the choice of ω. In fact, let g_i be the geometric genus of A_i, i.e. the genus of the desingularization of A_i. Then [31 p.75]

$$(1.3) \qquad A_i\cdot K = -A_i\cdot A_i + 2g_i - 2 + 2\delta_i$$

where δ_i is the "number" of nodes and cusps on A_i. Each singular point on A_i other than a node or cusp counts as at least two nodes. Fortunately, such more complicated singularities will not occur in this paper.

The minimal resolution of V is characterized by there being no A_i which is a non-singular rational curve with $A_i\cdot A_i = -1$ [7 p.364]. The intersection matrix $(A_i\cdot A_j)$ is negative definite [26] so by (1.3) we see the following.

Proposition 1.8: π is the minimal resolution of V if and only if $A_i\cdot K \geq 0$ for all A_i.

It follows immediately from (1.2) that if B and C are cycles then

$$(1.4) \qquad \chi(B+C) = \chi(B) + \chi(C) - B\cdot C.$$

Associated to π is a unique fundamental cycle Z [2, pp.131-132] such that $Z > 0$, $A_i \cdot Z \le 0$ all A_i, and such that Z is minimal with respect to those two properties. Z may be computed from the intersection matrix as follows [20 p.607] via what is called a computation sequence (in the sense of Laufer) for Z

$$Z_o=0,\ Z_1=A_{i_1}, Z_2 = Z_1+A_{i_2}, \ldots\ldots,\ Z_j=Z_{j-1}+A_{i_j}, \ldots Z_\ell=Z_{\ell-1}+A_{i_\ell} = Z$$

where A_{i_1} is arbitrary and $A_{i_j} \cdot Z_{j-1} > 0$, $1 < j \le \ell$. $\mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)$ represents the sheaf of germs of sections of a line bundle over A_{i_j} of chern class $-A_{i_j} \cdot Z_{j-1}$. So $H^o(M,\mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)) = 0$ for $j > 1$.

$$(1.5) \qquad 0 \to \mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j) \to \mathcal{O}_{Z_j} \to \mathcal{O}_{Z_{j-1}} \to 0$$

is an exact sequence. From the long exact homology sequence for (1.5), it follows by induction that

$$(1.6) \qquad H^o(M,\mathcal{O}_{Z_k}) = \mathbb{C} \qquad 1 \le k \le \ell$$

$$(1.7) \qquad \dim H^1(M,\mathcal{O}_{Z_k}) = \Sigma \dim H^1(M,\mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)),$$
$$1 \le j \le k$$

Since M is two dimensional and not compact,

$$(1.8) \qquad H^2(M,F) = 0.$$

for any coherent sheaf F on M [33].

<u>Lemma 1.9</u> Let Z_k be part of a computation sequence for Z and such that $\chi(Z_k) = 0$. Then $\dim H^1(M,\mathcal{O}_D) \le 1$ for all cycles D such that $0 \le D \le Z_k$. Also, $\chi(D) \ge 0$.

Definition 1.10 A cycle $E > 0$ is minimally elliptic if $\chi(E) = 0$ and $\chi(D) > 0$ for all cycles D such that $0 < D < E$.

Proposition 1.11 Let $Z_k > 0$ be part of a computation sequence for the fundamental cycle and such that $\chi(Z_k) = 0$. Let $B = \Sigma\, b_i A_i$ and $C = \Sigma\, c_i A_i$, $1 \leq i \leq n$, be any cycles such that $0 < B, C \leq Z_k$ and $\chi(B) = \chi(C) = 0$. Let $M = \Sigma \min(b_i, c_i) A_i$, $1 \leq i \leq n$. Then $M > 0$ and $\chi(M) = 0$. In particular, there exists a unique minimally elliptic cycle E with $E \leq Z_k$.

Wagreich [37] defined the singularity p to be elliptic if $\chi(D) \geq 0$ for all cycles $D > 0$ and $\chi(F) = 0$ for some cycles $F > 0$. He proved that this definition is independent of the resolution. It is easy to see that under this hypothesis, $\chi(Z) = 0$. The converse is also true [37], [24]. Henceforth, we will adopt the following definition.

Definition 1.12 p is said to be weakly elliptic if $\chi(Z) = 0$.

The following analogue to proposition 1.11 holds for weakly elliptic singularity.

Proposition 1.13 Suppose that $\chi(D) \geq 0$ for all cycles $D > 0$. Let $B = \Sigma\, b_i A_i$ and $C = \Sigma\, c_i A_i$, $1 \leq i \leq n$, be any cycles such that $0 < B, C$ and $\chi(B) = \chi(C) = 0$. Let $M = \Sigma \min(b_i, c_i) A_i$, $1 \leq i \leq n$. Then $M > 0$ and $\chi(M) = 0$. In particular, there exists a unique minimally elliptic cycle E.

Lemma 1.14 Let E be a minimally elliptic cycle. Then for $A_i \subset \operatorname{supp} E$, $A_i \cdot E = -A_i \cdot K$. Suppose additionally that π is the minimal resolution. Then E is the fundamental cycle for the singularity having supp E as its exceptional set. Also, if E_k is part of a computation sequence for E as a fundamental cycle and $A_j \subset \operatorname{supp}(E-E_k)$, then the computation sequence may be continued past E_k so as to terminate at $E = E_\ell$ with $A_{i_\ell} = A_j$.

Theorem 1.15 Let π: $M \rightarrow V$ be the minimal solution of the normal two dimensional variety V with one singular point p. Let Z be the fundamental cycle on the exceptional set $A = \pi^{-1}(p)$. Then the following are equivalent:

1) Z is a minimally elliptic cycle

2) $A_i \cdot Z = -A_i \cdot K$ for all irreducible components A_i in A

3) $\chi(Z) = 0$ and any connected proper subvariety of A is the exceptional set for a rational singularity.

Definition 1.16 Let p be a normal two-dimensional singularity p is *minimally elliptic* if the minimal resolution π: $M \rightarrow V$ of a neighborhood of p satisfies the conditions of theorem 1.15.

CHAPTER II

BASIC THEORY FOR WEAKLY ELLIPTIC SINGULARITIES AND MAXIMAL IDEAL CYCLE

§1 Minimal good resolution of weakly elliptic singularities

In this section, we study the minimal good resolution of weakly elliptic singularities. We want to understand the nature of the computation sequence for the fundamental cycle Z and what kind of curves can be in the exceptional fibre.

Lemma 2.1 Let $\pi: M \to V$ be a resolution of the normal two dimensional space V with p as its only singularity. Let $\pi^{-1}(p) = A = \cup A_i$, $1 \le i \le n$, be the decomposition of the exceptional set A into irreducible components. Suppose there exists a minimally elliptic cycle E on A. Then $\text{supp}E = A_1$, if and only if either A_1 is a nonsingular elliptic curve or A_1 is a singular rational curve with node or cusp singularity. If $\text{supp}E = \cup A_i$, $1 \le i \le k$, and $k \ge 2$, then $\chi(A_1) = \ldots = \chi(A_k) = 1$ and $A_1, \ldots, A_k$ are nonsingular rational curves.

Let Z be the fundamental cycle on A. If $\chi(Z) = 0$ and $n \ge 2$, then $\chi(A_{k+1}) = \ldots = \chi(A_n) = 1$. In particular, if suppE consists of more than one irreducible component, then all A_i, $1 \le i \le n$, are nonsingular rational curves. If $\text{supp}E = A_1$, then all A_i, $2 \le i \le n$, are nonsingular rational curves.

Proof We claim that suppE = A_1 if, and only if, $\chi(A_1) = 0$. Suppose suppE = A_1. Then $E = nA_1$ for some positive integer n.

$$\begin{aligned}\chi(nA_1) &= \chi(A_1) + \chi((n-1)A_1) - (n-1)A_1 \cdot A_1\\ &= n\ \chi(A_1) - \frac{n(n-1)}{2} A_1 \cdot A_1\end{aligned}$$

Since $\chi(E) = 0$, $\chi(A_1) = \frac{(n-1)}{2} A_1 \cdot A_1$. By definition of minimally elliptic cycle (Definition 1.10), $\chi(A_1) = \frac{(n-1)}{2} A_1 \cdot A_1 \geq 0$. However, $\chi(A_1) = \frac{(n-1)}{2} A_1 \cdot A_1 \leq 0$. Therefore $\chi(A_1) = 0$. Conversely, if $\chi(A_1) = 0$, then $E = A_1$. This completes the proof of our claim. By (1.2) and (1.3)

$$\chi(A_1) = -\frac{1}{2}(A_1 \cdot A_1 + A_1 \cdot K) \text{ where K is the canonical divisor on M}$$

$$= -\frac{1}{2}(A_1 \cdot A_1 - A_1 \cdot A_1 + 2g_1 - 2 + 2\delta_1) \text{ where } \delta_1 \text{ is the "number" of nodes and cusps on } A_1$$

$$= 1 - g_1 - \delta_1.$$

Therefore

$$\chi(A_1) = 0 \qquad 1 - g_1 - \delta_1 = 0$$

$$(1) \quad g_1 = 1 \text{ and } \delta_1 = 0$$

$$\text{or} \quad (2) \quad g_1 = 0 \text{ and } \delta_1 = 1$$

So suppE = A_1 if either A_1 is a nonsingular elliptic curve or A_1 is a singular rational curve with node or cusp singularity. If suppE = $A_1 \cup \ldots \cup A_k$, $k \geq 2$, then $\chi(A_i) > 0$ for $A_i \subsetneqq$ suppE by the definition of minimally elliptic cycle. On the other hand, $\chi(A_i) \leq 1$ by (1.1) and (1.6). So $\chi(A_i) = 1$ and hence $1 - g_i - \delta_i = 1$ for $1 \leq i \leq k$. This implies that $g_i = 0 = \delta_i$, i.e. A_i, $1 \leq i \leq k$, are nonsingular rational curves.

To prove the rest of the lemma, it suffices to show that if $\chi(Z) = 0$, then $\chi(A_i) = 1$ for $A_i \not\subseteq$ suppE. $\chi(Z) = 0$ implies that $\chi(D) \geq 0$ for $D > 0$ [24, Corollary 4.3]. By (1.1) and (1.6), we know that $\chi(A_i) \leq 1$. So $0 \leq \chi(A_i) \leq 1$. However, $\chi(A_i)$ cannot be equal to zero by Proposition 1.13. Therefore $\chi(A_i) = 1$ for $A_i \not\subseteq$ suppE, i.e., $A_i \not\subseteq$ suppE are nonsingular rational curves.

Proposition 2.2 Let π: $M \to V$ be the minimal resolution for a weakly elliptic singularity p. Let π': $M' \to V$ be the minimal resolution such that A_i' are nonsingular and have normal crossings, i.e. the A_i' meet transversely and no three meet at a point. Then $\pi = \pi'$ and all the A_i are rational curves except for the following cases.

(1) A_1 is a nonsingular elliptic curve. $A_2, \ldots A_n$ are nonsingular rational curves. In this case, $\pi = \pi'$. In fact, $0 \leq A_i \cdot A_j \leq 1$ for $i \neq j$.

(2) A_1 is a rational curve with a node singularity. $A_2, \ldots A_n$ are nonsingular rational curves and have normal crossings. In fact, $0 \leq A_i \cdot A_j \leq 1$ for $i \neq j$.

(3) A_1 is a rational curve with a cusp singularity. $A_2, \ldots A_n$ are nonsingular rational curves and have normal crossings. In fact $0 \leq A_i \cdot A_j \leq 1$ for $i \neq j$.

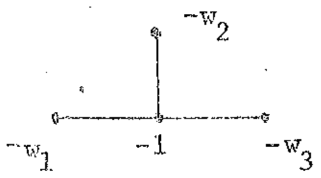
(4) All A_i are nonsingular rational curves and have normal crossings except A_1 and A_2 having first order tangential contact at one point. In fact $A_1 \cdot A_2 = 2$ and $0 \leq A_i \cdot A_j \leq 1$ for $i \neq j$, $(i,j) \neq (1,2)$ and $(i,j) \neq (2,1)$.

(5) All A_i are nonsingular rational curves and have normal crossings except A_1, A_2, A_3 all meeting transversely at the same point. In fact, if $n \geq 4$, then $0 \leq A_i \cdot A_j \leq 1$ for $1 \leq i \leq n$, $j \geq 4$ $i \neq j$ and $A_1 \cdot A_2 = 1$, $A_3 \cdot (A_1 + A_2) = 2$.

In case (2), π' has the following weighted eual graph as its subgraph

with $w_1 \geq 5$

In case (3) - (5), π' has the following weighted dual graph as its subgraph



with $w_i \geq 2$ $1 \leq i \leq 3$

The proof is long but straightforward with many cases.

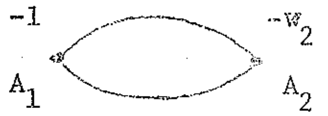
Corollary 2.3 Let π be the minimal resolution with nonsingular A_i and normal crossings for a weakly elliptic singularity.

Let E be the minimally elliptic cycle, $E \leq Z$, the fundamental cycle. Then E may be chosen as part of a computation sequence for Z and $E = Z_k$. Moreover, if $Z_j < E$ is part of a computation sequence for Z and $A_m \subset \mathrm{supp}(E-Z_j)$, then the computation sequence may be continued past Z_j so that $A_{i_k} = A_m$.

Proof: The proof is the same as Corollary 3.6 of [24].

Proposition 2.4 Let π be the minimal good resolution for a minimally elliptic singularity. Suppose π is not the minimal resolution. Then the fundamental cycle is one of the following forms

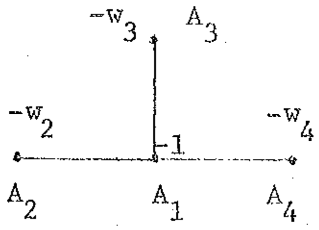
(I)



with $w_2 \geq 5$

$Z = 2A_1 + A_2$

(II)



with $w_i \geq 2$

$2 \leq i \leq 4$

(1) If $A_2 \cdot A_2 \leq -3$, $A_3 \cdot A_3 \leq -3$, $A_4 \cdot A_4 \leq -3$, then

$$Z = 3A_1 + A_2 + A_3 + A_4$$

(2) If $A_2 \cdot A_2 = -2$, $A_3 \cdot A_3 = -3$, $A_4 \cdot A_4 < -6$, then

$$Z = 6A_1 + 3A_2 + 2A_3 + A_4$$

(3) If $A_2 \cdot A_2 = -2$, $A_3 \cdot A_3 \leq -4$, $A_4 \cdot A_4 < -4$, then

$$Z = 4A_1 + 2A_2 + A_3 + A_4$$

Proof: An easy case by case checking.

Proposition 2.5 Let π: $M \to V$ be the minimal good resolution of a normal two dimensional Stein space with p as its only weakly elliptic singular point.

Case 1: If suppE has at least two irreducible components, then for any computation sequence of the following form $Z_o = 0$, $Z_1 = A_{i_1}$, $\ldots, Z_k = E, \ldots, Z_\ell = Z$. We have $A_{i_j} \cdot Z_{j-1} = 1$ for $j \neq k$ and $A_{i_k} \cdot Z_{k-1} = 2$. If $\overline{\text{suppZ} - \text{suppE}} \neq \phi$, then for any $A_{i_1} \subseteq \overline{\text{suppZ}-\text{suppE}}$ we can choose a computation sequence of the following form $Z_o = 0$, $Z_1 = A_{i_1}, \ldots, Z_r, Z_{r+1}, \ldots Z_{r+k} = E + Z_r, \ldots Z_\ell = Z$ such that $\text{supp}Z_r \subseteq \overline{\text{suppZ}-\text{suppE}}$ and $Z_{r+1} - Z_r, \ldots, Z_{r+k} - Z_r = E$, is part of a computation sequence for Z. Moreover, any computation sequence of the above form has the following properties: $A_{i_j} \cdot Z_{j-1} = 1$ for $j \neq r + k$ and $A_{i_{r+k}} \cdot Z_{r+k-1} = 2$.

Case 2: If suppE has only one irreducible components, then for any computation sequence of the following form $Z_0 = 0$, $Z_1 = A_{i_1} = E$, $\ldots$, $Z_\ell = Z$, we have $A_{i_j} \cdot Z_{j-1} = 1$ for all j. If $\overline{\text{suppZ}-\text{suppE}} \neq \phi$, then for any $A_i \subseteq \overline{\text{suppZ}-\text{suppE}}$, we can choose a computation sequence of the following from $Z_o = 0$, $Z_1 = A_{i_1}, \ldots Z_r$, $Z_{r+1} = Z_r + A_{i_{r+1}}, \ldots Z_\ell = Z$ where $A_{i_{r+1}} = E$. Moreover, any computation sequence of the above form has $A_{i_j} \cdot Z_{j-1} = 1$ for all j.

Proof: Case 1

$$
\begin{aligned}
0 < A_{i_k}\cdot Z_{k-1} &= A_{i_k}\cdot(E - A_{i_k})\\
&= -A_{i_k}\cdot K - A_{i_k}\cdot A_{i_k} \quad \text{by Lemma 1.14}\\
&= -2g_{i_k}+2
\end{aligned}
$$

So $g_{i_k} = 0$ and $A_{i_k}\cdot Z_{k-1} = 2$. Since $\chi(Z) = 0$, $H^1(M,\mathcal{O}_Z) = \mathbb{C}$ by (1.1) and (1.6). As all A_i are nonsingular rational curve, therefore (1.7) and Riemann - Roch Theorem will show that $A_{i_j}\cdot Z_{j-1} = 1$ for $j \neq k$.

From the above proof, we know that for any $A_i \subseteq \text{suppE}$ such that there exists $A_j \nsubseteq \text{suppE}$ and $A_i\cdot A_j > 0$, then $e_i = 1$ and $A_i\cdot A_j = 1$, where e_i is the coefficient of A_i in E. It is easy to see that the computation sequence in case 1 of the proposition can be chosen. Now we are going to prove the last statement of case 1. By the above argument, we know $A_{i_{r+k}}(Z_{r+k-1} - Z_r) = 2$ and hence $A_{i_{r+k}}\cdot Z_{r+k-1} \geq 2$ because $A_{i_{r+k}} \subseteq \text{suppE}$ and $Z_r \subseteq \overline{\text{suppZ}-\text{suppE}}$. Since $H^1(M,\mathcal{O}_Z) = \mathbb{C}$, by (1.7) and Riemann - Roch Theorem, there is at most one $A_{i_j}\cdot Z_{j-1} = 2$. So $A_{i_{r+k}}\cdot Z_{r+k-1} = 2$ and $A_{i_j}\cdot Z_{j-1} = 1$ for $j \neq r + k$.

Case 2: Since $\chi(E) = \chi(Z) = 0$, (1.1) and (1.6) imply that $H^1(M,\mathcal{O}_Z) = \mathbb{C} = H^1(M,\mathcal{O}_E)$. So by (1.7) and Riemann - Roch Theorem, it follows immediately that $A_{i_j}\cdot Z_{j-1} = 1$, for all j.

Now let us prove the last statement of case 2. By Lemma 2.1 we know that $A_{i_{r+1}}$ is a nonsingular elliptic curve.

Moreover, for any $A_j \neq A_{i_{r+1}}$, A_j is a nonsingular rational curve. By (1.7) and $H^1(M,\mathcal{O}_Z) = \mathbb{C}$, we have $\dim H^1(M, \mathcal{O}(-Z_r)/\mathcal{O}(-Z_r-A_{i_{r+1}})) \leq 1$. The chern class of the line bundle associated to $\mathcal{O}(-Z_r)/\mathcal{O}(-Z_r-A_{i_{r+1}})$ on $A_{i_{r+1}}$ is $-A_{i_{r+1}}\cdot Z_r \leq -1$. By Serre duality Theorem and Riemann Roch Theorem $\dim H^1(M, \mathcal{O}(-Z_r)/\mathcal{O}(-Z_r-A_{i_{r+1}})) = 2g_{i_{r+1}} - 2 + A_{i_{r+1}}\cdot Z = A_{i_{r+1}}\cdot Z$. So $A_{i_{r+1}}\cdot Z_r = 1 = \dim H^1(M, \mathcal{O}(-Z_r)/\mathcal{O}(-Z_r-A_{i_{r+1}}))$. By (1.7), Serre duality theorem and Riemann - Roch Theorem, we know that $A_{i_j}\cdot Z_{j-1} = 1$ for all j. Moreover, A_{i_j} are nonsingular rational curves for $j \neq r + 1$.

Corollary 2.6 Let π: $M \to V$ be the minimal good resolution of a normal two dimensional Stein space V with p as its only weakly elliptic singular point. Suppose supp$E = A_1$. Let $Z = \Sigma z_i A_i$. Then $z_1 = 1$.

Proof: This is contained in the proof of case 2 of the above Proposition.

§2 Laufer-type vanishing Theorem

Proposition 2.7: Let p be a weakly elliptic singularity. Let π: $M \to V$ be the minimal good resolution of a Stein neighborhood V of p having p as its only singular point. Let $Y > 0$ be a cycle on the exceptional set A such that $A_i\cdot Y \leq 0$ for all irreducible components A_i of A. Let Z be the fundamental cycle and E the minimally elliptic cycle. Let $0 = Z_o,\ldots,Z_\ell = Z$ be a computation sequence for Z with $E = Z_k$ and A_{i_k} such that $A_{i_k}\cdot Y < 0$. Then

$H^1(M, \mathcal{O}(-Y-Z_j)) = 0$ for $0 \leq j \leq \ell$.

Proof: The proof is similar to the proof of Lemma 3.11 in [24].

Proposition 2.8: Let p, π, M, V, Y, Z and E be as in Propisition 2.8. Let $E = \sum_{i=1}^{t} e_i A_i$. Suppose $E \cdot Y < 0$. Let A_1 be an arbitrary $A_i \subseteq \text{supp}E$. Then $\rho\colon H^0(M, \mathcal{O}(-Y)) \to H^0(M, \mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))$ is surjective if A_1 is an elliptic curve or if there exists $A_j \subseteq \text{supp}E$, $A_j \neq A_1$ with $A_j \cdot Y < 0$ or if $e_1 > 1$. If A_1 is a rational curve, $A_j \cdot Y = 0$ for $A_j \neq A_1$, $A_j \subseteq \text{supp}E$, and $e_1 = 1$, then the image of ρ is a subspace S of codimension 1 in $H^0(M, \mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))$. If $\dim S \geq 2$, then the elements of S have no common zeroes as sections of line bundle L on A_1 associated to $\mathcal{O}(-Y)/\mathcal{O}(-Y-A_1)$. If $\dim S = 1$, then there is one common zero at a point $q \varepsilon A$ with $q \notin A_j$ where $A_j \cdot Y = 0$ and $A_j \subseteq \text{supp}E$.

Proof: The proof is similar to the proof of lemma 3.12 in [24]

§3 Structure Theorem for weighted dual graphs of weakly elliptic singularities

For weighted dual graphs of weakly elliptic singularities, we can obtain some information from the following two propositions. Much more complete information is given in Chapter III.

Proposition 2.9: Let $\pi\colon M \to V$ be a resolution of a normal 2 dimensional Stein space V with p as its only weakly elliptic singularity. Let E be the minimally elliptic cycle on $A = \pi^{-1}(p)$. Suppose B is a connected subvariety of A such that $B \not\supseteq \text{supp}E$.

Then B is the exceptional set of a rational singularity.

Proof: The fact that B is exceptional in M follows from [19 p. 89 Lemma 5.11]. Let Z_B denote the fundamental cycle on B. It follows by [2, p.132 Theorem 3] that $\chi(Z_B) \leq 1$. On the other hand, since p is a weakly elliptic singular point, $\chi(Z_B) \geq 0$, $\chi(Z_B)$ cannot equal to zero. Otherwise it will contradict the minimality of the minimally elliptic cycle by Proposition 1.13 since $B \not\supset \text{supp}E$. Therefore $\chi(Z_B) = 1$. Apply Theorem 3 of [2], our result follows.

Proposition 2.10: Let $\pi: M \to V$ be a resolution of normal two dimensional Stein space with p as its only weakly elliptic singular point. Let E be the minimally elliptic cycle on the exceptional set $A = \pi^{-1}(p)$. Suppose B is a connected subvariety of A containing $|E|$. Then B is the exceptional set for weakly elliptic singularity. In particular, if B = suppE, then B is a minimally elliptic singularity.

Proof: As in Proposition 2.10, we know that B is exceptional in M. Let Z_B be the fundamental cycle on B. Then $\chi(Z_B) \leq 1$ by Theorem 3 of [2]. Since p is a weakly elliptic singularity, so $\chi(Z_B) \geq 0$. Hence, $0 \leq \chi(Z_B) \leq 1$. $\chi(Z_B)$ cannot equal to one. Otherwise it will imply that B is an exceptional set of rational singularity by Theorem 3 of [2]. Since $B \supseteq |E|$ Theorem 1 of [2], says that $\chi(E) \geq 1$. This is a contradiction so $\chi(Z_B) = 0$ and B is the exceptional set for a weakly elliptic singularity.

§4 Maximal Ideal Cycle

Let $\pi: M \to V$ be the resolution of normal two dimensional space V with p as its only singularity. Let m be the maximal ideal in ${}_V\mathcal{O}_p$. One important question in normal two dimensional singularity is the "identification of m ". In this section, we define the maximal ideal cycle which serves partially to identify the maximal ideal.

Definition 2.11: Let A be the exceptional set in the resolution $\pi: M \to V$ of a 2-dimensional space V with p and its only singularity. Suppose that $\{A_i\}_{1 \leq i \leq n}$ are the irreducible components of A. Let m be the maximal ideal in ${}_V\mathcal{O}_p$. If $f \in m$, then the divisor of f, $(f) = [f] + D$ where $[f] = \Sigma\, n_i A_i$ and D does not involve any of A_i. Let Y be the positive cycle such that $Y = \inf\limits_{f \in m} [f]$. Then Y is called the maximal ideal cycle.

Proposition 2.12: Use the notation of Definition 2.12. The maximal ideal cycle is a positive cycle s.t. $Y \cdot A_i \leq 0$ for all $A_i \subseteq A$. In particular $Y \geq Z$. In fact if $f_1, \ldots f_r \in m$ such that $f_1, \cdots f_r$ generate m, then $Y = \inf\limits_{1 \leq i \leq r} [f_i]$.

Proof: Easy.

Proposition 2.13: Use the notation of Definition 2.12. Let Y be the maximal ideal cycle, then $m\mathcal{O} \subseteq \mathcal{O}(-Y)$. Moreover, if $m\mathcal{O}$ is locally principal, i.e. $m\mathcal{O} = \mathcal{O}(-D)$ for some positive divisor D, then $D = Y$ and $m\mathcal{O} = \mathcal{O}(-Y)$.

Proof: Easy.

Definition 2.14: Let $\sigma: M' \to M$ be a monodial transformation with center $q \in M$. We associate with the curve $C \subseteq M$, $q \notin C$ the curve C^* the proper transform of C in M'. If q is a point of multiplicity n of the curve C, we associate with this curve the curve $C^* + n L \subset M'$ where $L = \sigma^{-1}(q)$. With the divisor $Z = \Sigma k_i C_i$, we associate the divisor $\sigma^*(Z) = \Sigma k_i C_i^* + k_i n_i L$, where n_i is the multiplicity of the point q on the curve C_i.

Lemma 2.15: Let $\pi: M \to V$ be a resolution of normal two dimensional analytic space with p as its only singularity. Let $A = \pi^{-1}(p) = \bigcup A_i$ be the decomposition of A into irreducible components. Suppose W is a positive cycle on A such that $W \cdot A_j \leq 0$ for all $A_j \subseteq A$. For any positive cycle X on A such that $X \geq W$, $X^2 \leq W^2$. Also, $X^2 = W^2$ if and only if $X = W$.

Proof: Let $X = W + \Sigma n_i A_i$ where $n_i \geq 0$. Then $X^2 = W^2 + 2\,\Sigma n_i (A_i \cdot W) + \sum_{i,j} n_i\, n_j\, (A_i \cdot A_j)$. Now $A_i \cdot W \leq 0$ by the hypothesis. The last expression is nonpositive since $(A_i \cdot A_j)$ is negative definite. Moreover, this expression is zero if and only if $n_i = 0$ for all i by the definiteness.

Lemma 2.16: Let $\pi: M \to V$ be a normal two dimensional analytic space with p as its only singularity. Let $A = \pi^{-1}(p) = \bigcup_{i=1}^{t} A_i$ be the decomposition of A into irreducible components. Let $\sigma: M' \to M$ be a monoidal transformation with point q as center. Let $D = \pi^{-1}(q)$ and A_i' be the proper transform of A_i by σ. Then $(\pi\cdot\sigma)^{-1}(p) = D \cup (\bigcup_{i=1}^{t} A_i')$. Suppose X is a positive cycle on A such that $A_i \cdot X \leq 0$ for all $A_i \subseteq A$. Then $D\cdot\sigma^*(X) = 0$ and $A_i' \cdot\sigma^*(X) \leq 0$ for all $1 \leq i \leq t$.

Proof: Since A_i is linearly equivalent to some divisor not passing through q, hence X is also linearly equivalent to some divisor not passing through q. It follows that $\pi^*(X)\cdot D = 0$. By p. 421 of [37] $X\cdot A_i = \sigma^*(X)\cdot\sigma^*(A_i)$. So $0 \geq X\cdot A_i$ implies that $0 \geq \sigma^*(X)\cdot\sigma^*(A_i) = \sigma^*(X)(A_i' + m_i D) = \sigma^*(X)\cdot A_i'$.

Theorem 2.17: Let $\pi: M \to V$ be a normal two dimensional analytic space with p as its only singularity. Let $A = \pi^{-1}(p) = \bigcup_{i=1}^{t} A_i$ be the decomposition of A into irreducible components. Let Y be the maximal ideal cycle associated to π. Then the multiplicity of ${}_V\mathcal{O}_p \geq -Y\cdot Y$. If $m\mathcal{O}$ is locally principle, then the multiplicity of ${}_V\mathcal{O}_p = -Y\cdot Y$.

Proof: If $m\mathcal{O}$ is locally principal, then $m\mathcal{O} = \mathcal{O}(-Y)$ by Proposition 2.14. In this case Theorem 2.7 of [37] says that multiplicity of ${}_V\mathcal{O}_p$ is equal to $-Y\cdot Y$.

In the general case, let $\pi': M' \to M$ be the monoidal transformation with center $m\mathcal{O}$. The map π' is a composition of monoidal transformations σ with points as center [see 42, lemma, p.538]. Let $A^1 = (\pi\cdot\pi')^{-1}(p) = \bigcup_{i=1}^{s} A'_i$. Then the lemma 2.16 says that $A'_i \cdot \pi'^*(Y) \leq 0$ for all $1 \leq i \leq s$. Let $\mathcal{O}'$ be the structure sheaf on M'. Let Y' be the maximal ideal cycle relative to $\pi\cdot\pi'$. Then $m\mathcal{O}' = \mathcal{O}'(-Y')$. But $m\mathcal{O}' \subseteq \mathcal{O}'(-\pi'^*(Y))$. So $Y' \geq \pi'^*(Y)$. Theorem 2.7 of [37] and lemma 2.15 will show that the multiplicity $({}_V\mathcal{O}_p) = -Y'\cdot Y' \geq -[\pi'^*(Y)]^2$. However, for any proper modification σ and divisor L, we know that $[\sigma^*(L)]^2 = L^2$. So $(\pi'^*(L))^2 = L^2$. In particular $(\pi'^*(Y))^2 = Y^2$. Therefore multiplicity of ${}_V\mathcal{O}_p \geq -Y^2$.

Definition 2.18 Let p be the only singularity of the normal two dimensional space V. Let $\pi: M \to V$ be the resolution of V. Let $A = \bigcup A_i$, $1 \leq i \leq n$ be the decomposition of $A = \pi^{-1}(p)$ into irreducible components. Let K be the canonical divisor on M. We define the negative cycle $K' = \Sigma k_i A_i$ on A where $k_i \in \mathbb{Z}$, the set of integers, to be a cycle such that $A_i \cdot K' = A_i \cdot K$ for all $A_i \subseteq A$. (K' does not always exist).

The following Theorem gives a "non-lower" estimate of the maximal ideal cycle in terms of the cycle K'.

Theorem 2.19: Let $\pi: M \to V$ be the minimal resolution of a normal two dimensional Stein space with p as its only singular point. Suppose K' exists and $\dim H^1(M,\mathcal{O}) \geq 2$, then the maximal ideal cycle Y relative to π cannot be greater than or equal to $-K'$.

Proof: By Theorem 3.2, p.603 of [20], we know that $H^1(M, \mathcal{O}(K'))=0$. The following cohomology exact sequence

$$H^1(M, \mathcal{O}(K')) \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_{-K'}) \to 0$$

shows that $H^1(M, \mathcal{O}_{-K'}) \cong H^1(M, \mathcal{O})$. Since $\chi(-K') = -\frac{1}{2}[(-K')\cdot K + (-K')\cdot(-K')] = -\frac{1}{2}[(-K')\cdot K' + (-K')(-K')] = 0$ by (1.2), hence (1.1) says that $\dim H^0(M, \mathcal{O}_{-K'}) = \dim H^1(M, \mathcal{O}_{-K'}) = \dim H^1(M, \mathcal{O}) \geq 2$.

Suppose on the contrary that $Y \geq -K'$. Since π is the minimal resolution $A_i \cdot K' \geq 0$ for all A_i, so $-K' \geq Z$ by the definition of the fundamental cycle Z. It follows that there is a natural injective map $H^0(M, \mathcal{O}(K')) \to H^0(M, \mathcal{O}(-Z))$. We claim that this map is actually surjective. Given any $g \in H^0(M, \mathcal{O}(-Z))$, we know that g is actually a function on V which vanishes at p. By Proposition 2.13, $g \in H^0(M, \mathcal{O}(-Y))$. However $Y \geq -K'$ implies that $H^0(M, \mathcal{O}(-Y)) \subseteq H^0(M, \mathcal{O}(K'))$. So g can also be considered as an element in $H^0(M, \mathcal{O}(K'))$. This proves our claim. Look at the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & H^0(M, \mathcal{O}(K')) \arrow[r] \arrow[d, "\wr"] & H^0(M, \mathcal{O}) \arrow[r] \arrow[d, "\wr"] & H^0(M, \mathcal{O}_{-K'}) \arrow[r] \arrow[d] & H^1(M, \mathcal{O}(K')) \cong 0 \arrow[d] \\
0 \arrow[r] & H^0(M, \mathcal{O}(-Z)) \arrow[r] & H^0(M, \mathcal{O}) \arrow[r] & H^0(M, \mathcal{O}_Z) \cong \mathbb{C} \arrow[r] & H^1(M, \mathcal{O}(-Z)).
\end{tikzcd}$$

Since $H^o(M, \mathcal{O}_Z) \simeq \mathbb{C}$ by (1.6), so $H^o(M,\mathcal{O}) \to H^o(M,\mathcal{O}_Z)$ is surjective. We have $H^o(M, \mathcal{O}_{-K'})$ is isomorphic to $H^o(M,\mathcal{O}_Z)$. However, $\dim H^o(M, \mathcal{O}_Z) = 1 < \dim H^o(M, \mathcal{O}_{-K'})$. This leads to a contradiction. Q.E.D.

If ${}_V\mathcal{O}_p$ is a Gorenstein ring, i.e. there is some neighborhood Q of p in V and a holomorphic 2-form ω on Q-p such that ω has no zeros on Q-p, then K' exists.

Theorem 2.20: If we assume ${}_V\mathcal{O}_p$ is Gorenstein in Theorem 2.19, then the same result holds even if π is not necessarily the minimal resolution.

Proof: As ${}_V\mathcal{O}_p$ is Gorenstein, there exists $\omega \in H^o(M-A,\Omega)$ having no zeros near A. Serre duality gives $H^1(M,\mathcal{O})$ as dual to $H^1_*(M,\Omega)$, where Ω is the canonical sheaf, i.e. the sheaf of germs of holomorphic 2-forms. By Theorem 3.4, p.604 of [20], for suitable M, which can be chosen to be arbitrarily small neighborhoods of $A = \pi^{-1}(p)$, $H^1_*(M,\Omega)$ may be identified with

$$H^o(M-A,\Omega)/H^o(M,\Omega).$$

So

$$\dim H^o(M-A,\Omega)/H^o(M,\Omega) = n \geq 2.$$

There exists $\omega_2,\ldots,\omega_n$ in $H^o(M-A,\Omega)$ such that the image of $\omega,\omega_2,\ldots,\omega_n$ in $H^o(M-A,\Omega)/H^o(M,\Omega)$ forms a basis. Since ω is non-zero in a neighborhood of A, we may assume that $\omega_i = f_i\omega$ $2 \leq i \leq n$

where $f_i \in H^o(M,\mathcal{O})$. Moreover we can assume that f_i are vanishing at p, i.e., $f_i \in H^o(M,m\mathcal{O})$. Otherwise we simply replace f_i by $f_i - f_i(p)$, $2 \leq i \leq n$.

Suppose our theorem is false. Then the maximal ideal cycle $Y \geq [\omega]$. Since $m\mathcal{O} \subseteq \mathcal{O}(-Y)$ by Proposition 2.13, we have $\omega_i = f_i\omega$, $2 \leq i \leq n$ all in $H^o(M,\Omega)$. This contradicts the fact that the image of ω, $\omega_2,\ldots\omega_n$ forms a basis for

$$H^o(M-A,\Omega)/H^o(M,\Omega).$$

CHAPTER III

ELLIPTIC SEQUENCES AND MAXIMALLY ELLIPTIC SINGULARITIES

One might classify hypersurface singularities by $h = \dim H^1(M, \mathcal{O})$. If $h = 0$, then the singularity is rational [20]. If $h = 1$, then the singularity is minimally elliptic [24]. Let us consider the condition $h = 2$. All hypersurface singularities as well as complete intersection are Gorenstein, so the following theorem applies.

Theorem 3.1: Let $\pi: M \to V$ be a resolution of the normal two dimensional Stein space V with p as its only singularity. Suppose ${}_V\mathcal{O}_p$ is Gorenstein and $H^1(M, \mathcal{O}) = \mathbb{C}^2$. Then p is a weakly elliptic singularity.

Proof: Let $\pi^{-1}(p) = A = \cup A_i$, $1 \le i \le n$, be the decomposition of the exceptional set A into irreducible components and Z be the fundamental cycle on A. Since $H^1(M, \mathcal{O})$ is independent of the choice of the resolution [20, Lemma 3.1, p.599] and [2,p.124], we may assume that π is the minimal good resolution. By (1.6), $H^o(M, \mathcal{O}_Z) = \mathbb{C}$. So we have the following exact cohomology sequence:

$$0 \to H^1(M, \mathcal{O}(-Z)) \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_Z) \to 0.$$

Since $H^1(M, \mathcal{O}) = \mathbb{C}^2$, $\dim H^1(M, \mathcal{O}_Z)$ is either 0, 1 or 2. If $H^1(M, \mathcal{O}_Z) = 0$, then $\chi(Z) = \dim H^o(M, \mathcal{O}_Z) - \dim H^1(M, \mathcal{O}_Z) = 1$, i.e. $p(Z) = 1 - \chi(Z) = 0$. By Theorem 3 of [2], p is a rational singularity.

However, as $H^1(M,\mathcal{O}) = \mathbb{C}^2$, the first direct image $R^1\pi{*}\mathcal{O}_V$ is not zero by Lemma 3.1 of [20]. This leads to a contradiction. If $H^1(M,\mathcal{O}_Z) = \mathbb{C}^2$, then $H^1(M, \mathcal{O}(-Z)) = 0$. As ${}_V\mathcal{O}_p$ is Gorenstein, there exists $\omega \varepsilon H^0(M-A,\Omega)$ having no zeros near A, where Ω is the canonical sheaf, i.e. the sheaf of germs of holomorphic 2-forms. By Theorem 3.4, p.604 of [20], for suitable M, which can be chosen to be artibrarily small neighborhoods of $A = \pi^{-1}(p)$, $H^1_*(M,\Omega)$ may be identified with $H^0(M-A,\Omega)/H^0(M,\Omega)$. So $\dim H^0(M-A,\Omega)/H^0(M,\Omega) = 2$ and there exists $\omega' \varepsilon H^0(M-A,\Omega)$ such that the image of ω, ω' in $H^0(M-A,\Omega)/H^0(M,\Omega)$ form a basis. Since ω is non-zero in a neighborhood of A, we may assume that $\omega' = f\omega$ where $f \varepsilon H^0(M,\mathcal{O})$. Furthermore, replacing f by $f - f(p)$ if necessary, we can assume that $f \varepsilon H^0(M,m\mathcal{O})$. Let w_i be the order of the pole of ω on A_i. Consider a cover as in Lemma 3.8 of [24]. On P_1

$$\omega = \frac{\omega_1(x_1,y_1)}{y_1^{w_1}} \; dx_1 \wedge dy_1$$

where $\omega_1(x_1,y_1)$ is a holomorphic function, $\omega_1(x_1,0) \not\equiv 0$. There is a holomorphic function $f(x_1)$, $r \leq x_1 \leq R$ such that

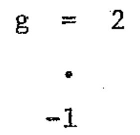
$$\int_{\substack{|x_1|=R\\ |y_1|=R}} y_1^{w_1-1} f(x_1) \frac{\omega_1(x_1,y_1)}{y_1^{w_1}} \; dx_1 \wedge dy_1 \neq 0$$

Let $\lambda_{o1} = y_1^{w_1-1} f(x_1)$ and $\lambda_{oj} = 0$ for $2 \leq j \leq t$. Then by Lemma 3.8 of [24] $c\ell s[\lambda] \neq 0$ in $H^1(M',\mathcal{O})$. Let $Z = \Sigma z_i A_i$, $1 \leq i \leq n$, be the fundamental cycle. If $w_1-1 \geq z_1$, then λ may be thought of as also a cocycle in $H^1(N(\mathfrak{U}), \mathcal{O}(-Z))$. So $c\ell s[\lambda] = 0$ in $H^1(M', \mathcal{O}(-Z))$ and necessarily in $H^1(M',\mathcal{O})$. Thus $w_1-1 \geq z_1$ is impossible, i.e. $w_1 \leq z_1$. As $m\mathcal{O} \subseteq \mathcal{O}(-Z)$, p.133 of [2], we have $\omega' = f\omega \in H^o(M,\Omega)$, i.e., ω, ω' cannot form a basis for $H^o(M-A,\Omega)/H^o(M,\Omega)$. This is a contradiction. So the only possible case is $H^1(M, \mathcal{O}_Z) = \mathbb{C}$. Hence $\chi(Z) = \dim H^o(M, \mathcal{O}_Z) - \dim H^1(M, \mathcal{O}_Z) = 0$, i.e. p is a weakly elliptic singularity. Q.E.D.

However, that $\dim H^1(M,\mathcal{O}) = 3$ and ${}_V\mathcal{O}_p$ is Gorenstein do not imply p is a weakly elliptic singularity.

Example: Let V be the locus in $\mathbb{C}^3$ of $z^2 = x^6 + y^6$. Then the dual weighted graph is

1+2g [illegible] -2 +2g +
2h



It can be calculated by [23] that $\dim H^1(M,\mathcal{O}) = 3$.

Theorem 3.2: Let V be a normal two dimensional Stein space with p as its only singularity. Suppose ${}_V\mathcal{O}_p$ is Gorenstein, i.e. there is some neighborhood Q of p in V and a holomorphic 2 form ω on Q-p such that ω has no zeros on Q-p. If there exists $f \in \mathcal{O}_V$ such that ω, $f\omega$, $f^2\omega, \ldots, f^{n-1}\omega$ is a basis for

$H^1_*(M,\Omega)$, then p is a weakly elliptic singular point.

Proof: Replacing f by f − f(p) if necessary, we may assume that $f \in H^0(M,m\mathcal{O})$. By (1.6), $H^0(M,\mathcal{O}_Z) = \mathbb{C}$. So we have the following exact cohomology sequence.

$$0 \to H^1(M,\mathcal{O}(-Z)) \to H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}_Z) \to 0.$$

By Theorem 3.1, we need only consider the case $n \geq 3$. It is easy to see that $\dim H^1(M,\mathcal{O}_Z) > 0$. Otherwise, as observed in the proof of Theorem 3.1, p will be a rational singular point. To prove that p is a weakly elliptic singular point, it suffices to show that $H^1(M,\mathcal{O}_Z) = \mathbb{C}$. Suppose on the contrary that $\dim H^1(M,\mathcal{O}_Z) \geq 2$. Then $\dim H^1(M,\mathcal{O}(-Z)) \leq n-2$. Let the notation be as the proof of Theorem 3.1. We know that there exists $A_1 \subseteq A$ such that on P_1

$$(3.1) \qquad f^i\,\omega = \frac{\omega_1(x_1,y_1)}{y_1^{-ia_1+w_1}}\,dx_1 \wedge dy_1, \quad w_1 - i\,a_1 > 0, \quad 0 \leq i \leq n-1$$

where $\omega_1(x_1,y_1)$ is a holomorphic function, $\omega_1(x_1,0) \not\equiv 0$. $(\omega) = -\Sigma\, w_i Ai$ and $(f) = \Sigma\, a_i A_i + D = [f] + D$. D does not involve any A_i. There are holomorphic functions $g_i(x_1)$, $r \leq x_1 \leq R$ such that

$$\int_{\substack{|x_1|=R\\ |y_1|=R}} y_1^{w_1-ia_1-1}\, g_i(x_1)\, \frac{\omega_1(x_1,y_1)}{y_1^{w_1-ia_1}}\, dx_1 \wedge dy_1 \neq 0$$

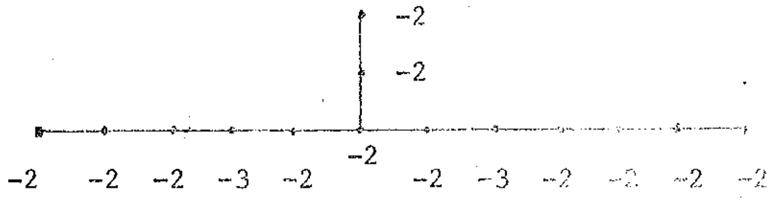
Let $\lambda^i_{o1} = y_1^{w_1-i\,a_1-1}\ g_i(x_1)$ and $\lambda^i_{oj} = 0$ for $2 \le j \le t$. Then by Lemma 3.8 of [24], $c\ell s[\lambda^i] \neq 0$ in $H^1(M',\mathcal{O})$. In fact, $\{\lambda^i\}$ forms a basis for $H^1(M,\mathcal{O})$ because $\langle\lambda^i, f^j\omega\rangle = 0$ for $i \neq j$. Since $\dim H^1(M,\mathcal{O}) - \dim H^1(M, \mathcal{O}(-Z)) = \dim H^1(M, \mathcal{O}_Z) \geq 2$, there are at least two λ^{i_1}, λ^{i_2} which are not in $H^1(M,\mathcal{O}(-Z))$. Hence, $w_1 - i_1a_1-1 < z_1$ and $w_1 - i_2a_1 - 1 < z_1$, i.e., $w_1 \le z_1 + i_1a_1$, $w_1 \le z_1 + i_2a_1$. Since $i_1 \neq i_2$ and $1 \le i_1, i_2 \le n-1$, we may assume that $w_1 \le z_1 + (n-2)a_1$. But $[f]\cdot A_i \le 0$ for all $A_i \subseteq A$ by p.133 of [2]. So $[f] \geq Z$, by the definition of fundamental cycle Z. In particular $z_1 \le a_1$. So, $w_1 \le (n-1)a_1$. This contradicts (3.1). Q.E.D.

A partial converse of Theorem 3.2 will be proved later. Weakly elliptic singularities can be effectively studied by the following method of elliptic sequences.

Definition 3.3: Let A be the exceptional set of the minimal good resolution $\pi: M \to V$ where V is a normal two dimensional Stein space with p as its only weakly elliptic singularity. If $E\cdot Z < 0$, we say that the elliptic sequence is $\{Z\}$ and the length of elliptic sequence is equal to one. Suppose $E\cdot Z = 0$. Let B_1 be the maximal connected subvariety of A such that $B_1 \supseteq \text{supp}E$ and $A_i\cdot Z = 0 \quad \forall\ A_i \subseteq B_1$. Since A is an exceptional set, $Z\cdot Z < 0$. So B_1 is properly contained in A. Suppose $Z_{B_1}\cdot E = 0$. Let B_2 be the maximal connected subvariety of B_1 such that $B_2 \supseteq |E|$ and

$A_i \cdot Z_{B_1} = 0 \quad \forall \; A_i \subseteq B_2$. By the same argument as above, B_2 is properly contained in B_1. Continuing this process, we finally obtain B_m with $Z_{B_m} \cdot E < 0$. We call $\{Z_{B_o} = Z,\ Z_{B_1}, \ldots, Z_{B_m}\}$ the elliptic sequence and the length of elliptic sequence is $m + 1$.

Example 1: Let p be a weakly elliptic singularity whose weight dual graph is of the following form



$$Z = \begin{array}{cccccccccccc} &&&&&1&&&&&& \\ &&&&&2&&&&&& \\ 1&1&1&1&2&3&2&1&1&1&1&1 \end{array}$$

$$Z_{B_1} = \begin{array}{cccccccccccc} &&&&&1&&&&&& \\ &&&&&2&&&&&& \\ 0&1&1&1&2&3&2&1&1&1&1&0 \end{array}$$

$$Z_{B_2} = \begin{array}{cccccccccccc} &&&&&1&&&&&& \\ &&&&&2&&&&&& \\ 0&0&1&1&2&3&2&1&1&1&0&0 \end{array}$$

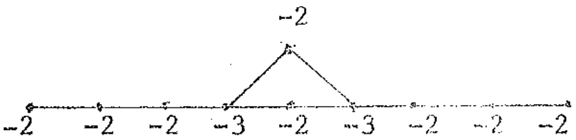
$$Z_{B_3} = \begin{array}{cccccccccccc} &&&&&1&&&&&& \\ &&&&&2&&&&&& \\ 0&0&0&1&2&3&2&1&1&0&0&0 \end{array}$$

$$E = \begin{array}{cccccccccccc} &&&&&1&&&&&& \\ &&&&&2&&&&&& \\ 0&0&0&1&2&3&2&1&0&0&0&0 \end{array}$$

The elliptic sequence is $\{Z = Z_{B_o}, Z_{B_1}\ \ Z_{B_2}, Z_{B_3}\}$ and the length of elliptic sequence is 4.

Remark 3.4: The elliptic sequence is defined purely topologically.

Example 2: Let p be a weakly elliptic singularity whose weight dual graph is of the following form



$$Z = 1\ \ 1\ \ 1\ \ 1\ \ \overset{1}{1}\ \ 1\ \ 1\ \ 1\ \ 1$$

$$Z_{B_1} = 0\ \ 1\ \ 1\ \ 1\ \ \overset{1}{1}\ \ 1\ \ 1\ \ 1\ \ 0$$

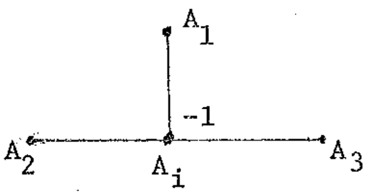
$$Z_{B_2} = 0\ \ 0\ \ 1\ \ 1\ \ \overset{1}{1}\ \ 1\ \ 1\ \ 0\ \ 0$$

$$Z_{B_3} = 0\ \ 0\ \ 0\ \ 1\ \ \overset{1}{1}\ \ 1\ \ 0\ \ 0\ \ 0$$

The elliptic sequence is $\{Z = Z_{B_0}, Z_{B_1}, Z_{B_2}, Z_{B_3} = Z_E\}$ and the length of the elliptic sequence is 4.

Proposition 3.5: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only weakly elliptic singularity. Then for any $A_i \not\subseteq |E|$, A_i are nonsingular rational curves with self intersection number less than or equal to -2.

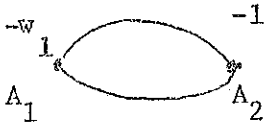
Proof: The fact that $A_i \not\subseteq |E|$ are nonsingular rational curves follows from Lemma 2.1. Suppose there exists $A_i \not\subseteq |E|$ such that $A_i \cdot A_i = -1$. It follows easily from Proposition 2.2 that A_i is a "star" in the dual weighted graph Γ of exceptional set $\pi^{-1}(p) = A$, i.e., there exist $A_1, A_2, A_3 \subseteq A$ such that Γ has the following graph as its subgraph



Then $\chi(A_1 + A_2 + A_3 + 2A_i) = 0$. This is impossible by Proposition 1.13 and that $A_i \not\subseteq \text{supp}E$.

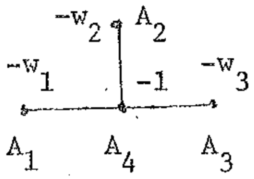
Lemma 3.6: Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of the normal two dimensional weakly elliptic singularity p. Suppose p is not a minimally elliptic singularity, then $-K' \geq Z + E$ whenever K' exists. If $E \cdot Z < 0$ and $|E| \subsetneq A$, then K' does not exist.

Proof: If π is the minimal resolution, then $A_i \cdot K' \geq 0$ for all $A_i \subseteq A$ by Proposition 1.8. So $-K' \geq Z > E$ by the definition of fundamental cycle. Suppose π is not the minimal resolution. Then the corresponding dual weighted graph Γ consists of either (a)



$w_1 \geq 5$

or (b)



$w_i \geq 2, \quad 1 \leq i \leq 3$

as its proper subgraph. In case (a), $E = A_1+A_2$. We claim that $k_1' \neq 0$ where $K' = \Sigma\, k_i'\, A_i$. For if $k_1' = 0$, then $A_1 \cdot K' \leq 0$ since K' is a negative cycle. On the other hand, $A_1 \cdot K' = A_1 \cdot K = -A_1 \cdot A_1 + 2g_1 - 2 \geq 3 > 0$. This is a contradiction. Hence $k_1' \neq 0$. We claim that k_2' cannot be zero also. For if $k_2' = 0$, then $A_2 \cdot K' \leq -2$ since $A_2 \cdot A_1 = 2$. On the other hand, $A_2 \cdot K' = A_2 \cdot K = -A_2 \cdot A_2 + 2g_2 - 2 = 1 - 2 = -1$. This is a contradiction Hence, $k_2' \neq 0$. It follows that $-K' \geq E$. In case (b), $E = A_1 + A_2 + A_3 + 2A_4$. We claim that one of k_i', $1 \leq i \leq 3$ cannot equal to zero. For if $k_1' = k_2' = k_3' = 0$, then $A_4 \cdot K' \geq 0$. This is because there exists no $A_i \subsetneqq |E|$, $A_i \cdot A_4 > 0$ by the proof of Proposition 3.5. However, $A_4 \cdot K' = A_4 \cdot K = -A_4 \cdot A_4 + 2g_4 - 2 = 1 - 2 = -1$. This is a contradiction. So we may assume $k_1' \neq 0$. If $k_2' = 0$, then $A_2 \cdot K' \leq 0$. On the other hand, $A_2 \cdot K' = A_2 \cdot K = -A_2 \cdot A_2 - 2 \geq 0$. Hence, $A_2 \cdot K' = 0$ and $A_2 \cdot A_2 = -2$. If k_3' also equals to 0, then similar argument will show $A_3 \cdot A_3 = -2$. The intersection matrix cannot be negative definite. So we may assume that $k_3' \neq 0$.

We claim that $k_4' \neq 0$. For if $k_4' = 0$, then $A_4 \cdot (K') \leq -2$. On the other hand, $A_4 \cdot K' = A_4 \cdot K = A_4 \cdot A_4 + 2g_4 - 2 = -1$. This is a contradiction So $k_4' \neq 0$. We claim that $k_2' \neq 0$. For if $k_2' = 0$, then $A_2 \cdot (K') \leq -1$. On the other hand $A_2 \cdot K' = A_2 \cdot K = A_2 \cdot A_2 + 2g_2 - 2 \geq 0$. This is a contradiction We claim that $k_4' \leq -2$. For if $k_4' = -1$, then $A_4 \cdot K' = k_1' + k_2' + k_3' + 1 \leq -2$. On the other hand, $A_4 \cdot K' = A_4 \cdot K = -A_4 \cdot A_4 + 2g_4 - 2 = -1$. This is a contradiction. So $k_4' \leq -2$. We have proved in both case (a) and (b) $-K' \geq E$. We claim that actually $-K' \gneq E$. Since p is not a minimally elliptic singularity, there exists $A_i \not\subseteq |E|$, $A_i \cap E \neq \phi$. It suffices to prove $k_i' \neq 0$. For if $k_i' = 0$, then $A_i \cdot K' < 0$. On the other hand, $A_i \cdot K' = A_i \cdot K = -A_i \cdot A_i + 2g_i - 2 = -A_i \cdot A_i - 2 \geq 0$. This is a contradiction. Therefore $-K' = E + D$ where D is a non-zero positive cycle. We claim that $A_i \cdot D \leq 0$ for all $A_i \subseteq A$. If $A_i \subseteq |E|$, then $A_i(-K') = A_i(-K) = A_i \cdot E$ by Lemma 1.14. So $A_i \cdot D = 0$. If $A_i \not\subseteq |E|$, then $A_i \cdot A_i \leq -2$ and hence $A_i(-K') = A_i(-K) = A_i \cdot A_i + 2 \leq 0$. However, $A_i \not\subseteq |E|$, so $A_i \cdot E \geq 0$. It follows that $A_i \cdot D = A_i(-K') - A_i E \leq 0$. This proves our claim. By definition of the fundamental cycle, $D \geq Z$. So in particular $-K' \geq Z + E$.

Suppose $E \cdot Z < 0$, we want to prove K' does not exist. Suppose on the contrary that K' exists. By the above proof $-K' = Z + D$ where D is a positive cycle. By (1.2), $\chi(-K') = -\frac{1}{2}[(-K') \cdot K + (-K') \cdot (-K')] = -\frac{1}{2}[(-K') \cdot (K') + K' \cdot K'] = 0$, so $0 =$

$\chi(Z+D) = \chi(Z) + \chi(D) - Z \cdot D$. Since p is a weakly elliptic singularity, $\chi(Z) \geq 0$ and $\chi(D) \geq 0$. Also $Z \cdot D \leq 0$ because Z is the fundamental cycle and D is a positive cycle. It follows easily that $\chi(D) = 0$ and $Z \cdot D = 0$. Since $Z \cdot E < 0$, $|D| \not\supseteq |E|$. By Proposition 1.13, we conclude that $D = 0$. But then $Z = -K' \geq Z + E$ which is absurd.

Theorem 3.7: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as it's only weakly elliptic singularity. Suppose p is not a minimally elliptic singularity and K' exists. Then the elliptic sequence is of the following form

$$Z_{B_0} = Z, \quad Z_{B_1}, \ldots\ldots, Z_{B_\ell}, \quad Z_{B_{\ell+1}} = Z_E \qquad \ell \geq 0.$$

Moreover, $-K' = \sum_{i=0}^{\ell} Z_{B_i} + E$.

Proof: Lemma 3.6 says that length of the elliptic sequence is greater than or equal to 2 and $-K' \geq Z + E$. So $-K' = Z + D_1$ where D_1 is a non-zero positive cycle on A. By (1.2), $\chi(-K') = 0$. So $0 = \chi(Z + D_1) = \chi(Z) + \chi(D_1) - Z \cdot D_1$. Since p is a weakly elliptic singularity, $\chi(Z) = 0$, $\chi(D_1) \geq 0$. Because Z is the fundamental cycle and D_1 is a positive cycle, so $Z \cdot D_1 \leq 0$. Consequently, $\chi(D_1) = 0$ and $Z \cdot D_1 = 0$. By Proposition 1.13, $\chi(D_1) = 0$ implies that $|D_1|$ is connected and contains $|E|$. We claim that $|D_1| = B_1$. Since $D_1 \cdot Z = 0$ and $|D_1|$ is connected and contains $|E|$, we have $|D_1| \subsetneq B_1$. Suppose $|D_1| \neq B_1$. Then there exists $A_i \not\subseteq |D_1|$, $A_i \subseteq B_1$ and $A_i \cap |D_1| \neq \phi$. Hence, $A_i(-K') = A_i(Z + D_1) = A_i \cdot D_1 > 0$. On the other hand, since $A_i \not\subseteq |E|$, $A_i \cdot (-K') = A_i A_i + 2 \leq 0$ by Proposition 3.5. This is a contradiction. Hence, $|B_1| = |D_1|$. Let U_1 be a holomorphically convex neighborhood of B_1 such that $\Phi_1: U_1 \to V_1$ represents B_1 as exceptional set where V_1 is a normal two-dimensional Stein space with $\Phi_1(B_1)$ as its only singularity. We claim that the K' cycle on U_1 which is denoted by K'_{U_1} exists and $K'_{U_1} = -D_1$. In fact for any $A_i \subsetneq B_1$

$$\begin{aligned} A_i \cdot (-D_1) &= A_i \cdot (-Z - D_1) \\ &= A_i \cdot K \text{ where } K \text{ is the canonical divisor on } M \\ &= -A_i \cdot A_i + 2g_i - 2 \\ &= A_i \cdot K_{U_1} \text{ where } K_{U_1} \text{ is the canonical divisor on } U_1. \end{aligned}$$

So $-D_1 = K'_{U_1}$. By induction on the length of elliptic sequence, the proof reduces to the following Proposition.

Proposition 3.8: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only weakly elliptic singularity. Suppose p is not a minimally elliptic singularity and K' exists. If the length of elliptic sequence is equal to two. Then the elliptic sequence is $\{Z, Z_E\}$. Moreover $-K' = Z + E$.

Proof: Lemma 3.6 says that $-K' > Z$. So $-K' = Z + D$ where D is a non-zero positive cycle on A. By (1.2), $\chi(-K') = 0$. So $0 = \chi(Z + D) = \chi(Z) + \chi(D) - Z \cdot D$. Since p is a weakly elliptic singularity, $\chi(Z) = 0$ and $\chi(D) \geq 0$. As Z is the fundamental cycle and D is a positive cycle, we have $Z \cdot D \leq 0$. Consequently, $\chi(D) = 0$ and $Z \cdot D = 0$. Arguing as above, we know that $|B_1| = |D|$. Moreover K'_{U_1} exists and $K'_{U_1} = -D$ where U_1 is a holomorphic convex neighborhood of B_1. By Lemma 3.6, $B_1 \neq |E|$ cannot occur since the length of the elliptic sequence is equal to two. So $|D| = B_1 = |E|$. We claim that $D = E$. Since $\chi(D) = 0$, we have $D \geq E$, i.e., $D = E + D'$ where D' is a positive cycle with $|D'| \subseteq |E|$. Since $A_i \cdot D = A_i \cdot (-K'_{U_1}) = A_i \cdot E$ for all $A_i \subseteq |E|$, so $A_i \cdot D' = 0$ for all $A_i \subseteq |E|$. It follows that $D' \cdot D' = 0$. Therefore $D' = 0$ and $D = E$. We have proved the elliptic sequence is $\{Z, Z_E\}$ and $-K' = Z + E$. Q.E.D.

Let p be the only singularity of the two dimensional hypersurface Stein space V. Let $\pi: M \to V$ be a resolution of V. Let $A = \bigcup_i A_i$, $1 \leq i \leq n$, be the decomposition of $A = \pi^{-1}(p)$ into irreducible components. Let μ be the Milnor number of p. Then Laufer [23] proved that

$$(3.2) \qquad \mu = n + K' \cdot K' - \dim H^1(A; \mathbb{C}) + 12 \dim H^1(M, \mathcal{O}).$$

Formula (3.2) has various applications. One is that it gives a means of calculating $\dim H^1(M, \mathcal{O})$ for hypersurface singularities. This calculation is very difficult if not impossible in general. However, given a weighted dual graph corresponding to a singularity, we have to solve a system of linear equations in order to find K'. For weakly elliptic singularities, Theorem 3.7 provides us a quick method to find K'.

§2 Maximally Elliptic Singularities

The length of the elliptic sequence gives information about $h = \dim H^1(M,\mathcal{O})$.

Theorem 3.9: Let $\pi: M \to V$ be the minimal good resolution of a normal two dimensional Stein space with p as its only weakly elliptic singularity. Then $\dim H^1(M,\mathcal{O})$ is less than or equal to the length of the elliptic sequence if K' exists.

Proof: If the length of the elliptic sequence is equal to 1, i.e. the elliptic sequence consists of the fundamental cycle Z only, then $Z \cdot E < 0$. By Theorem 4.1 of [24], $H^1(M,\mathcal{O}) = \mathbb{C}$. So from now on, we may assume that the elliptic sequence is of the following form: $Z_{B_o} = Z$, $Z_{B_1}, \ldots, Z_{B_\ell}$, $Z_{B_{\ell+1}} = Z_E$, $\ell \geq 0$, and $K' = -(\sum_{i=0}^{\ell} Z_{B_i} + E)$ by Theorem 3.7. Choose a computation sequence for the fundamental cycle Z of the following form: $Z_o = 0$, $Z_1, \ldots, Z_k = E, \ldots, Z_{r_1} = Z_{B_\ell}, \ldots,$ $Z_{r_j} = Z_{B_{\ell-j+1}}, \ldots Z_{r_{\ell+1}} = Z_{B_o} = Z$. Consider the following sheaf exact sequence

$$\begin{gathered}
0 \to \mathcal{O}(-Z)/\mathcal{O}(-Z-Z_1) \to \mathcal{O}_{Z+Z_1} \to \mathcal{O}_Z \to 0 \\
\vdots \\
0 \to \mathcal{O}(-Z-Z_{k-1})/\mathcal{O}(-Z-E) \to \mathcal{O}_{Z+E} \to \mathcal{O}_{Z+Z_{k-1}} \to 0 \\
\vdots \\
0 \to \mathcal{O}(-Z-Z_{r_\ell -1})/\mathcal{O}(-Z-Z_{r_\ell}) \to \mathcal{O}_{Z+Z_{B_1}} \to \mathcal{O}_{Z+Z_{r_\ell -1}} \to 0 \\
\vdots \\
0 \to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-Z_1) \to \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}+Z_1} \to \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}} \to 0 \\
\vdots \\
0 \to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-Z_{k-1})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-E) \to \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}+E} \\
\to \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}+Z_{k-1}} \to 0 \\
\vdots \\
0 \to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-Z_{r_{\ell-j}-1})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-Z_{B_{j+1}}) \\
\to \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}+Z_{B_{j+1}}} \to \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}+Z_{r_{\ell-j}-1}} \to 0 \\
\vdots \\
0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_1) \to \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+Z_1} \\
\to \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}} \to 0
\end{gathered} \tag{3.3}$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1}) / \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)$$

$$\to \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E} \to \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + Z_{k-1}} \to 0$$

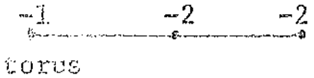
Let $Z_{B_i} = \sum_j {}_{B_i}z_j A_j$. We remark that if $E = A_1$ is an elliptic curve, then ${}_{B_i}z_1 = 1$ for all i by corollary 2.6. The usual long cohomology exact sequence for (3.3), (1.6) and Riemann Roch Theorem will show that $\dim H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) \leq \ell + 2 =$ length of the elliptic sequence because $H^o(M,{}_{j}\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{h-1}) / \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_h))$ and $H^1(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{h-1}) / \mathcal{O}(-\sum_{i=0} Z_{B_i} - Z_h))$ are non-zero only if $b = k$. Since $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) = H^1(M, \mathcal{O}(K'))= 0$ by Corollary 3.3 of [20], the exact sequence

$$H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) \to 0$$

shows that $\dim H^1(M, \mathcal{O}) = \dim H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) \leq \ell + 2$. Q.E.D.

The following example which is due to Laufer shows that $\dim H^1(M, \mathcal{O})$ can be strictly less than the length of the elliptic sequence even for hypersurface singularities. As far as the author's knowledge is concerned, this is the first known example for double-point singularity with maximal ideal cycle strictly greater than the fundamental cycle.

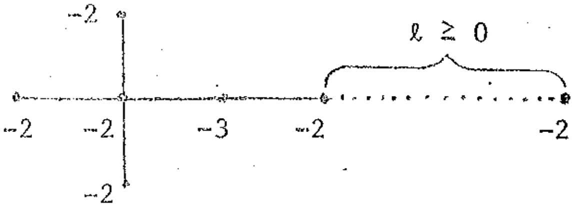
Example 3 (Laufer): Let V be the locus in $\mathbb{C}^3$ of $z^2 = y(x^4+y^6)$. Then the dual weighted graph is



This is a weakly elliptic singularity and the length of the elliptic sequence is equal to three. It can be calculated that $\mu = 22$, where μ is the Milnor number. By Theorem 3.7, $-K' = Z_{B_o}+Z_{B_1}+E$; $K'\cdot K' = Z_{B_o}^2+Z_{B_1}^2+E^2 = -3$. By (3.2), we know $\dim H^1(M,\mathcal{O}) = 2$.

The following two examples show that $\dim H^1(M,\mathcal{O})$ can actually equal to the length of the elliptic sequence.

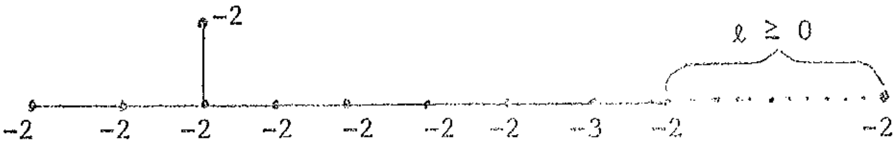
Example 4: Let V be the locus in $\mathbb{C}^3$ of $z^2 = y^3+x^{9+6\ell}$. Then the dual weighted graph is



This is a weakly elliptic singularity and the length of the elliptic sequence is equal to $\ell + 1$. It can be calculated that the Milnor number $\mu = 16 + 12\ell$. By Theorem 3.7 $-K' = \sum_{i=0}^{\ell-1} Z_{B_i} + E$, $K'^2 = \sum_{i=0}^{\ell-1} Z_{B_i}^2 + E^2 = -(\ell+1)$. By (3.2), we know that $\dim H^1(M, \mathcal{O}) = \ell + 1$ = length of the elliptic sequence.

Example 5: Let V be the locus in $\mathbb{C}^3$ of $z^2 = y^3 + x^{11+6\ell}$.

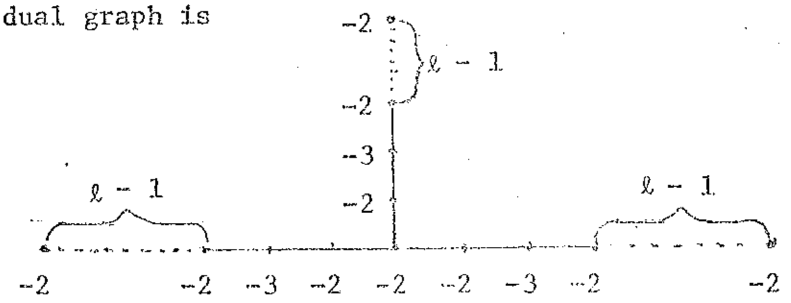
Then the dual weighted graph is



This is a weakly elliptic singularity and the length of the elliptic sequence is equal to $\ell + 1$. It can be calculated that the Milnor number $\mu = 20 + 12\ell$. By Theorem 3.7, $-K' = \sum_{i=0}^{\ell-1} Z_{B_i} + E$, $K'^2 = \sum_{i=0}^{\ell-1} Z_{B_i}^2 + E^2 = -(\ell+1)$. By (3.2), we know that $\dim H^1(M, \mathcal{O}) = \ell + 1$ = length of the elliptic sequence.

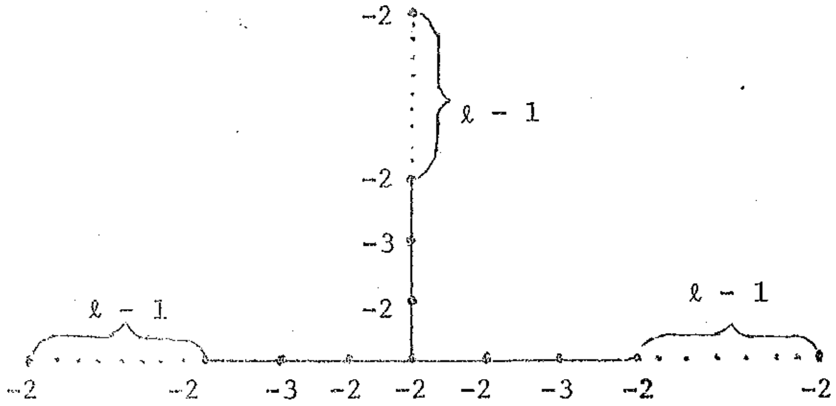
Example 6 (Wagreich): Let V be the locus in $\mathbb{C}^3$ of $z^3 = x^3 + y^{3\ell+1}$.

Then the dual graph is



This is a weakly elliptic singularity and the length of the elliptic sequence is equal to ℓ. It can be calculated that the Milnor number $\mu = 12\ell$. By Theorem 3.7, $-K' = \sum_{i=0}^{\ell-1} Z_{B_i} + E$, $K'^2 = -(3\ell+1)$. By (3.2) we know that $\dim H^1(M, \mathcal{O}) = \ell =$ length of the elliptic sequence.

Example 7 (Wagreich): Let V be the locus in $\mathbb{C}^3$ of $z^3 = x^3+y^{3\ell+2}$. Then the dual weighted graph is



This is a weakly elliptic singularity and the length of the elliptic sequence is equal to ℓ. It can be calculated that the Milnor number $\mu = 12\ell+4$. By Theorem 3.7, $-K' = \sum_{i=0}^{\ell-1} Z_{B_i} + E$. $K'^2 = -3\ell$. By (3.2) we know that $\dim H^1(M, \mathcal{O}) = \ell =$ length of the elliptic sequence.

Definition 3.10: Let V be a normal 2-dimensional Stein space with p as its only weakly elliptic singularity. Let $\pi: M \to V$ be the minimal good resolution. Suppose K' exists. If $\dim H^1(M,\mathcal{O})$ = length of the elliptic sequence, then p is called a maximally elliptic singularity.

Theorem 3.11: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only maximally elliptic singular point. Then ${}_V\mathcal{O}_p$ is Gorenstein.

Proof: If length of the elliptic sequence is equal to one, then Lemma 3.6 says that p is a minimally elliptic singularity. By Theorem 3.10 of [24], ${}_V\mathcal{O}_p$ is Gorenstein. Therefore we may suppose that the length of the elliptic sequence is greater than or equal to two. By Theorem 3.7, we know that the elliptic sequence is of the following form $Z_{B_0} = Z$, $Z_{B_1}, \ldots, Z_{B_\ell}$, $Z_{B_{\ell+1}} = Z_E$, $\ell \geq 0$ and $-K' = \sum_{i=0}^{\ell} Z_{B_i} + E$.

Serre duality gives $H^1(M,\mathcal{O})$ as dual to $H^1_*(M,\Omega)$ where Ω is the canonical sheaf, i.e. the sheaf of germs of holomorphic 2-forms. By Theorem 3.4 p.604 of [20], for suitable M, which can be arbitrarily small neighborhoods of $A = \pi^{-1}(p)$, $H^1_*(M,\Omega)$ may be identified with $H^0(M-A,\Omega)/H^0(M,\Omega)$. Let U_1 be a holomorphically convex neighborhood of B_1 such that Φ_1: $U_1 \to V_1$ represents B_1 as an exceptional set where V_1 is a normal two dimensional Stein space with $\Phi_1(B_1)$ as its only weakly elliptic singularity. We claim that K'_{U_1}, the K' cycle on U_1, exists. In fact $-K'_{U_1} = Z_{B_1} + \dots + Z_{B_\ell} + E$ because for all $A_i \subseteq B_1$. $A_i \cdot (-\sum_{i=1}^{\ell} Z_{B_i} - E) = A_i(-\sum_{i=0}^{\ell} Z_{B_i} - E) = A_i \cdot (K') = 2g_i - 2 - A_i \cdot A_i$. So the length of the elliptic sequence relative to Φ_1 is $\ell + 1$. Let $\omega_1, \dots, \omega_{\ell+2} \in H^0(M-A, \Omega)$ such that its images form a basis for $H^0(M-A,\Omega)/H^0(M,\Omega)$. Suppose on the contrary that ${}_v\mathcal{O}_p$ is not Gorenstein. We claim that the pole sets of ω_i, $1 \leq i \leq \ell + 2$, are contained in B_1. For if there exists ω_i, say ω_1, has a pole set which is not contained in B_1, then the divisor of ω_1 has the following form

$$(\omega_1) = -\sum_{i=1}^{t} a_i A_i + \sum_{j=t+1}^{n} b_j A_j + \sum_{k=1}^{n_1} d_k X_k \, .$$

$$a_i > 0, \quad b_j \geq 0, \quad d_k \geq 0$$

where $t \geq 1$, $X_r \not\subseteq A$, $X_r \cap A \neq \phi$, $\forall \; 1 \leq r \leq n_1$, and there exists $1 \leq i \leq t$ such that $A_i \not\subseteq B_1$. For any $A_h \subseteq A$,

$$A_h \cdot (K') = A_h \cdot (\omega_1)$$

$$A_h \cdot ((\omega_1) - K') = 0$$

Let $[\omega_1] = -\sum_{i=1}^{t} a_i A_i + \sum_{j=t+1}^{n} b_j A_j$. Then $A_h \cdot ([\omega_1] - K') \leq 0$ for all $A_h \subseteq A$. Since ${}_v\mathcal{O}_p$ is not a Gorenstein ring, either there exists $0 < b_j, t+1 \leq j \leq n$ or there exists $d_k > 0$, $0 \leq k \leq n_1$ by Lemma 3.6. If the former case occurs, then $[\omega_1] - K' \neq 0$

because K' is a negative cycle. If the latter case occurs, we claim that $[\omega_1] - K' \neq 0$ also. For let $0 \leq r \leq n_1$ such that $d_r > 0$. There exists $A_r \subseteq A$ such that $A_r \cdot X_r > 0$. Then

$$\begin{aligned} A_r \cdot ([\omega_1] - K') &= A_r \cdot [\omega_1] - A_r \cdot K' \\ &= A_r \circ [\omega_1] - A_r \cdot (\omega_1) \\ &= A_r \cdot [\omega_1] - A_r \cdot ([\omega_1] + \sum_{k=0}^{n_1} d_k X_k) \\ &= -A_r \cdot \sum_{k=0}^{n} d_k X_k \leq -d_r < 0 \end{aligned}$$

Therefore $[\omega_1] - K'$ is not zero in any cases. Notice that some coefficient of $Ai \nsubseteq B_1$ in $[\omega_1] - K'$ is strictly less than the corresponding coefficient of that component in the fundamental cycle Z because $-K' = \sum_{i=0}^{\ell} Z_{B_i} + E$. If $[\omega_1] - K'$ is a positive cycle, we let $Z^1 = \inf([\omega_1] - K', Z)$. It follows from M. Artin's argument, p.131-p.132 of [2] that Z^1 is also a positive cycle and $Z^1 \cdot A_k \leq 0$ for all $A_k \subseteq A$. However, $Z^1 < Z$. This contradicts the definition of the fundamental cycle Z. So $[\omega_1] - K'$ cannot be a positive cycle. Let

$$Z_o = [\omega_1] - K' = \sum_{i=1}^{s} h_i Ai - \sum_{j=s+1}^{n} c_j A_j$$

$$h_i \geq 0 \qquad c_j > 0 \qquad \text{and } s < n.$$

Without lose of generality, we may assume that $c_{s+1} = \max(c_j)$,

$s+1 \le j \le n$. Consider $Z_1 = c_{s+1} Z + z_{s+1} Z_o = \sum_{i=0}^{n} z_i^1 A_i$, where $Z = \sum_{i=1}^{n} z_i A_i$. Since $Z_o \cdot A_i \le 0$ for all $A_i \subseteq A$, we have $A_i \cdot Z_1 \le 0$ for all $A_i \subseteq A$. Also $z_i^1 \ge 0$ for $1 \le i \le s$ and $z_{s+1}^1 = 0$. By changing the index if necessary, we may assume $z_{s+2}^1 = \min(z_i^1)$, $s+2 \le i \le n$. If $z_{s+2}^1 \ge 0$, then Z_1 is a positive cycle with supp$Z_1 \subsetneqq A$ because $z_{s+1}^1 = 0$. If $z_{s+2}^1 < 0$, consider $Z_2 = -z_{s+2}^1 Z + z_{s+2} Z_1 = \sum_{i=1}^{n} z_i^2 A_i$, then $A_i \cdot Z_2 \le 0$ for all $A_i \subseteq A$, $z_i^2 \ge 0$ for $1 \le i \le s+1$ and $z_{s+2}^2 = 0$. Continuing this process, we finally get a positive cycle D on A with supp$D \subsetneqq A$ and $A_i \cdot D \le 0$ for all $A_i \subseteq A$. But this is impossible by previous argument. We conclude that the pole set of ω_i, $1 \le i \le \ell+2$, are contained in B_1. It follows that ω_i/U_1, the restriction of ω_i to U_1 are in $H^o(U_1 - B_1, \Omega)$ for all $1 \le i \le \ell+2$. Since the length of the elliptic sequence on U_1 is $\ell+1$, by Theorem 3.10 $\dim H^1(U_1, \mathcal{O}) \le \ell+1$. Hence $\dim H^o(U_1 - B_1, \Omega)/H^o(U_1, \Omega) \le \ell+1$ and there exists $\lambda_1, \ldots, \lambda_{\ell+2} \in \mathbb{C}$, not all $\lambda_i = 0$ such that $\lambda_1 \omega_1/U_1 + \ldots + \lambda_{\ell+2}\, \omega_{\ell+2}/U_1 \in H^o(U_1, \Omega)$ where ω_i/U_1 is the restriction of ω_i on U_1. It follows that $\lambda_1 \omega_1 + \ldots + \lambda_{\ell+2}\, \omega_{\ell+2} \in H^o(M, \Omega)$ which contradicts to our assumption that images of $\omega_1, \ldots, \omega_{\ell+2}$ form a basis for $H^o(M-A, \Omega)/H(M, \Omega)$.

Theorem 3.12: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only maximally elliptic singularity. If $Z_E \cdot Z_E \le -2$, then $m\mathcal{O} = \mathcal{O}(-Z)$.

Proof: The proof is long and is essentially an application of Laufer-type vanishing Theorems.

Proposition 3.13: Let $\pi: M \to V$ be the minimal good resolution of a normal two-dimensional Stein space with p as its only maximally elliptic singularity. Let $Z_{B_0} = Z, Z_{B_1}, \ldots, Z_{B_\ell}$, $Z_E = Z_{B_{\ell+1}}$ be the elliptic sequence. Then for any $0 \leq h \leq \ell$, there exists $f \in H^0(M, \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i}))$ such that $f \notin H^0(M, \mathcal{O}(-\sum_{i=0}^{h+1} Z_{B_i}))$. In fact, the vanishing order of f on A_j is precisely $\sum_{i=0}^{h} {}_{B_i}z_j$ where $Z_{B_i} = \sum_k {}_{B_i}z_k A_k$ and $A_j \subseteq B_{h+1}$.

Proof: By the definition of maximally elliptic singularity, $\dim H^1(M, \mathcal{O})$ = the length of the elliptic sequence. By the proof of Theorem 3.9, we know that maximal ellipticity implies $H^1(M, \mathcal{O}_{\sum_{i=0}^{h} Z_{B_i}}) = \mathbb{C}^{h+1}$ for all $0 \leq h \leq \ell$. Moreover, $H^0(M, \mathcal{O}_{\sum_{i=0}^{h+1} Z_{B_i}}) \to H^0(M, \mathcal{O}_{\sum_{i=0}^{h} Z_{B_i}})$ are surjective. Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{h+1} Z_{B_i})) \arrow[r] \arrow[d] & H^o(M,\mathcal{O}) \arrow[r] \arrow[d] & H^o(M,\mathcal{O}_{\sum_{i=0}^{h+1} Z_{B_i}}) \arrow[d] \\
0 \arrow[r] & H^o(M, \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i})) \arrow[r] & H^o(M,\mathcal{O}) \arrow[r] & H^o(M, \mathcal{O}_{\sum_{i=0}^{h} Z_{B_i}})
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & H^1(M, \mathcal{O}(-\sum_{i=0}^{h+1} Z_{B_i})) \arrow[r] \arrow[d] & H^1(M,\mathcal{O}) \arrow[r] \arrow[d] & H^1(M, \mathcal{O}_{\sum_{i=0}^{h+1} Z_{B_i}}) \arrow[r] \arrow[d] & 0 \\
\arrow[r] & H^1(M, \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i})) \arrow[r] & H^1(M,\mathcal{O}) \arrow[r] & H^1(M, \mathcal{O}_{\sum_{i=0}^{h} Z_{B_i}}) \arrow[r] & 0
\end{tikzcd}$$

Since $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)) = 0$ by Proposition 2.8,
$H^0(M,\mathcal{O}) \to H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + Z_E})$ is surjective. It follows that
$H^0(M,\mathcal{O}) \to H^0(M, \mathcal{O}_{\sum_{i=0}^{h} Z_{B_i}})$ are surjective for all $0 \leq h \leq \ell$.

An easy diagram chase will show that there exists
$f \in H^0(M, \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i}))$ but $f \notin H^0(M, \mathcal{O}(-\sum_{i=0}^{h+1} Z_{B_i}))$. Let

$A_j \subsetneq B_{h+1}$. Choose a computation sequence for $Z_{B_{h+1}}$ of the following form: $Z_o = 0$, $Z_1 = A_j, \ldots Z_{r_{\ell-h}} = Z_{B_{h+1}}$. Look at the following sheaf exact sequence

$$0 \to \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i} - Z_1) \to \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i}) \to \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{h} Z_{B_i} - Z_1) \to 0$$

$$\vdots$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i} - Z_k) \to \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i} - Z_{k-1}) \to \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i} - Z_{k-1})/\mathcal{O}(-\sum_{i=0}^{h} Z_{B_i} - Z_k) \to 0$$

$$\vdots$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i} - Z_{B_{h+1}}) \to \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i} - Z_{r_{\ell-h}-1}) \to \mathcal{O}(-\sum_{i=0}^{h} Z_{B_i} - Z_{r_{\ell-h}-1})/\mathcal{O}(-\sum_{i=0}^{h} Z_{B_i} - Z_{B_{h+1}}) \to 0$$

If the vanishing order of f of A_j is larger than $\sum_{i=0}^{\ell} {}_{B_i} z_j$, then the usual cohomology exact sequence argument will show that $f \in H^o(M, \mathcal{O}(-\sum_{i=0}^{h+1} Z_{B_i}))$, which is a contradiction. Q.E.D.

The following corollary is a partial converse of Theorem 3.2.

Corollary 3.14: Let V be a normal two dimensional Stein space with p as its only maximally elliptic singular point. Let $Z_{B_o} = Z$, $Z_{B_1}, \ldots, Z_{B_\ell}$, $Z_{B_{\ell+1}} = Z_E$ be the elliptic sequence. If there exists $A_1 \subseteq |E|$ such that the coefficients of A_1 in Z_{B_i}, $0 \leq i \leq \ell$ are equal, then there exists $f \in H^o(M, \mathcal{O})$, $\omega \in H^o(M-A, \Omega)$ such that ω, $f\,\omega, \ldots, f^{\ell+1}\omega$ forms a basis of $H^1_*(M, \mathcal{O})$.

Proof: An easy consequence of Theorem 3.7, Theorem 3.11 and Proposition 3.13.

The following Theorem will be useful in calculating of Hilbert function dim m^n/m^{n+1}.

Theorem 3.15: Let π: $M \to V$ be the minimal good resolution of a normal 2-dimensional Stein space with p as its only maximally elliptic singularity. If $Z_E \cdot Z_E \leq -3$, then

$$H^o(M, \mathcal{O}(-Z)) \underset{C}{\otimes} H^o(M, \mathcal{O}(-nZ)) \to H^o(M, \mathcal{O}(-(n+1)Z))$$

is surjective for all $n > 1$. If we assume further that the length of the elliptic sequence is equal to two, then the above map is surjective for all $n \geq 1$. In this case, $m^n \simeq H^o(A, \mathcal{O}(-nZ))$ for all $n \geq 0$ where $A = \pi^{-1}(p)$.

Proof: The proof is long and tedious.

CHAPTER IV

ALMOST MINIMALLY ELLIPTIC SINGULARITIES

Although the title of this chapter is "Almost Minimally Elliptic Singularities", our main interest is to build up a theory for those singularities which has $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ Gorenstein. We will prove that if p is an almost minimally elliptic singularity and ${}_V\mathcal{O}_p$ is Gorenstein then $H^1(M,\mathcal{O}) = \mathbb{C}^2$. But it is not true that $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ Gorenstein will imply that p is an almost minimally elliptic singularity.

§1. General Theory for Almost Minimally Elliptic Singularities

Definition 4.1: Let π: $M \to V$ be the minimal good resolution of a normal two dimensional Stein space with p as its only weakly elliptic singular point. Suppose p is not a minimally elliptic singularity, i.e., $|E| \subsetneqq \pi^{-1}(p)$. If for all $A_i \not\subseteq |E|$ and $A_i \cap |E| \neq \phi$, then $A_i \cdot Z < 0$. We call p an almost minimally elliptic singularity.

Theorem 4.2: Let π: $M \to V$ be the minimal good resolution of a normal two dimensional Stein space with p as its only almost minimally elliptic singularity. Suppose ${}_V\mathcal{O}_p$ is Gorenstein. Then $H^1(M,\mathcal{O}) = \mathbb{C}^2$.

Proof: If $\dim H^1(M,\mathcal{O}) = 0$, then p is a rational singularity, which implies $\chi(Z) = 1$. This is a contradiction. If $\dim H^1(M,\mathcal{O}) = 1$, then p is a minimally elliptic singularity by Theorem 3.10 of [24]. This contradicts our definition of almost minimal elliptic singularity. Therefore $\dim H^1(M,\mathcal{O}) \geq 2$. On the other hand $\dim H^1(M,\mathcal{O}) \leq 2$ by Theorem 3.9. We conclude that $\dim H^1(M,\mathcal{O}) = 2$. Q.E.D.

Example 3 in Chapter III shows that $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ Gorenstein do not imply that p is an almost minimal elliptic singularity. However, a partial converses of Theorem 4.2 will be shown later.

<u>Lemma 4.3</u>: Let $\pi: M \to V$ be the minimal good resolution of a normal two dimensional Stein space with p as its only weakly elliptic singularity. If $\dim H^1(M,\mathcal{O}) \neq 1$, then one of the following cases hold:

(1) $H^o(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) \simeq \mathbb{C} \simeq H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))$

(2) $H^o(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) \simeq 0 \simeq H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))$

<u>Proof</u>: Since $H^1(M,\mathcal{O}) \neq 1$, we have $E\cdot Z = 0$ by Theorem 4.1 of [24]. Choose a computation sequence for Z as follows. $Z_o = 0,\ldots,$ $Z_k = E,\ldots.$ Consider the following sheaf exact sequences.

$$0 \to \mathcal{O}(-Z-Z_1)/\mathcal{O}(-Z-E) \to \mathcal{O}(-Z)/\mathcal{O}(-Z-E) \to \mathcal{O}(-Z)/\mathcal{O}(-Z-Z_1) \to 0$$

$$0 \to \mathcal{O}(-Z-Z_2)/\mathcal{O}(-Z-E) \to \mathcal{O}(-Z-Z_1)/\mathcal{O}(-Z-E) \to \mathcal{O}(-Z-Z_1/\mathcal{O}(-Z-Z_2) \to 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$0 \to \mathcal{O}(-Z-Z_{k-1})/\mathcal{O}(-Z-E) \to \mathcal{O}(-Z-Z_{k-2})/\mathcal{O}(-Z-E) \to \mathcal{O}(-Z-Z_{k-2})/\mathcal{O}(-Z-Z_{k-1}) \to 0$$

By Riemann Roch Theorem, the usual long cohomology sequence argument will show that either (1) or (2) holds.

<u>Theorem 4.4</u>: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only weakly elliptic singularity. Suppose $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Then p is an almost minimally elliptic singularity if and only if $H^o(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) = \mathbb{C}$.

<u>Proof</u>: "$\Longrightarrow$" By (1.1) and (1.6), $H^o(M, \mathcal{O}_Z) \simeq \mathbb{C} \simeq H^1(M, \mathcal{O}_Z)$. The long exact cohomology sequence

$$0 \to H^o(M, \mathcal{O}(-Z)) \to H^o(M,\mathcal{O}) \to H^o(M, \mathcal{O}_Z) \to H^1(M, \mathcal{O}(-Z)) \to H^1(M,\mathcal{O})$$
$$\to H^1(M, \mathcal{O}_Z) \to 0$$

will show that $H^1(M, \mathcal{O}(-Z)) = \mathbb{C}$. Since p is an almost minimally elliptic singularity, $-K' = Z + E$. By (1.2), $H^1(M, \mathcal{O}(-Z-E)) = 0$. Now the following exact sequence

$$H^1(M, \mathcal{O}(-Z-E)) \to H^1(M, \mathcal{O}(-Z)) \to H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) \to 0$$

will show that $H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) \simeq \mathbb{C}$.

"$\Longleftarrow$" Conversely, suppose $H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) = \mathbb{C}$. Let $Z_{B_o}, Z_{B_1}, \ldots, Z_{B_\ell}, Z_E$ be the elliptic sequence. Then $-K' = \sum_{i=0}^{\ell} Z_{B_i} + E$. Choose a computation sequence for Z as follows:

$$Z_o = 0, Z_1, \ldots, Z_k = E, \ldots, Z_{r_1} = Z_{B_\ell}, \ldots, Z_{r_\ell} = Z_{B_1}, \ldots Z_{r_{\ell+1}} = Z_{B_o} = Z.$$

Suppose p is not an almost minimal elliptic singularity, then $\ell \geq 1$. Look at the following exact cohomology sequence

$$0 \to \underset{\mathbb{C}}{\underset{\wr\wr}{H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E))}} \to H^0(M, \mathcal{O}_{Z+E}) \to \underset{\mathbb{C}}{\underset{\wr\wr}{H^0(M, \mathcal{O}_Z)}}$$

$$\to \underset{\mathbb{C}}{\underset{\wr\wr}{H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E))}} \to H^1(M, \mathcal{O}_{Z+E}) \to \underset{\mathbb{C}}{\underset{\wr\wr}{H^1(M, \mathcal{O}_Z)}} \to 0$$

It is easy to see that $H^0(M, \mathcal{O}_{Z+E}) \to H^0(M, \mathcal{O}_Z)$ is surjective. Therefore $H^1(M, \mathcal{O}_{Z+E}) \simeq \mathbb{C}^2 \simeq H^0(M, \mathcal{O}_{Z+E})$. Since the following two sequences

$$H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \to H^1(M, \mathcal{O}_{Z+E}) \to 0$$

$$H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \to 0$$

are exact, $H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}})$ is an isomorphism by dimension considerations. It follows that $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to H(M, \mathcal{O})$ is a zero map. As ${}_V\mathcal{O}_p$ is Gorenstein, there exists $\omega \in H^0(M-A, \Omega)$ having no zeros near A. Let (ω) be the divisor of ω. Then $(\omega) = -\sum_{i=0}^{\ell} Z_{B_i} - E$. Let w_1 be the order of the pole of ω on $A_1 \subseteq |E|$. Consider a cover as in Lemma 3.8 of [24]. On P_1

$$\omega = \frac{\omega_1(x_1,y_1)}{y_1^{w_1}} \quad dx_1 \wedge dy_1$$

where $\omega_1(x_1,y_1)$ is a holomorphic function, $\omega_1(x_1,0) \not\equiv 0$. There is a holomorphic function $f(x_1)$, $r \le x_1 \le R$ such that

$$\int_{\substack{|x_1|=R \\ |y_1|=R}} y_1^{w_1-1} \; f(x_1) \; \frac{\omega_1(x_1,y_1)}{y_1^{w_1}} \, dx_1 \wedge dy_1 \neq 0$$

Let $\lambda_{o1} = y_1^{w_1-1} \; f(x_1)$ and $\lambda_{oj} = 0$ for $j \neq 1$. Then by Lemma 3.8 of [24], $\mathrm{cls}[\lambda] \neq 0$ in $H^1(M,\mathcal{O})$. However, $w_1-1 \geq \sum_{i=0}^{\ell} B_i z_1$. Hence λ may be thought of as also a cocycle in $H^1(N(\mathfrak{U}), \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}))$. Consequently, $\mathrm{cls}[\lambda] = 0$ in $H^1(M,\mathcal{O})$ because $H^1(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to H^1(M,\mathcal{O})$ is a zero map. This leads to a contradiction. Q.E.D.

Theorem 4.5: Let $\pi\colon M \to V$ be the minimal good resolution of normal two dimensional Stein space V with p as its only weakly elliptic singularity. If $\mathcal{O}(-Z)/\mathcal{O}(-Z-E)$ corresponds to a trivial line bundle L over $(|E|,\mathcal{O}_E)$, then $H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E)) \simeq \mathbb{C}$. Conversely, suppose $H^1(M,\mathcal{O}) \simeq \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. If $H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E)) = \mathbb{C}$, then $\mathcal{O}(-Z)/\mathcal{O}(-Z-E)$ corresponds to a trivial line bundle L over $(|E|,\mathcal{O}_E)$.

Proof: Suppose $\mathcal{O}(-Z)/\mathcal{O}(-Z-E)$ corresponds to a trivial line bundle L over $(|E|,\mathcal{O}_E)$. Let U be an holomorphically convex neighborhood of $|E|$ such that $\Phi\colon U \to V_1$ represents $|E|$ as an exceptional set where V_1 is a normal 2-dimensional Stein space with $\Phi(|E|)$ as its only minimally elliptic singularity. The group of sections of B is isomorphic to $H^o(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E))$. However, L is a trivial bundle over $(|E|, \mathcal{O}_E)$. So the group of sections of L is isomorphic to $H^o(U, \mathcal{O}_E) \simeq \mathbb{C}$. Therefore $H^o(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) = \mathbb{C}$.

Conversely, suppose $H^1(M, \mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Then $H^o(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) \simeq \mathbb{C}$ implies that p is an almost minimally elliptic singularity by Theorem 4.4. There exists $f \in H^o(M, \mathcal{O}(-Z))$ such that the image of f in $H^o(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E))$ viewed as section of the line bundle L is nowhere zero by Proposition 3.13. Hence L is a trivial bundle over $(|E|, \mathcal{O}_E)$. Q.E.D.

With notation as above, let $\phi: \mathcal{O} \to \mathcal{O}_E = \mathcal{O}/\mathcal{O}(-E)$ be the quotient map. Define $\mathcal{O}_E^* = \phi(\mathcal{O}^*) \subseteq \mathcal{O}_E$. Let $\alpha: \mathbb{Z} \to \mathcal{O}_E$ be $\phi \cdot i$ where $i: \mathbb{Z} \to \mathcal{O}$ is the obvious inclusion map. $\beta: \mathcal{O}_E \to \mathcal{O}_E^*$ is defined as follows. For a germ f in a stalk of $\mathcal{O}_E$, let F be a germ in $\mathcal{O}$ such that $\phi(F) = f$. Then we set $\beta(f) = \phi$ $(\exp 2\pi i\ F)$. We claim that β is well defined. Let F_1 be another germ in $\mathcal{O}$ such that $\phi(F_1) = f$. Then $F_1 = F + g$ where g can be considered as germ in $\mathcal{O}(-E)$. Hence

$$\phi(\exp(2\pi i F_1)) = \phi(\exp(2\pi i F+2\pi i g)) = \phi\, (1 + \frac{2\pi i F+2\pi i g}{1!}$$

$$+ \frac{(2\pi i F+2\pi i g)^2}{2!} + \ldots + \frac{(2\pi i F+2\pi i g)^n}{n!} + \ldots .)$$

$$= \phi[(1 + \frac{2\pi i F}{1} + \frac{(2\pi i F)^2}{2!} + \ldots + \frac{(2\pi i F)^n}{n!} + \ldots) + gh]$$

$$= \phi(\exp 2\pi i F).$$

Lemma 4.6: $0 \to \mathbb{Z} \overset{\alpha}{\to} \mathcal{O}_E \overset{\beta}{\to} \mathcal{O}_E^* \to 0$ is an exact sheaf sequence.

Proposition 4.7: Let π: $M \to V$ be the minimal good resolution of normal 2-dimensional Stein space V with p as its only weakly elliptic singularity. Let $\mathcal{O}(-Z)/\mathcal{O}(-Z-E)$ correspond to a line bundle L over $(|E|, \mathcal{O}_E)$. Suppose $H^1(M, \mathcal{O}) \simeq \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_o}=Z, \ldots, Z_{B_\ell}$, $Z_{B_{\ell+1}}=Z_E$ be the elliptic sequence. Let D be the subvariety of B_ℓ consisting of those irreducible components $A_i \subseteq B_\ell$ such that $A_i \cap |E| \neq \phi$. Suppose Z/D, the restriction of Z to D, is equal to Z_{B_ℓ}/D, the restriction of Z_{B_ℓ} to D. Then $L^{\ell+1}$ is a trivial line bundle over $(|E|, \mathcal{O}_E)$.

Note: Let $A = \bigcup_{i=1}^{n} A_i \supseteq D = \bigcup_{i=1}^{t} A_i$. If $Z = \sum_{i=1}^{n} z_i A_i$, then $Z/D = \sum_{i=1}^{t} z_i A_i$.

Proof: Let $Z_o = 0, \ldots, Z_k=E, \ldots$ be a computative sequence for Z. Look at the following sheaf exact sequences:

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_1)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)$$

$$\to \mathcal{O}(\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_1) \to 0$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_2)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_1)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)$$

$$\to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_1)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_2) \to 0$$

$$\vdots$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-2})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)$$

$$\to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-2})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1}) \to 0$$

By Riemann Roch Theorem, the usual long exact cohomology sequence will show that either $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \cong \mathbb{C} \cong H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$ or $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \cong 0 \cong H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$. We claim that the latter case cannot occur. Otherwise $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}))$ will be an isomorphism. However, by Theorem 4.9, which will be proved later, we have $m\mathcal{O} \subseteq \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$. It follows that the maximal ideal cycle $Y \gtrsim \sum_{i=0}^{\ell} Z_{B_i} + E = -K'$. This is absurd and our claim is proved. Hence we have the following exact sequence

$$0 \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}))$$

$$\to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \simeq \mathbb{C} \to 0$$

Let $f \varepsilon H^o(M, \mathcal{O}(\sum_{i=0}^{\ell} Z_{B_i}))$ be such that the image of f in $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$ is not zero. Then $f \notin H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$. We are going to prove that actually $f \notin H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - A_1))$ for any $A_1 \lneqq |E|$. Choose a computation sequence of the following form: $Z_o=0$, $Z_1=A_1, \ldots, Z_k=E, \ldots.$

Consider the following sheaf exact sequences

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_2) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_1) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_1)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_2) \to 0$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_3) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_2) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_2)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_3) \to 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1}) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E) \to 0$$

By Riemann Roch Theorem, the usual long cohomology exact sequence will show that $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_j)) \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{j-1}))$, $2 \le j \le k$ are isomorphism. By composing the maps, we get

$H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - A_1))$ is an isomorphism. Hence $f \notin H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - A_1))$. The image of f in $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$ viewed as section of the line bundle N over $(|E|, \mathcal{O}_E)$ corresponding to the sheaf $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)$ is nowhere zero. Hence N is a trivial bundle over $(|E|, \mathcal{O}_E)$.

Let us prove that for any $A_i \nsubseteq B_\ell$, $A_i \cap |E| = \phi$. First observe that $Z/D = Z_{B_\ell}/D$ implies $Z_{B_i}/D = Z_{B_\ell}/D$ for all $0 \leq i \leq \ell$. Suppose first that $A_i \subsetneq B_{\ell-1}$ and $A_i \nsubseteq B_\ell$. If $A_i \cap |E| \neq \phi$, then there exists $A_j \subseteq |E|$ such that $A_i \cap A_j \neq \phi$. Since $Z_{B_{\ell-1}}/D = Z_{B_\ell}/D$ and $A_j \cdot Z_{B_\ell} = 0$, $A_j \cdot Z_{B_{\ell-1}} \geq A_j \cdot (Z_{B_{\ell-1}}/D + A_i) = A_j \cdot (Z_{B_\ell}/D + A_i)$ $= A_j(Z_{B_\ell} + A_i) = 1 > 0$. This is a contradiction. Suppose that if $A_i \subseteq B_h$ and $A_i \nsubseteq B_\ell$, then $A_i \cap |E| = \phi$. We want to prove that it is also true for B_{h-1}. Then the decreasing induction argument will complete the proof. Let $A_i \subseteq B_{h-1}$ and $A_i \nsubseteq B_\ell$. If $A_i \cap |E| \neq \phi$, then there exists $A_j \subseteq |E|$ such that $A_i \cap A_j \neq \phi$. By induction hypothesis, $0 = A_j \cdot Z_{B_h} = A_j \cdot Z_{B_h}/D$. Hence $A_j \cdot Z_{B_{h-1}} \geq A_j \cdot (Z_{B_{h-1}}/D + A_i)$ $= A_j \cdot (Z_{B_h}/D + A_i) = 1 > 0$. This is absurd. Our claim is proved.

Hence

$$L^{\ell+1} = \mathcal{O}(-Z)/\mathcal{O}(-Z-E) \underbrace{\otimes_{\mathcal{O}_E} \cdots \otimes_{\mathcal{O}_E} \mathcal{O}(-Z)/\mathcal{O}(-Z-E)}_{\ell+1}$$

$$\cong \mathcal{O}(-(\ell+1)Z)/\mathcal{O}(-(\ell+1)Z-E) \cong \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)$$

It follows that $L^{\ell+1} \cong N$ is a trivial bundle over $(|E|, \mathcal{O}_E)$.

Theorem 4.8: Let $\pi: M \to V$ be the minimal good resolution of normal 2-dimensional Stein space V with p as its only weakly elliptic singularity. Suppose $H^1(M, \mathcal{O}) = \mathbb{C}^2$, $H^1(|E|, \mathbb{Z}) = 0$, and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_o}, Z_{B_1}, \ldots Z_{B_\ell}$, $Z_E = Z_{B_{\ell+1}}$ be the elliptic sequence. Let D be the subvariety of B_ℓ consisting of those irreducible components $A_i \subseteq B_\ell$ such that $A_i \cap |E| \neq \phi$. If $Z/D = Z_{B_\ell}/D$, then $\ell = 0$, i.e., p is an almost minimally elliptic singularity.

Proof: Let L be a line bundle over $(|E|, \mathcal{O}_E)$ corresponding to $\mathcal{O}(-Z)/\mathcal{O}(-Z-E)$. Consider the following commutative diagram

$$\begin{tikzcd}
H^1(|E|, \mathcal{O}_E^*) \arrow[r, "c^*"] \arrow[d, "\phi_1"] & H^2(|E|, \mathbb{Z}) \arrow[d, "\wr"] \\
H^1(|E|, \mathcal{O}_{|E|}^*) \arrow[r] \arrow[d, "\phi_2"] & H^2(|E|, \mathbb{Z}) \arrow[d, "\wr"] \\
\bigoplus_{A_i \subseteq |E|} H^1(A_i, \mathcal{O}_{A_i}^*) \arrow[r, "c"] & H^2(|E|, \mathbb{Z}) \simeq \bigoplus_{A_i \subseteq |E|} H^2(A_i, \mathbb{Z})
\end{tikzcd}$$

Since $A_i \cdot Z = 0$, $c(\phi_2 \cdot \phi_1 (L)) = 0$. Therefore $c^*(L) = 0$.

Look at the following exact sequence:

$$0 \simeq H^1(|E|, \mathbb{Z}) \to H^1(|E|, \mathcal{O}_E) \to H^1(|E|, \mathcal{O}_E^*) \xrightarrow{c^*} H^2(|E|, \mathbb{Z}).$$

Since $H^1(|E|, \mathcal{O}_E) = \mathbb{C}$, $c^*(L) = 0$ and the fact that $L^{\ell+1}$ is a trivial bundle by Proposition 4.7, it follows that L is a trivial bundle itself. By Theorem 4.4 and Theorem 4.5, p is an almost minimally elliptic singularity, i.e., $\ell = 0$.

§2 Calculation of Multiplicities

Suppose $H^1(M,\mathcal{O}) \simeq \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. In this section we identify the maximal ideal and in particular, we get a formula for the multiplicity of a singularity.

Theorem 4.9: Let $\pi\colon M \to V$ be the minimal good resolution of normal 2-dimensional Stein space with p as its only weakly elliptic singularity. Suppose $H^1(M,\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0} = Z, \ldots, Z_{B_1}, \ldots, Z_{B_\ell}, Z_{B_{\ell+1}} = Z_E$ be the elliptic sequence. Then $m\mathcal{O} \subseteq \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$. If $Z_E \cdot Z_E \leq -2$, then $m\mathcal{O} = \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$.

Proof: Since $\chi(\sum_{i=0}^{\ell} Z_{B_i}) = 0$ by (1.4), $\dim H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) = \dim H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}})$.

The following two exact sequences

$$H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \to H^1(M,\mathcal{O}_Z) \simeq \mathbb{C} \to 0$$

$$H^1(M,\mathcal{O}) \to H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \to 0$$

say that $\dim H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}})$ is either two or one. If $H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) = \mathbb{C}^2$, then $H^1(M,\mathcal{O}) \to H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}})$ is an isomorphism by dimensional consideration. It follows that $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to H^1(M,\mathcal{O})$

is a zero map. As ${}_V\mathcal{O}_p$ is Gorenstein, by the proof of Theorem 4.4, we will get a contradiction. We conclude that $H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) = \mathbb{C}$.

Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \arrow[r] \arrow[d] & H^0(M, \mathcal{O}) \arrow[r] \arrow[d, "\wr"] & H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \cong \mathbb{C} \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & H^0(M, \mathcal{O}(-Z)) \arrow[r] & H^0(M, \mathcal{O}) \arrow[r] & H^0(M, \mathcal{O}_Z) \cong \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, $H^0(M, (-\sum_{i=0}^{\ell} Z_{B_i})) \to H^0(M, \mathcal{O}(-Z))$ is an isomorphism. Since $m\mathcal{O} \subseteq \mathcal{O}(-Z)$, it follows easily that $m\mathcal{O} \subseteq \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$.

Suppose $Z_E \cdot Z_E \leq -2$, we want to prove $m\mathcal{O} = \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$. It suffices to prove $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}) \subseteq m\mathcal{O}$. Let us first show that:

$$(4.1) \qquad \rho: \ H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - A_1))$$

is surjective for all $A_1 \subseteq |E|$. If $E = A_1$ is a nonsingular elliptic curve, then $-K' = \sum_{i=0}^{\ell} Z_{B_i} + A_1$. Since $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - A_1)) = 0$ by Theorem 3.2 of [20], ρ is surjective by the usual long cohomology exact sequence argument. If $|E|$ has at least two irreducible components, then $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - A_1)) = 0$ by Riemann Roch Theorem. We are going to show $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - A_1)) \simeq \mathbb{C} \simeq H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}))$. The exact sequence

$$0 \to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to \overset{\simeq \mathbb{C}^2}{H^1(M, \mathcal{O})} \to H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \simeq \mathbb{C} \to 0$$

shows that we indeed have $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) = \mathbb{C}$. Choose a computation sequence for Z of the following form, $Z_o=0$, $Z_1=A_{i_1}=A_1, \ldots,$ Z_{k-1}, $Z_k=E, \ldots.$ The long exact cohomology sequence

$$0 \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) \simeq \mathbb{C} \to H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+A_1})$$

$$\to H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \simeq \mathbb{C} \to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) = 0$$

$$\to H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+A_1}) \to H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \to 0$$

will show that $H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+A_1}) = \mathbb{C}^2$ and $H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) = \mathbb{C}$.

Consider the following long exact cohomology sequence

$$0 \to H^o(M, (-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) \to H^o(M, \mathcal{O}) \to H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+A_1}) \simeq \mathbb{C}^2$$

$$\to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+A_1}) = \mathbb{C} \to 0.$$

We claim that $H^0(M,\mathcal{O}) \to H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+A_1})$ is surjective. Otherwise the image R of $H^0(M,\mathcal{O}) \to H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+A_1})$ will be isomorphic to $\mathbb{C}$. The five lemma together with the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) \arrow[r] \arrow[d] & H^0(M,\mathcal{O}) \arrow[r] \arrow[d, "\wr"] & R \simeq \mathbb{C} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & H^0(M, \mathcal{O}(-Z)) \arrow[r] & H^0(M,\mathcal{O}) \arrow[r] & H^0(M,\mathcal{O}_Z) \simeq \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

will show that $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} -A_1)) \to H^0(M, \mathcal{O}(-Z))$ is an isomorphism. The following sheaf exact sequences

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_2) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_1) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_1)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_2) \to 0$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_3) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_2) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_2)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_3) \to 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-E) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_{k-1}) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_{k-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_k) \to 0$$

will show that $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} -Z_j)) \to H^0(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} -Z_{j-1}))$ are isomorphism for $2 \leq j \leq k$. By composing the maps, we get $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} -E)) \to H^0(M, \mathcal{O}(-Z))$ is an isomorphism. Since $m\mathcal{O} \subseteq \mathcal{O}(-Z)$, the maximal ideal cycle $Y \geq \sum_{i=0}^{\ell} Z_{B_i} + E = -K'$. This contradicts Theorem 2.21. We conclude that $H^0(M,\mathcal{O}) \to H^0(M,\mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+A_1})$ is surjective. It follows that $H^1(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) = \mathbb{C}$.

Look at the following exact cohomology sequence

$$0 \to H^0(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) \to H^0(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}))$$

$$\to H^0(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) \to H^1(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1))$$

$$\to H^1(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to H^1(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) \to 0.$$

Since $H^1(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) = 0$, $H^1(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)) \to H^1(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}))$ is an isomorphism by dimension considerations. Therefore ρ in (4.1) is surjective. Given a point $a_1 \in A_1$, let $F \in H^0(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1))$ be nonzero near a_1 as a section of the line bundle associated to $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-A_1)$. $f \in H^0(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}))$ projecting onto F will generate $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$ near a_1 since it must vanish to the prescribed orders on the A_1 near a_1 and will have no other zeros near a_1.

In order to prove $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}) \subseteq m\mathcal{O}$, it remains to prove $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}) \subseteq m\mathcal{O}$ near A-supp$E$. There are two subcases.

Case (1): There exists $A_i \subseteq |E|$ such that $E \cdot Z_E + 1 \leq A_i \cdot Z_E \leq -1$ or $E = A_i$ is a nonsingular elliptic curve. For any

$A_1 \not\subseteq |E|$, choose a computation sequence for Z of the following form, $Z_o=0$, $Z_1=A_{i_1}=A_1, \ldots, Z_r, Z_{r+1}, \ldots, Z_{r+k} = Z_r+E, \ldots Z_{r_{\ell+1}} = Z$ where supp $Z_r \subseteq \overline{A - |E|}$ and $Z_{r+1}-Z_r, \ldots, Z_{r+k}-Z_r = E$ is part of a computation sequence for Z. Our hypothesis guarantees that the computation sequence can be so chosen such that $A_{i_{r+k}} \cdot Z_E<0$ by Corollary 2.3. Consider the following exact sheaf sequence for $n \geq 0$.

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_1) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ)$$

$$\to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ) / \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_1) \to 0$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_j) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_{j-1})$$

$$\to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_{j-1}) / \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_j) \to 0$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_{r_{\ell+1}}) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_{r_{\ell+1}-1})$$

$$\to \frac{\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_{r_{\ell+1}-1})}{\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z)} \to 0 . \tag{4.2}$$

We recall that $(\sum_{i=0}^{\ell} Z_{B_i}+Z_E)\cdot A_i \leq 0$ for all $A_i \subseteq A$.

$\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_{j-1}) / \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_j)$ is the sheaf of germs of sections of a line bundle over A_{i_j} of chern chern class $-A_{i_j}(\sum_{i=0}^{\ell} Z_{B_i}+Z_E+nZ+Z_{j-1})$. If $|E|$ has at least two irreducible components, from Proposition 2.5, $A_{i_{r+k}}\cdot Z_{r+k-1} = 2$ and $A_{i_j}\cdot Z_{j-1} = 1$ for $j \neq r+k$. So $A_{i_j} \cdot (\sum_{i=0}^{\ell} Z_{B_i}+Z_E+nZ+Z_{j-1}) \leq 1$ for all j and all n. Thus $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_{j-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_j)) = 0$ and the maps $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_j)) \to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_{j-1}))$ in (4.2) are surjective. Composing the maps, we see that

$$\phi: H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_j)) \to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-Z_j))$$

is surjective for all $n \geq 0$. For sufficiently large n, ϕ is the 0 map by [7, §4 Satz 1, p.355]. Hence $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-Z_j)) = 0$. If $|E| = A_i$ is a nonsingular elliptic curve, then $A_i \cdot A_i \leq -2$. By Corollary 2.6, we know that $e_i = z_i = 1$. Since $A_{i_j}\cdot Z_{j-1} = 1$ for all j by Proposition 2.5, $A_{i_j}\cdot(\sum_{i=0}^{\ell} Z_{B_i}+Z_E+nZ+Z_{j-1}) \leq 1$ for all $j \neq r+1$ and $A_{i_{r+1}}(\sum_{i=0}^{\ell} Z_{B_i}+Z_E+nZ+Z_r) \leq -1$. Thus $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_{j-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-nZ-Z_j)) = 0$ for all j and n. A similar argument as above will show that $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-Z_j)) = 0$. In particular $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-A_1)) = 0$. Therefore, $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E)) \to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_E-A_1))$ is surjective. We remark that the above argument is also applicable to the following situation. With notation as above, there exists $A_j \subseteq \mathrm{supp} E$ such that $A_j \neq A_{i_{r+1}}$ and $A_j \cdot Z_E < 0$.

Case (2): $|E|$ has at least two irreducible components and there exists $A_i \subseteq |E|$ such that $e_i = 1$, $A_i \cdot Z_E < 0$ and $A_j \cdot Z_E = 0$ for all $A_j \subseteq |E|$ where $A_j \neq A_i$. The proof of case (1) fails only because $A_{i_{r+k}} \neq A_i$, i.e., $A_{i_{r+1}} \cdot Z_E < 0$. Suppose first that $A_1 \cap A_{i_{r+1}} = A_1 \cap A_i \neq \phi$, $A_1 \nsubseteq |E|$. Choose a computation sequence for Z with $E = Z_k$, $A_{i_k} = A_i$, and $A_{i_{k+1}} = A_1$. By Proposition 2.7, $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_j)) = 0$ for all j. Therefore,

$$H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)) \to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+1}))$$

is surjective. It follows that $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E))$ and $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+1}))$ have the same image R in $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1))$.

$$0 \to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_k)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+1}))$$

$$\to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+1}))$$

$$\to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_k)) \to 0$$

is an exact sequence. Thus the image of $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_k)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+1}))$ which is injected into $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1))$ via the natural map is contained in R.

If $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_k)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+1})) \neq 0$, then the elements of R have no common zeros on $A_1 - A_1 \cap A_i$ as section of the line bundle L on A_1 associated to $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1)$.

If $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_k)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+1})) = 0$, then $A_1 \cdot (\sum_{i=0}^{\ell} Z_{B_i} + Z_E) = 0$.

Hence $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1)) = \mathbb{C}$. It suffices to prove that $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)) \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1))$ is not a zero map. Since $A_1 \not\subseteq |E|$, $A_1 \cdot Z_E = 1$ and $A_1 \cap A_i \neq \phi$, the coefficient of A_i in Z_E is equal to 1. Hence $A_i \cdot Z_E = Z_E \cdot Z_E \leq -2$.

It follows that $A_i(\sum_{i=0}^{\ell} Z_{B_i} + Z_E) \leq -2$ and $\dim H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{j=0}^{\ell} Z_{B_j} - Z_E - A_i)) \geq 3$. The image of $\rho: H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)) \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{j=0}^{\ell} Z_{B_j} - Z_E - A_i))$ is a subspace of codimension 1 in $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{j=0}^{\ell} Z_{B_j} - Z_E - A_i))$ and the elements of S have no common zeros as sections of the line bundle L_i on A_i associated to $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{j=0}^{\ell} Z_{B_j} - Z_E - A_i)$ by Proposition 2.8.

It follows that $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)) \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1))$ is not a zero map.

In order to finish the proof of case (2), it remains to consider those $A_1 \not\subseteq |E|$ such that $A_1 \cap A_i = \phi$ and the computation sequence for Z starting from A_1 must first reach A_i in order to reach $|E|$. Choose a computation sequence for Z of the following form $E = Z_k$, $A_{i_k} = A_i$, $A_{i_{k+1}} \cap A_i \neq \phi$, $A_{i_{k+t}} = A_1$, and A_{i_j}, $k+1 \leq j \leq k+t$, are distinct to each other and not contained in $|E|$. Since $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_j)) = 0$ for all j by Proposition 2.7, $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)) \to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t}))$ is surjective. It follows that $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E))$ and $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t}))$ have the same image R in $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1))$

$$0 \to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t}))$$

$$\to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t}))$$

$$\to H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t-1})) \to 0$$

is an exact sequence. Thus the image of $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t}))$ which is injected into $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1))$ via natural map is contained in R. If $H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t})) \neq 0$, then the elements of R have no common zeros on $A_1 - (A_1 \cap A_{i_{k+t-1}})$ as sections of the line bundle L_1 on A_1 associated to $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1))$.

If $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - Z_{k+t})) = 0$, then $A_1 \cdot (\sum_{i=0}^{\ell} Z_{B_i} + Z_E) = 0$. Hence $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/ \quad (-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1)) = \mathbb{C}$. But by induction, we know that the elements of image of $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)) \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_{i_{k+t-1}}))$ have no common zeros on $A_{i_{k+t-1}} - (A_{i_{k+t-1}} \cap A_{i_{k+t-2}})$ as sections of the line bundle $L_{i_{k+t-1}}$ on $A_{i_{k+t-1}}$ associated to $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_{i_{k+t-1}})$. It follows that $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)) \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E - A_1))$ is surjective. Q.E.D.

Corollary 4.10: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only weakly elliptic singularity. Suppose $H^1(M, \mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_o} = Z, Z_{B_1}, \ldots, Z_{B_\ell}, Z_{B_{\ell+1}} = Z_E$ be the elliptic sequence. Suppose $Z_E \cdot Z_E = -1$. Let $A_i \subseteq |E|$ be such that $A_i \cdot Z_E = -1$. Let S be the image of $\rho: H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)) \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{j=0}^{\ell} Z_{B_j} - Z_E - A_i))$. Then $m\mathcal{O} = \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$ provided that the following condition holds. Let $A_1 \nsubseteq |E|$ and $A_1 \cap A_i \neq \phi$, then either $A_1 \cdot (\sum_{i=0}^{\ell} Z_{B_i} + Z_E) < 0$ or the elements of S have no common zeros at $A_1 \cap A_i$ as sections of the line bundle L_i on A_i associated to $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_E)/\mathcal{O}(-\sum_{j=0}^{\ell} Z_{B_j} - Z_E - A_i)$.

Proof: By the proof of Theorem 4.9.

Corollary 4.11: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only weakly elliptic singularity. Suppose $H^1(M, \mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_o} = Z, Z_{B_1}, \ldots, Z_{B_\ell}, Z_E$ be the elliptic sequence. Then the multiplicity $({}_V\mathcal{O}_p) \geq -\sum_{i=0}^{\ell} Z_{B_i}^2$. If $Z_E \cdot Z_E \leq -2$, then multiplicity $({}_V\mathcal{O}_p) = -\sum_{i=0}^{\ell} Z_{B_i}^2$.

Proof: Theorem 4.9 says that $m\mathcal{O} \subseteq \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})$. Hence the maximal ideal cycle Y relative to π is greater than or equal to $\sum_{i=0}^{\ell} Z_{B_i}$. By Theorem 2.17 multiplicity $({}_V\mathcal{O}_p) \geq -Y \cdot Y$. But $-Y \cdot Y \geq -(\sum_{i=0}^{\ell} Z_{B_i}) \cdot (\sum_{i=0}^{\ell} Z_{B_i}) = -\sum_{i=0}^{\ell} Z_{B_i}^2$ by Lemma 2.15. Hence multiplicity $({}_V\mathcal{O}_p) \geq -\sum_{i=0}^{\ell} Z_{B_i}^2$. The rest of the corollary is easy.

§3 Calculation of Hilbert Functions

Suppose $H^1(M, \mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. In this section we calculate the Hilbert function of ${}_V\mathcal{O}_p$. In particular, the dimension of the Zariski tangent space is computed. Hence we know the lowest possible embedding dimension of the singularity.

Theorem 4.12: Let V be a normal 2-dimensional Stein space with p as its only weakly elliptic singularity. Let $\pi: M \to V$ be the minimal good resolution. Suppose ${}_V\mathcal{O}_p$ is Gorenstein and $H^1(M,\mathcal{O}) = \mathbb{C}^2$. Let $Z_{B_o}=Z, Z_{B_1},\ldots,Z_{B_\ell}, Z_E$ be the elliptic sequence. If $Z_E \cdot Z_E \leq -3$, then $m^n \simeq H^o(A,\mathcal{O}(-n(\sum_{i=0}^{\ell} Z_{B_i})))$, $n \geq 0$.

Proof: The proof is long and tedious.

Theorem 4.13: Let V be a normal 2-dimensional Stein space with p as its only weakly elliptic singularity. Let π: $M \to V$ be the minimal good resolution. Suppose ${}_V\mathcal{O}_p$ is Gorenstein and $H^1(M,\mathcal{O}) = \mathbb{C}^2$. Let $Z_{B_o}=Z, Z_{B_1},\ldots,Z_{B_\ell}, Z_E$ be the elliptic sequence. If $Z_E \cdot Z_E \leq -3$, then $\dim m^n/m^{n+1} = -n(\sum_{i=0}^{\ell} Z_{B_i}^2)$, $n \geq 1$.

Proof: The following long cohomology exact sequence

$$0 \to H^o(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i})) \to H^o(M,\mathcal{O}_{(n+1)\sum_{i=0}^{\ell} Z_{B_i}}) \to H^o(M,\mathcal{O}_{n\sum_{i=0}^{\ell} Z_{B_i}})$$

$$\to H^1(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i})) \to H^1(M,\mathcal{O}_{(n+1)\sum_{i=0}^{\ell} Z_{B_i}})$$

$$\to H^1(M,\mathcal{O}_{n\sum_{i=0}^{\ell} Z_{B_i}}) \to 0$$

says that $\dim H^o(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i})) - \dim H^1(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i}))$

$$= \dim H^{o}(M, \mathcal{O}_{(n+1)\sum_{i=0}^{\ell} Z_{B_i}}) - \dim H^{1}(M, \mathcal{O}_{(n+1)\sum_{i=0}^{\ell} Z_{B_i}})$$

$$- \dim H^{o}(M, \mathcal{O}_{n\sum_{i=0}^{\ell} Z_{B_i}}) + \dim H^{1}(M, \mathcal{O}_{n\sum_{i=0}^{\ell} Z_{B_i}})$$

$$= \chi((n+1)\sum_{i=0}^{\ell} Z_{B_i}) - \chi(n\sum_{i=0}^{\ell} Z_{B_i})$$

$$= \chi(\sum_{i=0}^{\ell} Z_{B_i}) + \chi(n\sum_{i=0}^{\ell} Z_{B_i}) - n(\sum_{i=0}^{\ell} Z_{B_i})\cdot(\sum_{i=0}^{\ell} Z_{B_i}) - \chi(n\sum_{i=0}^{\ell} Z_{B_i})$$

$$= -n(\sum_{i=0}^{\ell} Z_{B_i}^{2}).$$

Consider the following cohomology exact sequence

$$0 \to H^{o}(M, \mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i})) \to H^{o}(M, \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}))$$

$$\to H^{o}(M, \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i})) \to H^{1}(M, \mathcal{O}(-n+1)\sum_{i=0}^{\ell} Z_{B_i}))$$

$$\to H^{1}(M, \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})) \to H^{1}(M, \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i})) \to 0$$

By theorem 4.12, $\dim m^{n}/m^{n+1} = \dim H^{o}(M, \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}))/H^{o}(M, \mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i}))$

$$= \dim H^{o}(M, \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i})) - \dim H^{1}(M, \mathcal{O}(-n+1)\sum_{i=0}^{\ell} Z_{B_i}))$$

$$+ \dim H^{1}(M, \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})) - \dim H^{1}(M, \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i}))$$

$$= -n(\sum_{i=0}^{\ell} Z_{B_i}^{2}) + \dim H^{1}(M, \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})) - \dim H^{1}(M, \mathcal{O}(-(n+1)\sum_{i=0}^{\ell} Z_{B_i})).$$

We claim that $H^1(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}) \simeq \mathbb{C}$ for all $n \geq 1$. Choose a computation sequence for Z of the following form $Z_o=0,\ldots,Z_k=E,\ldots,Z_{r_o}=Z_E,\ldots,$ $Z_{r_1}=Z_{B_\ell},\ldots,Z_{r_\ell}=Z_{B_1},\ldots,Z_{\gamma_{\ell+1}}=Z_{B_o}=Z$. Consider the following sheaf exact sequence.

$$0 \to \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_1)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-E) \to \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-E)$$

$$\to \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_1) \to 0$$

$$0 \to \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_2)/\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-E) \to \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_1)/\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-E)$$

$$\to \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_1)/\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_2) \to 0$$

$$0 \to \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_{k-1})/\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_k) \to \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_{k-2})/\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-E)$$

$$\to \mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_{k-2})/\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-Z_{k-1}) \to 0$$

By the proof of Theorem 4.9 , we know that there exists $f \in H^o(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}))$ such that the image of f in $H^o(M,\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-Z_1))$ is nonzero. The usual long cohomology exact sequence argument will show that $H^1(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-E)) \simeq \mathbb{C}$. Since $H^1(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-E)) = 0$, the following exact sequence

$$H^1(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-E)) \to H^1(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}))$$

$$\to H^1(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i}-E)) \to 0$$

will show that $H^1(M,\mathcal{O}(-n\sum_{i=0}^{\ell} Z_{B_i})) \simeq \mathbb{C}$. Hence $\dim m^n/m^{n+1} =$ $-n\sum_{i=0}^{\ell} Z_{B_i}^2$.

§4 Absolutely Isolatedness of Almost Minimally Elliptic Singularities.

The name absolutely isolated singularity is given in [5] and [16],[17] to a two-dimensional normal singularity, realized in $\mathbb{C}^3$, which can be resolved by means of a sequence of σ processes with centers at points. It is proved in [5] and [36] that double rational points are always absolutely isolated and, conversely, an arbitrary double absolutely isolated singularity in $\mathbb{C}^3$ is rational. In this paper we shall say that a two-dimensional isolated singularity is absolutely isolated if it can be resolved by means of a sequence of σ processes with centers at points, without requiring, in what follows, that it should be realized in $\mathbb{C}^3$. It is in this sense that Laufer proved that minimally elliptic singularities which are not double points are absolutely isolated.

Theorem 4.14: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only almost minimally elliptic singularity. If $Z_E \cdot Z_E \leq -3$ and ${}_V\mathcal{O}_p$ is Gorenstein, then p is absolutely isolated. Moreover, blow-up p at its maximal ideal yields exactly those curves A_i such that $A_i \cdot Z > 0$. The singularities remaining after the blow-up are the rational double points and a minimally elliptic singularity corresponding to delecting the A_i with $A_i \cdot Z > 0$ from the exceptional set. The self intersection number of the fundamental cycle of the minimally elliptic singularity is less than or equal to -3

Proof: The proof is long and tedious.

CHAPTER V

HYPERSURFACE WEIGHTED DUAL GRAPHS

One of the important questions in normal two dimensional singularities is "the classification of all weighted dual graphs for hypersurface singularities". It is known that in the weighted dual graphs for hypersurface singularities, the K' cycle must exist. In this chapter, we get a lower estimate of the dimension of Zariski tangent space in terms of the fundamental cycle, which will give us a necessary condition on hypersurface weighted dual graphs. In section 2, we give a complete topological classification of elliptic double points. Moreover, some of the defining equations are found. In section 3, we list all possible weighted dual graphs of hypersurface singularities with $h = \dim H^1(M, \mathcal{O}) = 2$.

§1 Lower Estimate of the Dimension of Zariski Tangent Space and Upper Estimate of Multiplicities of Hypersurface Singularities.

Theorem 5.1: Let $\pi\colon M \to V$ be a resolution of normal two dimensional Stein space V with p as its only singular point. Let Z be the fundamental cycle. Then

$$\dim m/m^2 \geq \chi(Z) - Z\cdot Z + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-2Z)).$$

If p is weakly elliptic, then $\dim m/m^2 \geq -Z\cdot Z + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-2Z))$. Suppose π is the minimal good resolution and p is a maximally elliptic singularity. Then dim $m/m^2 \geq -Z\cdot Z$. Moreover, if $Z_E\cdot Z_E \leq -3$, then dim $m^n/m^{n+1} = -nZ\cdot Z$ for all $n \geq 1$.

Proof: It is true that $H^0(A, \mathcal{O}(-Z)) = \text{dir lim } H^0(U, \mathcal{O}(-Z))$, U a neighborhood of A. Since Z is minimal, $m \cong H^0(A, \mathcal{O}(-Z))$. Since $m^2 \lneqq H^0(A, \mathcal{O}(-2Z))$, we have dim $m/m^2 \geq \dim H^0(A, \mathcal{O}(-Z))/H^0(A, \mathcal{O}(-2Z))$. The following cohomology exact sequence

$$
\begin{aligned}
0 \to H^0(A, \mathcal{O}(-2Z)) &\to H^0(A, \mathcal{O}(-Z)) \to H^0(A, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \\
&\to H^1(A, \mathcal{O}(-2Z)) \to H^1(A, \mathcal{O}(-Z)) \to H^1(A, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \to 0
\end{aligned}
$$

says that

$$
\begin{aligned}
\dim H^0(A, \mathcal{O}(-Z))/H^0(A, \mathcal{O}(-2Z)) &= \dim H^0(A, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \\
&\quad - \dim H^1(A, \mathcal{O}(-2Z)) + \dim H^1(A, \mathcal{O}(-Z)) - \dim H^1(A, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \\
&= \dim H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) - \dim H^1(M, \mathcal{O}(-2Z)) + \dim H^1(M, \mathcal{O}(-Z)) \\
&\quad - \dim H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z))
\end{aligned}
$$

by Lemma 3.1 of [20].

Look at the following cohomology exact sequence:

$$
\begin{aligned}
0 &\to H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \to H^0(M, \mathcal{O}_{2Z}) \to H^0(M, \mathcal{O}_Z) \to H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z) \\
&\to H^1(M, \mathcal{O}_{2Z}) \to H^1(M, \mathcal{O}_Z) \to 0.
\end{aligned}
$$

Since $H^0(M, \mathcal{O}_Z) \cong \mathbb{C}$ by (1.6) and $H^0(M, \mathcal{O}_{2Z}) \to H^0(M, \mathcal{O}_Z)$ is not a zero map, we have two short exact sequences

$$0 \to H^o(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \to H^o(M, \mathcal{O}_{2Z}) \to H^o(M, \mathcal{O}_Z) \to 0$$

$$0 \to H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \to H^1(M, \mathcal{O}_{2Z}) \to H^1(M, \mathcal{O}_Z) \to 0$$

Hence,

$$\begin{aligned}\dim m/m^2 \geq\ & \dim H^o(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) - \dim H^1(M, \mathcal{O}(-2Z)) \\ & + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \\ & = \dim H^o(M, \mathcal{O}_{2Z}) - \dim H^o(M, \mathcal{O}_Z) - \dim H^1(M, \mathcal{O}(-2Z)) \\ & + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) \\ & = \chi(2Z) - \chi(Z) + \dim H^1(M, \mathcal{O}_{2Z}) - \dim H^1(M, \mathcal{O}_Z) \\ & - \dim H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-2Z)) + \dim H^1(M, \mathcal{O}(-Z)) \\ & - \dim H^1(M, \mathcal{O}(-2Z)) \\ & = \chi(2Z) - \chi(Z) + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-2Z)) \\ & = \chi(Z) - Z \cdot Z + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-2Z)).\end{aligned}$$

If p is weakly elliptic, then $\chi(Z) = 0$. So $\dim m/m^2 \geq -Z \cdot Z + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-2Z))$.

Suppose π is the minimal good resolution and p is a maximally elliptic singular point. We claim that $H^1(M, \mathcal{O}(-nZ)) \simeq \mathbb{C}^{\ell+1}$ where $\ell+2$ is the length of elliptic sequence $Z_{B_o}, Z_{B_1}, \dots, Z_{B_\ell}, Z_E = Z_{B_{\ell+1}}$. Choose a computation sequence for Z of the following form

$Z_o = 0, \dots, Z_k = E, \dots, Z_{r_o} = Z_E, \dots, Z_{r_1} = Z_B, \dots, Z_{r_\ell} = Z_{B_1}, \dots,$ $Z_{r_{\ell+1}} = Z_{B_o} = Z$. Consider the following sheaf exact sequences

$$0 \to \mathcal{O}(-nZ-Z_1)/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i} - Z_E) \to \mathcal{O}(-nZ)/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i} - Z_E)$$

$$\to \mathcal{O}(-nZ)/\mathcal{O}(-nZ-Z_1) \to 0$$

$$0 \to \mathcal{O}(-nZ-Z_k)/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i} - Z_E) \to \mathcal{O}(-nZ-Z_{k-1})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i} - Z_E)$$

$$\to \mathcal{O}(-nZ-Z_{k-1})/\mathcal{O}(-nZ-Z_k) \to 0$$

$$0 \to \mathcal{O}(-nZ-Z_{B_1})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i} - Z_E) \to \mathcal{O}(-nZ-Z_{r_\ell - 1})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i} - Z_E)$$

$$\to \mathcal{O}(-nZ-Z_{r_\ell - 1})/\mathcal{O}(-nZ-Z_{B_1}) \to 0$$

$$0 \to \mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_1)/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E) \to \mathcal{O}(-nZ-\sum_{i-1}^{h} Z_{B_i})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E)$$

$$\to \mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i})/\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_1) \to 0$$

$$0 \to \mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_k)/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E) \to \mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_{k-1})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E)$$

$$\to \mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_{k-1})/\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_k) \to 0$$

$$0 \to \mathcal{O}(-nZ-\sum_{i=1}^{h+1} Z_{B_i})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E) \to \mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_{r_{\ell-h}-1})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E)$$

$$\to \mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_{r_{\ell-h}-1})/\mathcal{O}(-nZ-\sum_{i=1}^{h+1} Z_{B_i}) \to 0$$

$$0 \to \mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_1)/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E) \to \mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E)$$

$$\to \mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_1) \to 0$$

$$0 \to \mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_k)/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E) \to \mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_{k-1})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E)$$

$$\to \mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_{k-1})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_k) \to 0$$

$$0 \to \mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_{r_o-1})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E) \to \mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_{r_o-2})/$$

$$\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E) \to \mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_{r_0-2})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_{r_o-1}) \to 0$$

We claim that $H^0(M,\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_{j-1})/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E))$

$\to H^0(M,\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_{j-1})/\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_j))$ is surjective for all

$-1 \le h \le \ell-1$ and $0 \le j \le r_{h+1}$. The chern class of the line bundle associated to $\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_{j-1})/\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_j)$ is

$-A_{i_j}\cdot(nZ+\sum_{i=1}^{h} Z_{B_i}+Z_{j-1}) = -A_{i_j}\cdot Z_{j-1}$ which is < 0 for $j > 1$ and

0 for $j = 1$. For $j > 1$, the claim is trivially true because

$H^0(M,\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_{j-1})/\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_j)) = 0$. For $j = 1$,

by Proposition 3.13, we know that there exists $f \in H^0(M,\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}))$ such that the image of f in $H^0(M,\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i})/\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_1))$ is nonzero. Therefore, $H^0(M,\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i})/\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_1)) \simeq \mathbb{C}$ and $H^0(M,\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}))$ $\to H^0(M,\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i})/\mathcal{O}(-nZ-\sum_{i=1}^{h} Z_{B_i}-Z_1))$ is surjective. Now

the usual cohomology exact sequence argument will show that $H^1(M,\mathcal{O}(-nZ)/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E)) \simeq \mathbb{C}^{\ell+1}$. By Proposition 2.7, $H^1(M,\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E)) = 0$. So the exact sequence

$$H^1(M,\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E)) \to H^1(M,\mathcal{O}(-nZ)) \to H^1(M,\mathcal{O}(-nZ)/\mathcal{O}(-nZ-\sum_{i=1}^{\ell} Z_{B_i}-Z_E)) \to 0$$

shows that $H^1(M, \mathcal{O}(-nZ)) \cong \mathbb{C}^{\ell+1}$.

$$\begin{aligned} \dim m/m^2 &\geq -Z\cdot Z + \dim H^1(M, \mathcal{O}(-Z)) - \dim H^1(M, \mathcal{O}(-2Z)) \\ &= -Z\cdot Z. \end{aligned}$$

If $Z_E \cdot Z_E \leq -3$, then $-Z\cdot Z \geq 3$. In this case, all the inequalities above are actually equalities. In particular, $m^2 = H^0(A, \mathcal{O}(-2Z))$. By Theorem 3.15, we have $m^n = H^0(A, \mathcal{O}(-nZ))$, $n \geq 1$. Hence

$$\begin{aligned} \dim m^n/m^{n+1} &= \dim H^0(A, \mathcal{O}(-nZ))/H^0(A, \mathcal{O}(-(n+1)Z)) \\ &= \dim H^0(A, \mathcal{O}(-nZ)/\mathcal{O}(-nZ-Z)) \\ &\quad - \dim H^1(A, \mathcal{O}(-nZ-Z)) + \dim H^1(A, \mathcal{O}(-nZ)) \\ &\quad - \dim H^1(A, \mathcal{O}(-nZ)/\mathcal{O}(-nZ-Z)) \\ &= \dim H^0(M, \mathcal{O}(-nZ)/\mathcal{O}(-nZ-Z)) \\ &\quad - \dim H^1(M, \mathcal{O}(-nZ)/\mathcal{O}(-nZ-Z)) - (\ell+1) + (\ell+1) \\ &= \dim H^0(M, \mathcal{O}_{nZ+Z}) - \dim H^0(M, \mathcal{O}_{nZ}) - \dim H^1(M, \mathcal{O}_{nZ+Z}) \\ &\quad + \dim H^1(M, \mathcal{O}_{nZ}) = \chi((n+1)Z) - \chi(nZ) = -nZ\cdot Z \end{aligned}$$

Q.E.D.

Corollary 5.2: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space V with p as its only maximally elliptic singularity. Suppose p is a hypersurface singularity, then $Z \cdot Z \geq -3$.

The following theorem of Laufer and Lipman, gives an upper estimate of multiplicity in terms of $\dim H^1(M, \mathcal{O})$.

Theorem 5.3: Let $V = \{f(x,y,z) = 0\}$ have an isolated singularity at (0,0,0). Let n be the multiplicity of V. Then $\dim H^1(M, \mathcal{O}) \geq \frac{(n-1)(n-2)}{2}$ where M is a resolving manifold of V.

Proof: The proof is a refinement of the proof of [24, Theorem 3.14].

§2 Topological Classification of Weakly Elliptic Double Points

In 1964, M. Artin gave a complete topological classification of rational double points. In 1970, Wagreich proved that for double points, $Z \cdot Z \geq -2$. Using this fact he listed most of the possible weighted dual graphs of weakly elliptic double points. Using the fact that $-K'$ = the summation of an elliptic sequence and a combinatorial argument, we list all possible weighted dual graphs for weakly elliptic double points. Moreover, all these weighted dual graphs actually arise from weakly elliptic double points because we can find a defining equation for each of them. The defining equations have been found by an unpublished technique of Laufer.

<u>Proposition 5.4</u>: Let Γ be a weighted dual graph including genera for the vertices, associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z = Z_{B_0}, \ldots, Z_{B_\ell}, Z_E$ be the elliptic sequence. Then $Z \cdot Z \leq Z_{B_1} \cdot Z_{B_1} \leq \ldots \leq Z_{B_\ell} \cdot Z_{B_\ell} \leq Z_E \cdot Z_E$. If $Z_{B_i} \cdot Z_{B_i} = Z_{B_{i+1}} \cdot Z_{B_{i+1}}$, then $A_j \cdot A_j = -2$ for all $A_j \subseteq B_i$, $A_j \nsubseteq B_{i+1}$ $0 \leq i \leq \ell$.

<u>Proof</u>: For $0 \leq i \leq \ell$, let $A_j \subseteq B_i$ and $A_j \nsubseteq B_{i+1}$. If $A_j \cap B_{i+1} = \phi$, then $A_j \cdot (Z_{B_i} + Z_{B_{i+1}}) = A_j \cdot Z_{B_i} \leq 0$. If $A_j \cap B_{i+1} \neq \phi$, then $A_j \cdot Z_{B_i} < 0$ by the definition of elliptic sequence. Since $A_j \cdot Z_{B_{i+1}} = 1$ in this case, $A_j \cdot (Z_{B_i} + Z_{B_{i+1}}) \leq 0$. We observe that $Z_{B_i} \geq Z_{B_{i+1}}$, i.e., $Z_{B_i} - Z_{B_{i+1}}$ is a positive cycle. It follows that $(Z_{B_i} - Z_{B_{i+1}}) \cdot (Z_{B_i} + Z_{B_{i+1}}) \leq 0$. Hence $Z_{B_i} \cdot Z_{B_i} \leq Z_{B_{i+1}} \cdot Z_{B_{i+1}}$.

Suppose that $Z_{B_i} \cdot Z_{B_i} = Z_{B_{i+1}} \cdot Z_{B_{i+1}}$. We want to prove $A_j \cdot A_j = -2$ for all $A_j \subseteq B_i$ and $A_j \nsubseteq B_{i+1}$. Since $(Z_{B_i} - Z_{B_{i+1}}) \cdot (Z_{B_i} + Z_{B_{i+1}}) = Z_{B_i}^2 - Z_{B_{i+1}}^2 = 0$, we have $A_j \cdot (Z_{B_i} + Z_{B_{i+1}}) = 0$. Recall that $K' = -\sum_{i=0}^{\ell} Z_{B_i} - E$.

$$0 \leq A_j \cdot K' = -A_j \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E)$$

$$= -A_j (Z_{B_i} + Z_{B_{i+1}} + \ldots + Z_{B_\ell} + E)$$

$$= -A_j \cdot (Z_{B_{i+2}} + \ldots + Z_{B_\ell} + E)$$

$$\leq 0 \qquad \text{since } A_j \nsubseteq B_{i+2}$$

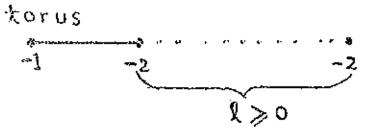
Therefore $0 = A_j \cdot K' = 2g_j - 2 - A_j \cdot A_j = -2 - A_j \cdot A_j$ and $A_j \cdot A_j = -2$. Q.E.D.

Proposition 5.5: Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z = Z_{B_o}, \ldots, Z_{B_\ell}, Z_E$ be the elliptic sequence. If $Z \cdot Z = -1$, then there exists a unique $A_1 \not\subseteq B_1$, $A_1 \cdot A_1 = -2$ such that $Z = Z_{B_1} + A_1$ and $A_1 \cap B_1 \neq \phi$. Moreover, if $A_2 \subseteq B_1$ such that $A_1 \cdot A_2 = 1$, then $A_2 \cdot Z_{B_1} = -1$ and $z_2 = 1$.

Proof: By the definition of the elliptic sequence and $Z \cdot Z = -1$, there exists a unique $A_1 \not\subseteq B_1$ such that $A_1 \cap B_1 \neq \phi$ and $z_1 = 1$. By proposition 5.4, we know that $A_1 \cdot A_1 = -2$. Since $z_1 = 1$, $A_1 \cdot Z < 0$ and $A_1 \cdot A_1 = -2$, we conclude that $z_2 = 1$ and A_1 cannot intersect any $A_o \not\subseteq B_1$ with $A_o \neq A_1$. Hence $A_2 \cdot Z_{B_1} = -1$. Otherwise $A_2 \cdot Z_{B_1} = 0$ would imply that $z_2 \geq 2$. So $Z = Z_{B_1} + A_1$.

Corollary 5.6: Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists and $Z = Z_{B_o}, \ldots, Z_{B_\ell} = Z_E$ be the elliptic sequence. If $Z \cdot Z = -1$, then Γ must be one of the following forms.

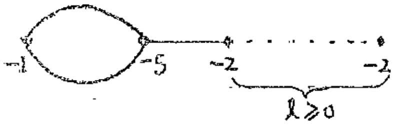
(1)



$$z^2 = y^3 + x^{6+6l}$$

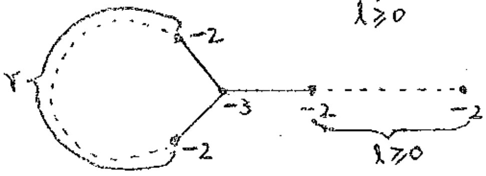
$Z = 1 \quad 1 \cdots\cdots 1$

(2)



$$z^2 = (y + x^{2+2l})(y^2 + x^{5+4l})$$

$Z = 2 \quad 1 \quad \underbrace{1 \cdots\cdots 1}_{l \geq 0}$



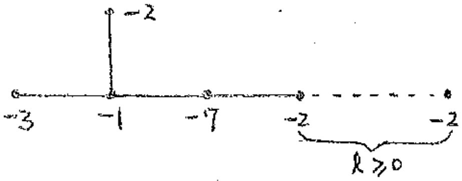
$$z^2 = (y + x^{2+2l})(y^2 + x^{r+5+4l})$$

$$r \geq 1$$

Z =

$1 \quad \underbrace{1 \cdots\cdots 1}_{l \geq 0}$

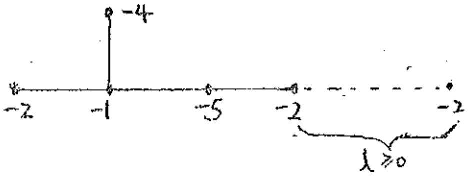
(3)



$$z^2 = y^3 + x^{7+6l}$$

$Z = 2 \quad \overset{3}{6} \quad 1 \quad \underbrace{1 \cdots\cdots 1}_{l \geq 0}$

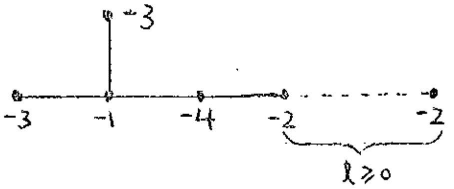
(4)



$$z^2 = y^3 + x^{5+4l} y$$

$Z = 2 \quad \overset{1}{4} \quad 1 \quad \underbrace{1 \cdots\cdots 1}_{l \geq 0}$

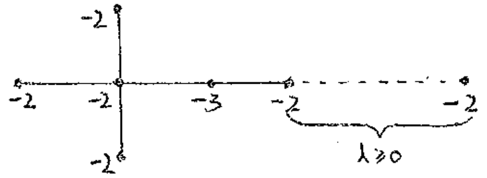
(5)



$$z^2 = y^3 + x^{8+6\ell}$$

$$Z = 1\ \overset{1}{3}\ 1\ 1 \cdots\cdots 1$$

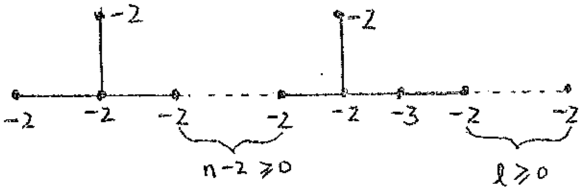
(6)



$$Z = 1\ \overset{1}{\underset{1}{2}}\ 1\ 1 \cdots\cdots 1$$

$$z^2 = y^3 + x^{9+6\ell}$$

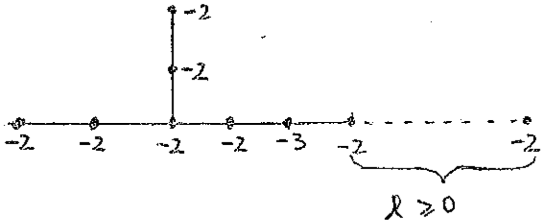
(7)



$$Z = 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ \overset{1}{2}\ 1\ 1 \cdots\cdots 1$$

$$z^2 = (y + x^{3+2\ell})(y^2 + x^{n+5+4\ell})$$

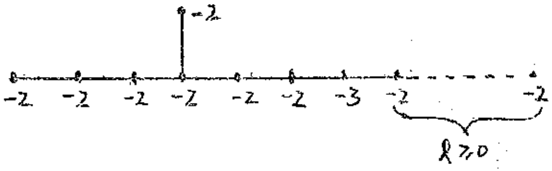
(8)



$$Z = 1\ 2\ \overset{\overset{1}{2}}{3}\ 2\ 1\ 1 \cdots\cdots 1$$

$$z^2 = y^3 + x^{10+6\ell}$$

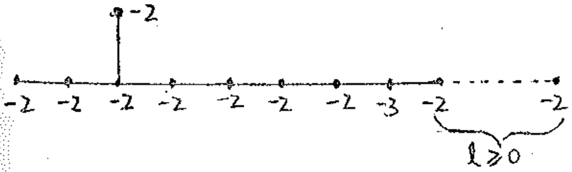
(9)



$$Z = 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots\cdots 1$$

$$z^2 = y^3 + x^{7+4\ell}y$$

(10)

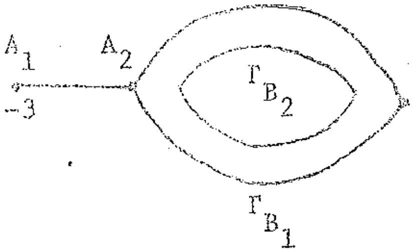


$$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2\ 1\ 1 \cdots\cdots 1$$

$$z^2 = y^3 + x^{11+6\ell}$$

Proposition 5.7: Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z = Z_{B_o}$, $\ldots, Z_{B_\ell}, Z_E$ be the elliptic sequence. If $Z \cdot Z = -2$, $Z_{B_1} \cdot Z_{B_1} = -1$, then Γ must be one of the following forms:

(1)

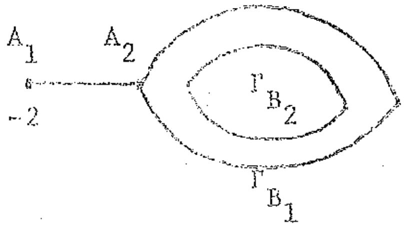


$A_2 \subseteq B_1$, $A_2 \nsubseteq B_2$

$Z = A + Z_{B_1}$, $A_2 \cdot Z_{B_1} = -1$

$z_2 = 1$

(2)



$Z = 2A_1 + D$, D is a positive cycle, $|D| = B_1$

$z_2 = 3$, $A_2 \cdot Z_{B_1} = 0$, $A_2 \subseteq B_1$, $A_2 \nsubseteq B_2$

where Γ_{B_i} is the graph of B_i.

Proof: By the definition of elliptic sequence and the fact that $Z \cdot Z = -2$ we have the following two cases.

(I) There exist $A_1, A_2 \subsetneq B_1$, $A_1 \cap B_1 \neq \phi \neq A_2 \cap B_1$ and $A_1 \neq A_2$. In this case, $A_1 \cdot Z = -1 = A_2 \cdot Z$ and $z_1 = 1 = z_2$. For $i = 1,2$, we have $0 \geq A_i \cdot (-K') = A_i(\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_i \cdot (Z + Z_{B_1}) = 0$.

So $0 = -A_i \cdot K' = 2 + A_i \cdot A_i$ and hence $A_i \cdot A_i = -2$, $i = 1,2$. Let $A_3, A_4 \subsetneq B_1$ such that $A_1 \cdot A_3 = 1$, $A_2 \cdot A_4 = 1$. Since

$A_1 \cdot A_1 = A_2 \cdot A_2 = -2$, $z_1 = z_2 = 1$, and $A_1 \cdot Z = A_2 \cdot Z = -1$, there is no $A_j \nsubseteq B_1$, $A_2 \neq A_j \neq A_1$ such that $A_j \cdot A_1 > 0$ or $A_j \cdot A_2 > 0$, i.e., $A = A_1 \cup A_2 \cup B_1$. Moreover, we know that $z_3 = z_4 = 1$. Hence $A_3 \cdot Z_{B_1} < 0$ and $A_4 \cdot Z_{B_1} < 0$. It follows that $A_3 = A_4$ and $A_3 \cdot Z_{B_1} = -1$ since $Z_{B_1} \cdot Z_{B_1} = -1$. As $z_3 = 1$, $Z = A_1 + A_2 + Z_{B_1}$, we have $A_3 \cdot Z_{B_1} = 1 > 0$, which is a contradiction. This case cannot occur.

(II) There exists a unique $A_1 \nsubseteq B_1$ such that $A_1 \cap B_1 \neq \phi$. In this case, we have either (A) $A_1 \cdot Z = -2$ and $z_1 = 1$, (B) $A_1 \cdot Z = -1$ and $z_1 = 2$, or (C) $A_1 \cdot Z = -1$ and $z_1 = 1$.

In (A), $0 \geq A_1 \cdot (-K') = A_1 \cdot (\sum_{i=0}^{\ell} Z_{B_i}) \geq A_1 \cdot (Z + Z_{B_1}) = -2+1 = -1$. So either $A_1 \cdot (K') = 0$ or $A_1 \cdot (-K') = -1$. If $A_1 \cdot (-K') = 0$, then $A_1 \cdot A_1 + 2 = 0$, i.e., $A_1 \cdot A_1 = -2$. It follows that $A_1 \cdot Z \geq -1$. This is a contradiction. If $A_1 \cdot (-K') = -1$, then $A_1 \cdot A_1 = -3$. Let $A_2 \subsetneq B_1$ such that $A_1 \cdot A_2 = 1$. Then $A_2 \nsubseteq B_2$. Since $A_1 \cdot Z = -2$, $A_1 \cdot A_1 = -3$ and $z_1 = 1$, there is no $A_i \nsubseteq B_1$, $A_i \neq A_1$ such that $A_i \cdot A_1 > 0$, i.e., $A = A_1 \cup B_1$. Moreover, we have $z_2 = 1$ and hence $Z = A_1 + Z_{B_1}$. So $A_2 \cdot Z_{B_1} = -1$ and we are in (1).

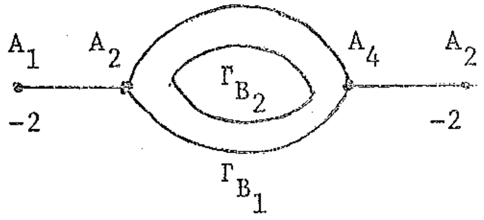
In (B), $0 \geq A_1 \cdot (-K') = A_1 (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_1 (Z + Z_{B_1}) = 0$. Then $A_1 \cdot K' = 0$ and $A_1 \cdot Z_{B_i} = 0$, $2 \leq i \leq \ell+1$. Let $A_2 \subsetneq B_1$ such that $A_1 \cdot A_2 = 1$. We have $A_1 \cdot A_1 = -2$ and $A_2 \nsubseteq B_2$. For any $A_i \nsubseteq B_1$, $A_i \neq A_1$, we have $A_i \cdot Z = 0 = A_i \cdot Z_{B_1}$. It follows that $2 + A_i \cdot A_i = A_i (-K') = A_i \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) = 0$, i.e., $A_i \cdot A_i = -2$. We claim that

$z_2 > 1$. For if $z_2 = 1$, then $\mathrm{supp}(Z-Z_{B_1})$ consists of those $A_i \not\subseteq B_1$. Consequently $Z^2-Z_{B_1}{}^2 = (Z-Z_{B_1}) \cdot (Z+Z_{B_1}) = 0$. However $Z^2 - Z_{B_1}{}^2 = -2+1 = -1$. This leads to a contradiction. Since $z_1 = 2$, $z_2 > 1$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, it is clear that $1 \leq \deg A_1 \leq 2$. If $\deg A_1 = 2$, then there exists a unique $A_3 \not\subseteq B_1$, $A_3 \cdot A_1 = 1$ $z_3 = 1$, and $z_2 = 2$. Let Γ_1 be the subgraph of Γ consisting of those $A_i \not\subseteq B_1$, $A_i \neq A_1$. Since $A_i \cdot A_i = -2$ for all A_i in Γ_1, Γ_1 is a graph of rational double point. Because $z_3 = 1$, it is easy to see that this case cannot occur. We conclude that $\deg A_1 = 1$, i.e., $A = A_1 \cup B_1$. Since $z_1 = 2$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, we have $z_2 = 3$. Then we are in (2).

In (C), $0 \geq A_1 \cdot (-K') = A_1 \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_1 \cdot (Z+Z_{B_1}) = 0$. Then $A_1 \cdot K' = 0$ and $A_1 \cdot Z_{B_i} = 0$, $2 \leq i \leq \ell+1$. Let $A_2 \subsetneqq B_1$ such that $A_1 \cdot A_2 = 1$. We have $A_1 \cdot A_1 = -2$. Since $z_1 = 1$ and $A_1 \cdot Z = -1$, $A = A_1 \cup B_1$ and $z_2 = 1$. So $Z = A_1 + Z_{B_1}$. But then $Z \cdot Z = (A_1+Z_{B_1}) \cdot (A_1 + Z_{B_1}) = A_1(A_1+Z_{B_1}) = -1$ which is absurd. Q.E.D.

Proposition 5.8: Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z = Z_{B_0}, \ldots, Z_{B_\ell}, Z_E$ be the elliptic sequence. If $Z \cdot Z = -2 = Z_{B_1} \cdot Z_{B_1}$, then Γ must be one of the following forms:

(1)



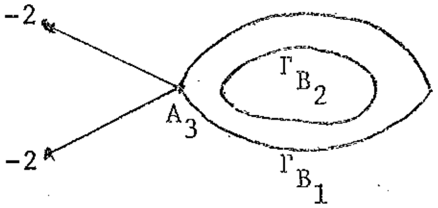
$Z=A_1+Z_{B_1}+A_2$, $A_3,A_4 \subseteq B_1$,

$A_3, A_4 \not\subseteq B_2$

$z_3 = 1 = z_4$, $A_3 \cdot Z_{B_1} = -1$

$= A_4 \cdot Z_{B_1}$

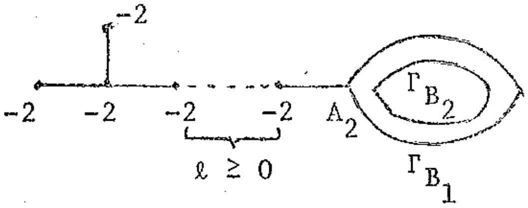
(2)



$Z = \overset{1}{1}\ Z_{B_1}$ $A_3 \subseteq B_1$, $A_3 \not\subseteq B_2$

$z_3 = 1$, $A_3 \cdot Z_{B_1} = -2$

(3)

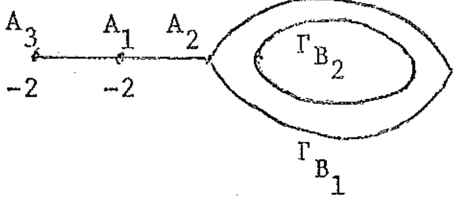


$Z = 1\ \overset{1}{2}\ 2 \ldots 2\ Z_{B_2}$, $A_2 \subseteq B_1$,

$A_2 \not\subseteq B_2$

$z_2 = 1$, $A_2 \cdot Z_{B_1} = -2$

(4)



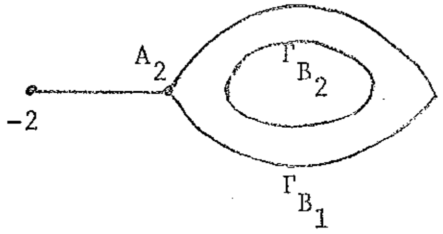
$Z = 1\ 2\ D$ $|D| = B_1$

D is a positive cycle

$z_2 = 2$, $A_2 \cdot Z_{B_1} = 0$

$A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$

(5)



$Z = 2 \quad D$

D is a positive cycle, $|D| = B_1$

$z_2 = 3, \quad A_2 \cdot Z_{B_1} = 0,$

$A_2 \subseteq B_1, \quad A_2 \not\subseteq B_2$

where Γ_{B_i} is the graph of B_i.

Proof: We firstly recall that by Proposition 5.4, $A_i \cdot A_i = -2$ for all $A_i \not\subseteq B_1$. By the definition of elliptic sequence and the fact that $Z \cdot Z = -2$, we have the following cases.

(I) There exist $A_1, A_2 \not\subseteq B_1$, $A_1 \neq A_2$ such that $A_1 \cap B_1 \neq \phi \neq A_2 \cap B_1$. In this case $A_1 \cdot Z = -1 = A_2 \cdot Z$ and $z_1 = z_2 = 1$. Let $A_3, A_4 \subseteq B_1$ such that $A_1 \cdot A_3 = 1 = A_2 \cdot A_4$. Since $z_1 = z_2 = 1$ and $A_1 \cdot Z = -1 = A_2 \cdot Z$, there is no $A_i \not\subseteq B_1$, $A_1 \neq A_i \neq A_2$ such that $A_i \cdot A_1 > 0$ or $A_i \cdot A_2 > 0$, i.e., $A = A_1 \cup A_2 \cup B_1$. Moreover $z_3 = 1 = z_4$ and $Z = A_1 + A_2 + Z_{B_1}$. If $A_3 \neq A_4$, then $A_3 \cdot Z_{B_1} = -1 = A_4 \cdot Z_{B_1}$ and $A_3, A_4 \not\subseteq B_2$. We are in (1). If $A_3 = A_4$, then $A_3 \cdot Z_{B_1} = -2$ and $A_3 \not\subseteq B_2$, we are in (2).

(II) There exists a unique $A_1 \not\subseteq B_1$ such that $A_1 \cap B_1 \neq \phi$. Since $Z \cdot Z = -2 = Z_{B_1} \cdot Z_{B_1}$, $(Z - Z_{B_1}) \cdot (Z + Z_{B_1}) = 0$. It follows that $A_i (Z + Z_{B_1}) = 0$ for all $A_i \not\subseteq B_1$. In particular, if $A_i \cap B_1 = \phi$, then $A_i \cdot Z = 0$. So we have either (A) $A_1 \cdot Z = -2$ and $z_1 = 1$, of (B) $A_1 \cdot Z = -1$ and $z_1 = 2$.

In (A) $A_1 \cdot A_1$ must be less than -2. But this is impossible because $A_1 \cdot A_1 = -2$.

In (B) Let $A_2 \subseteq B_1$ such that $A_1 \cdot A_2 = 1$. We claim that $A_2 \not\subseteq B_2$. Otherwise $0 \geq A_1(-K') = A_1 \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq -1 + 2 = 1$. This is absurd. The proof breaks up into four subcases.

(B1) There exist $A_3, A_4 \not\subseteq B_1$, $A_3 \neq A_4$ such that $A_3 \cdot A_1 = 1 = A_4 \cdot A_1$ and $z_3 = z_4 = 1 = z_2$. It follows that $A = A_1 \cup A_3 \cup A_4 \cup B_1$ and $Z = 2A_1 + A_3 + A_4 + Z_{B_1}$. We are in (3).

(B2) There exists $A_3 \not\subseteq B_1$ such that $A_1 \cdot A_3 = 1$ and $z_3 = 2$. Because $A_i \cdot Z = 0$ for $A_i \not\subseteq B_1$, $A_i \neq A_1$, it is easy to see that we are in (3).

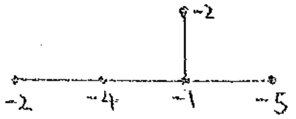
(B3) There exists $A_3 \not\subseteq B_1$ such that $A_1 \cdot A_3 = 1$, $z_3 = 1$ and $z_2 = 2$. Since $z_1 = 2$, $z_3 = 1$ and $A_3 \cdot Z = 0$, it follows that there is no $A_i \not\subseteq B_1$, $A_1 \neq A_i \neq A_3$ such that $A_i \cdot A_3 = 1$, i.e., $A = A_1 \cup A_3 \cup B_1$ and $Z = 2A_1 + A_3 + D$ where D is a positive cycle with support $= B_1$. We claim that $A_2 \cdot Z_{B_1} = 0$. Otherwise $Z = A_1 + A_3 + Z_{B_1}$ and hence $A_1 \cdot Z = 0$. This leads to a contradiction. We are in (4).

(B4) $z_2 = 3$. Then $A = A_1 \cup B_1$ and $Z = 2A_1 + D$ where D is a positive cycle with support $= B_1$. We claim that $A_2 \cdot Z_{B_1} = 0$. Otherwise $Z = A_2 + Z_{B_1}$. This leads to a contradiction. We are in (5). Q.E.D.

Definition 5.9: Let $\pi: M \to V$ be the minimal good resolution of weakly elliptic singularity p. Let $Z_{B_0} = Z, \ldots, Z_{B_\ell} = Z_E$ be the elliptic sequence. The set of self intersection numbers of the elliptic sequence is $\{Z_{B_o}^2, \ldots, Z_{B_\ell}^2\}$.

Corollary 5.10: Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists and the set of self intersection numbers of elliptic sequence consists of -2 and -1. Then Γ must be one of the following forms.

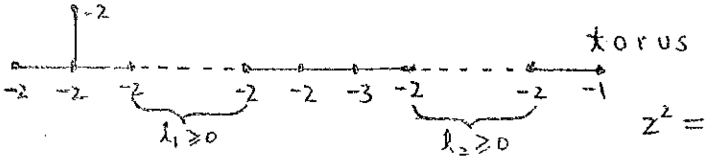
(1)



$$Z = 2\quad 3\quad \overset{5}{10}\quad 2$$

$$z^2 = x^5 + y^7$$

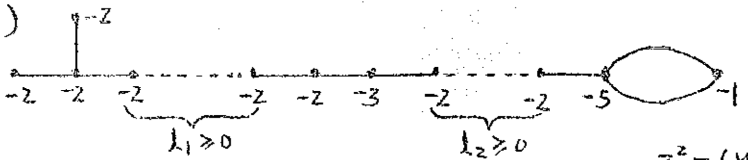
(2)



$$z^2 = (y^2 + x^{3+l_1})(x^3 + y^{12+6l_2})$$

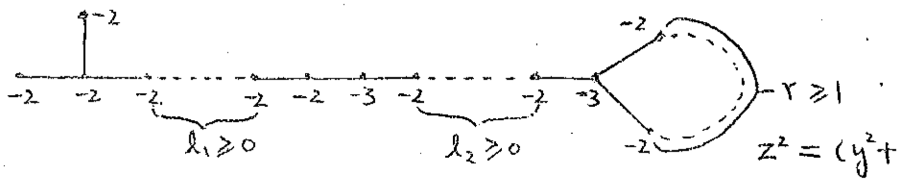
$$Z = 1\quad \overset{1}{2}\quad 2\cdots\cdots 2\quad 2\quad 1\quad 1\cdots\cdots 1\quad 1$$

(3)



$$z^2 = (y^2 + x^{3+l_1})(x + y^{4+2l_2}) \cdot (x^2 + y^{9+4l_2})$$

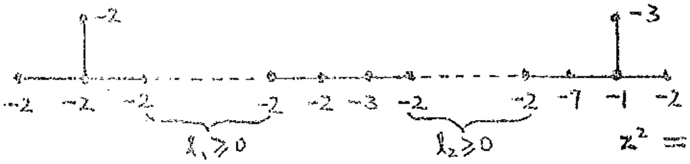
$$Z = 1\quad \overset{1}{2}\quad 2\cdots\cdots 2\quad 2\quad 1\quad 1\cdots\cdots 1\quad 1\quad 2$$



$$z^2 = (y^2 + x^{3+l_1})(x + y^{4+2l_2}) (x^2 + y^{9+4l_2+r})$$

$$Z = 1\quad \overset{1}{2}\quad 2\cdots\cdots 2\quad 2\quad 1\quad 1\cdots\cdots 1\quad 1\quad \overset{2}{6}\quad 3$$

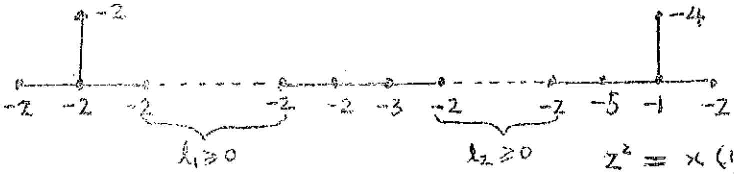
(4)



$$z^2 = (y^2 + x^{3+l_1})(x^3 + y^{13+6l_2})$$

$$Z = 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ 2\ 1\ 1 \cdots\cdots 1\ 1\ \overset{2}{6}\ 3$$

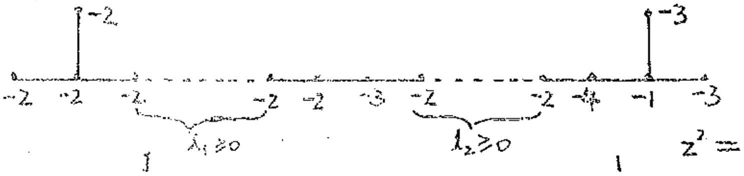
(5)



$$z^2 = x(y^2 + x^{l_1+3})(x^2 + y^{9+4l_2})$$

$$Z = 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ 2\ 1\ 1 \cdots\cdots 1\ 1\ \overset{1}{4}\ 2$$

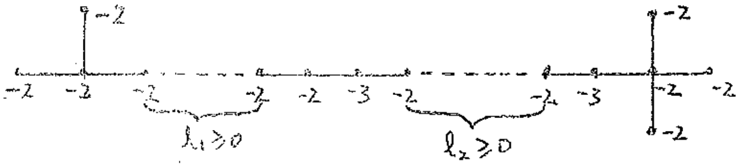
(6)



$$z^2 = (y^2 + x^{l_1+3})(x^3 + y^{14+6l_2})$$

$$Z = 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ 2\ 1\ 1 \cdots\cdots 1\ 1\ \overset{1}{3}\ 1$$

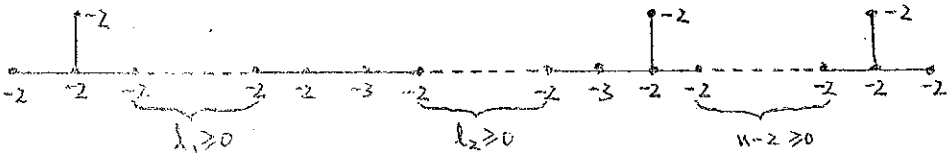
(7)



$$z^2 = (y^2 + x^{l_1+3})(x^3 + y^{15+6l_2})$$

$$Z = 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ 2\ 1\ 1 \cdots\cdots 1\ 1\ \overset{1}{\underset{1}{2}}\ 1$$

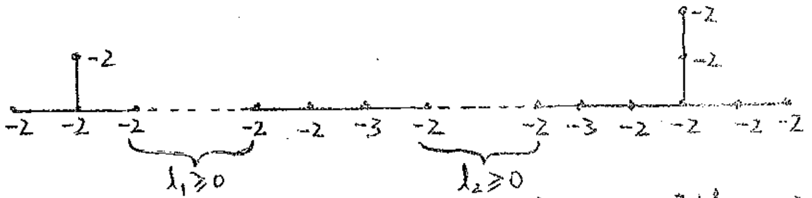
(8)



$$z^2 = (y^2 + x^{l_1+3})(x + y^{5+2l_2})(x^2 + y^{n+9+4l_2})$$

$$Z = 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ 2\ 1\ 1 \cdots\cdots 1\ 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ \overset{1}{2}\ 1$$

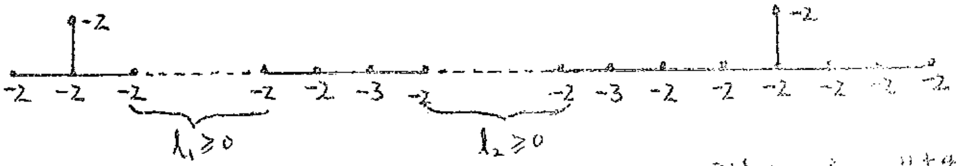
(9)



$z^2 = (y^2 + x^{3+l_1})(x^3 + y^{11+6l_2})$

$Z = 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ 2\ 1\ 1 \cdots\cdots 1\ 1\ 2\ \overset{1}{3}\ 2\ 1$

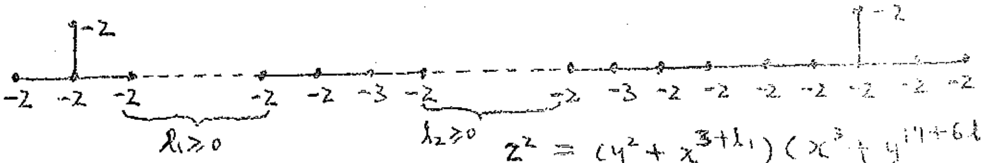
(10)



$z^2 = x(y^2 + x^{3+l_1})(y^2 + x^{11+4l_2})$

$Z = 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ 2\ 1\ 1 \cdots\cdots 1\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

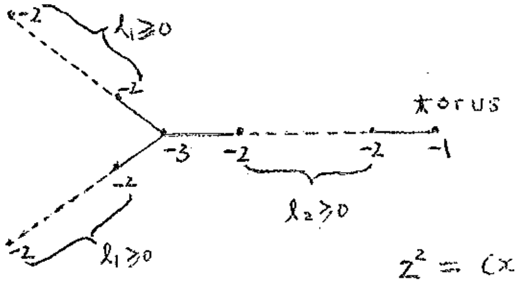
(11)

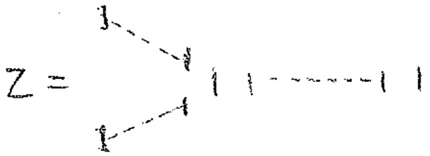


$z^2 = (y^2 + x^{3+l_1})(x^3 + y^{17+6l_2})$

$Z = 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ 2\ 1\ 1 \cdots\cdots 1\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

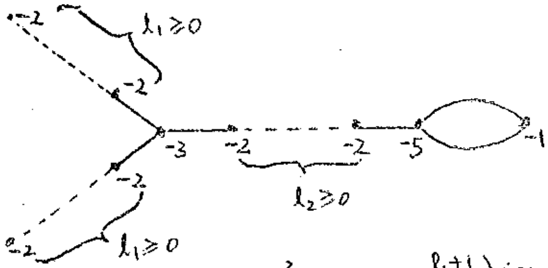
(12)

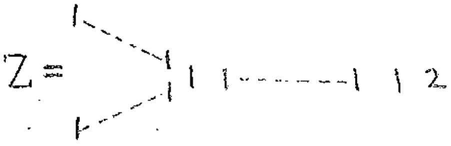




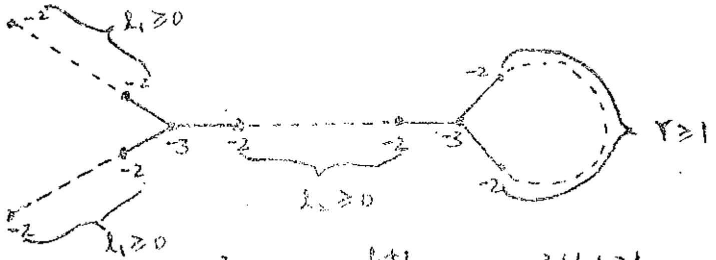
$z^2 = (x + y^{l_1+1})(x^3 + y^{9+6l_2+3l_1})$

(13)



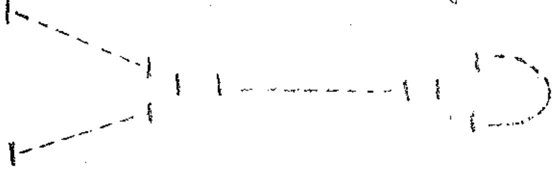


$z^2 = (x + y^{l_1+1})(x + y^{3+l_1+2l_2})(x^2 + y^{4+2l_1+4l_2})$

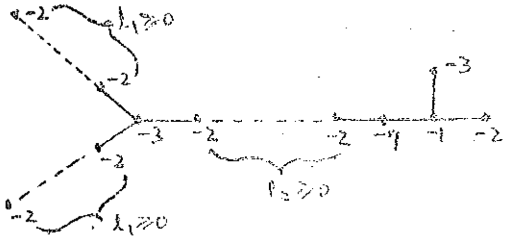


$$z^2 = (x + y^{l_1+1})(x + y^{3+l_1+2l_2})(x^2 + y^{r+7+2l_1+4l_2})$$

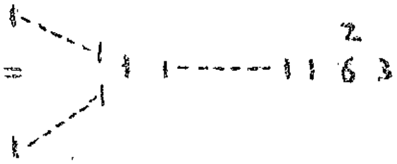
Z =



(14)

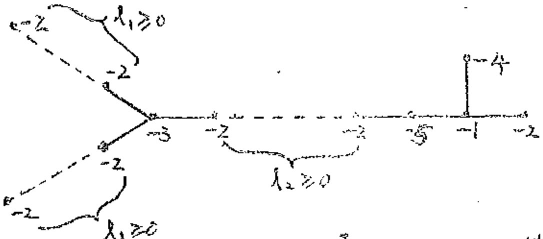


Z =

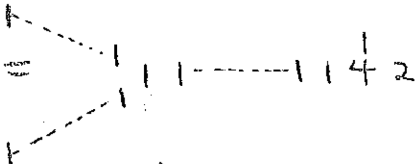


$$z^2 = (x + y^{1+l_1})(x^3 + y^{10+6l_2+3l_1})$$

(15)

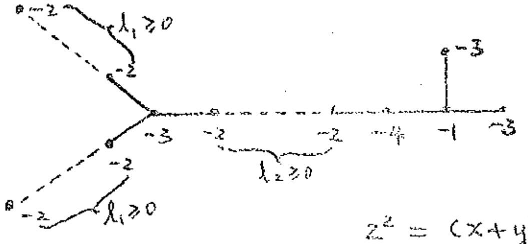


Z =

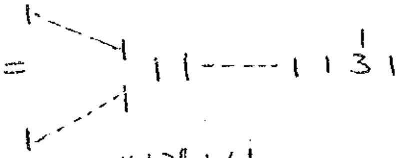


$$z^2 = x(x + y^{1+l_1})(x^2 + y^{7+2l_1+4l_2})$$

(16)

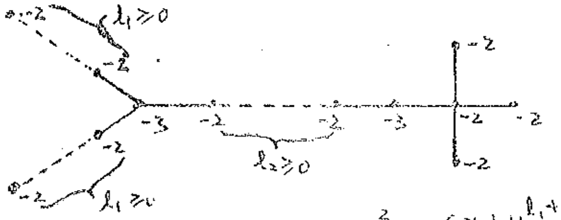


Z =

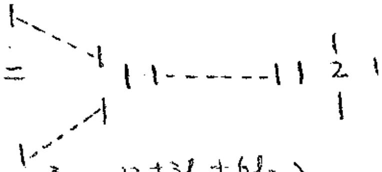


$$z^2 = (x + y^{1+l_1})(x^3 + y^{11+3l_1+6l_2})$$

(17)

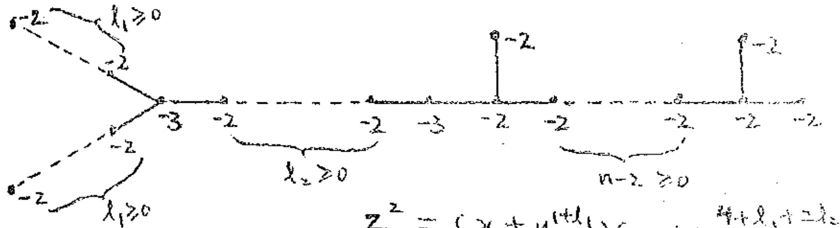


Z =



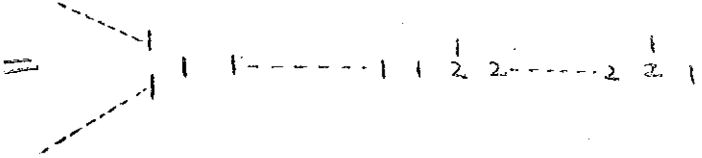
$$z^2 = (x + y^{l_1+1})(x^3 + y^{12+3l_1+6l_2})$$

(18)

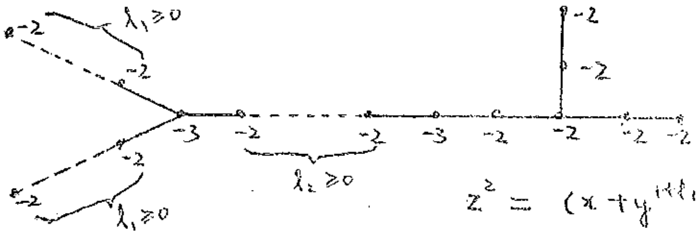


$$z^2 = (x+y^{1+l_1})(x+y^{4+l_1+2l_2})(x^2+y^{n+7+2l_1+4l_2})$$

Z =

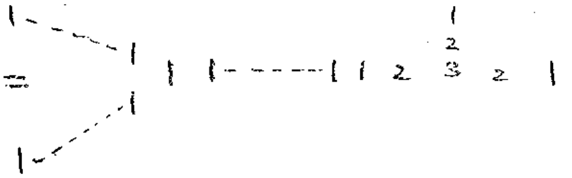


(19)

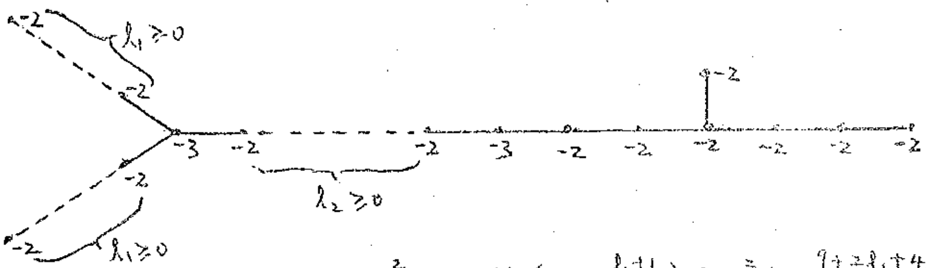


$$z^2 = (x+y^{1+l_1})(x^3+y^{13+3l_1+6l_2})$$

Z =

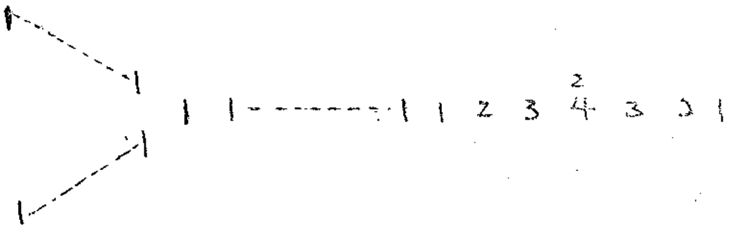


(20)

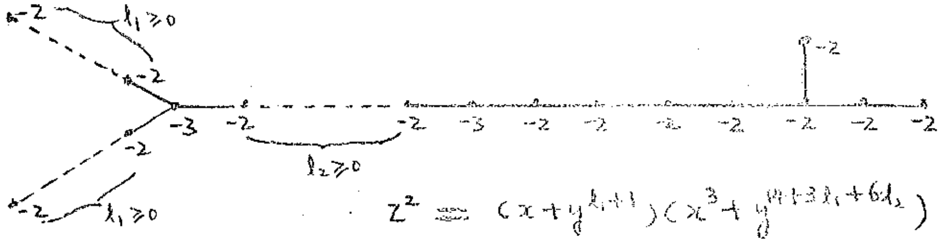


$$z^2 = x(x+y^{l_1+1})(x^2+y^{9+2l_1+4l_2})$$

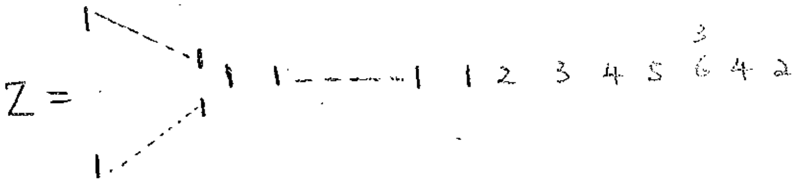
Z =



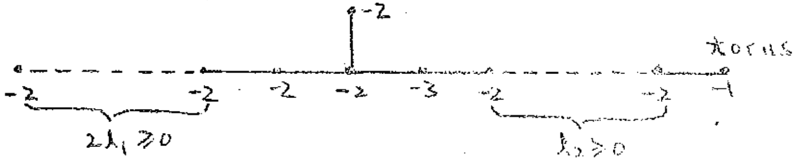
(21)



$$z^2 = (x+y^{l_1+1})(x^3+y^{14+3l_1+6l_2})$$



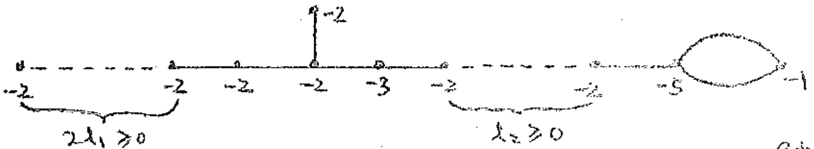
(22)



$$z^2 = y(x+y^{l_1+1})(x^3+y^{12+3l_1+6l_2})$$

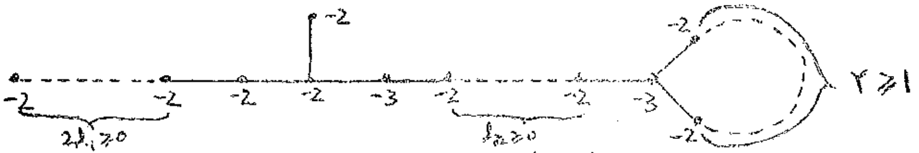
$$Z = 1\ \underbrace{2\cdots\cdots2}_{2l_1-1}\ 2\ \overset{1}{2}\ 1\ \underbrace{1\cdots\cdots1}_{l_2\geq 0}\quad 1$$

(23)



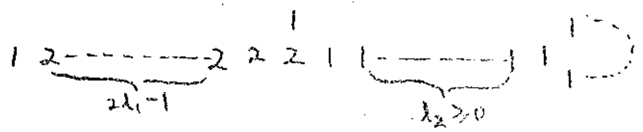
$$z^2 = y(x+y^{l_1+1})(x+y^{4+l_1+2l_2})(x^2+y^{9+2l_1+4l_2})$$

$$Z = 1\ \underbrace{2\cdots\cdots2}_{2l_1-1}\ 2\ \overset{1}{2}\ 1\ \underbrace{1\cdots\cdots1}_{l_2\geq 0}\ 1\ 2$$

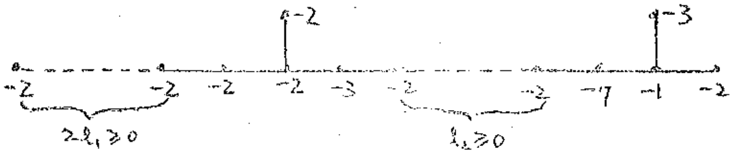


$$z^2 = y(x+y^{l_1+1})(x+y^{4+l_1+2l_2})(x^2+y^{r+9+2l_1+4l_2})$$

Z =



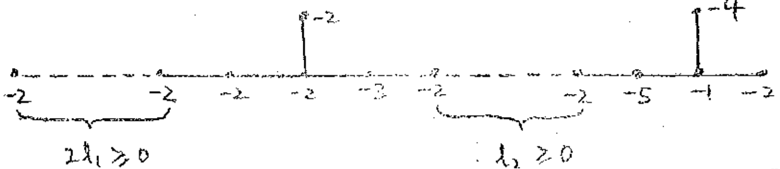
(24)



$$z^2 = y(x+y^{1+l_1})(x^3+y^{13+3l_1+6l_2})$$

$$Z_1 = 1\ \underbrace{2 \cdots\cdots 2}_{2l_1-1}\ 2\ \overset{1}{2}\ 1\ \underbrace{1 \cdots\cdots 1}_{l_2}\ 1\ \overset{2}{6}\ 3$$

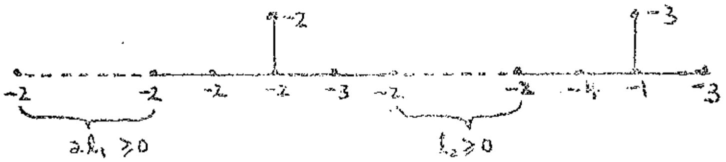
(25)



$$z^2 = yx(x+y^{1+l_1})(x^2+y^{9+2l_1+4l_2})$$

$$Z_1 = 1\ \underbrace{2 \cdots\cdots 2}_{2l_1-1}\ 2\ \overset{1}{2}\ 1\ \underbrace{1 \cdots\cdots 1}_{l_2 \geq 0}\ 1\ \overset{1}{4}\ 2$$

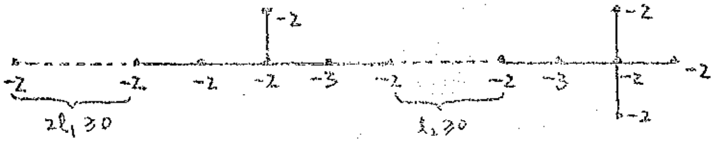
(26)



$$z^2 = y(x+y^{1+l_1})(x^3+y^{14+3l_1+6l_2})$$

$$Z = 1\ 2 \cdots\cdots 2\ 2\ \overset{1}{2}\ 1\ 1 \cdots\cdots 1\ 1\ \overset{1}{3}\ 1$$

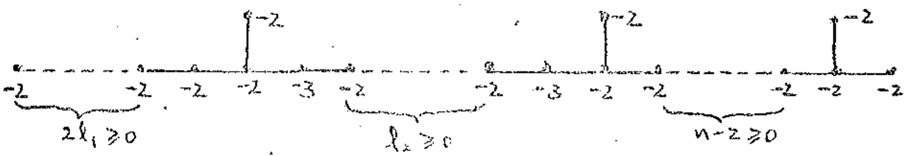
(27)



$$z^2 = y(x+y^{l_1+1})(x^3+y^{15+3l_1+6l_2})$$

$$Z_1 = 1\ 2 \cdots\cdots 2\ 2\ \overset{1}{2}\ 1\ 1 \cdots\cdots 1\ 1\ \underset{1}{\overset{1}{2}}\ 1$$

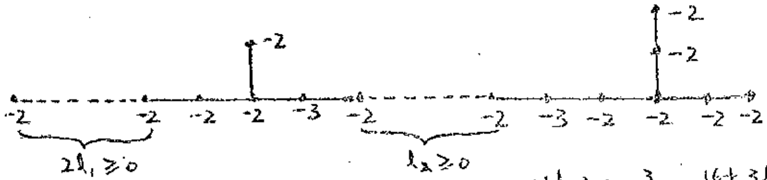
(28)



$$z^2 = y(x+y^{1+l_1})(x+y^{5+l_1+2l_2})(x^2+y^{n+9+2l_1+4l_2})$$

$$Z_1 = 1\ 2 \cdots\cdots 2\ 2\ \overset{1}{2}\ 1\ 1 \cdots\cdots 1\ 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ \overset{1}{2}\ 1$$

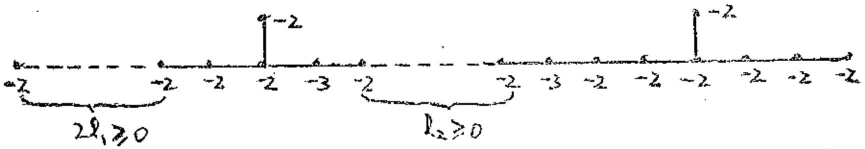
(29)



$$z^2 = y(x + y^{1+l_1})(x^3 + y^{16+3l_1+6l_2})$$

$Z = 1\ 2 \cdots 2\ 2\ \overset{1}{2}\ 1\ 1 \cdots 1\ 1\ 2\ \overset{\overset{1}{2}}{3}\ 2\ 1$

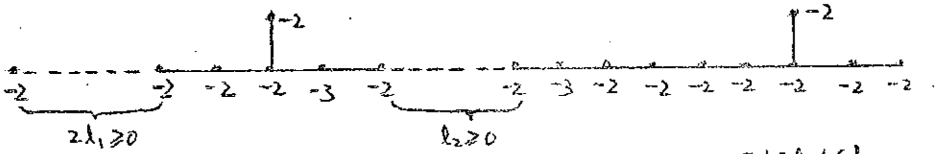
(30)



$$z^2 = yx(x + y^{l_1+1})(x^2 + y^{11+2l_1+4l_2})$$

$Z = 1\ 2 \cdots 2\ 2\ \overset{1}{2}\ 1\ 1 \cdots 1\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

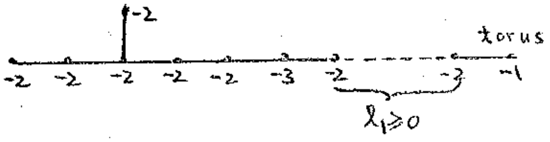
(31)



$$z^2 = y(x + y^{l_1+1})(x^3 + y^{17+3l_1+6l_2})$$

$Z = 1\ 2 \cdots 2\ 2\ \overset{1}{2}\ 1\ 1 \cdots 1\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

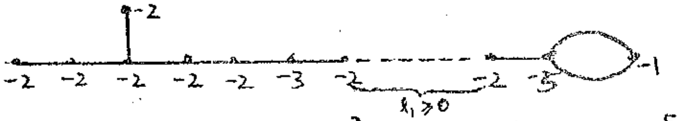
(32)



$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots 1\ 1$

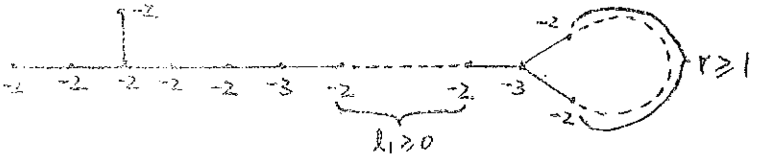
$$z^2 = (x^2 + y^3)(x^3 + y^{15+6l_1})$$

(33)



$$z^2 = (x^2 + y^3)(x + y^{5+2l_1})(x^2 + y^{11+4l_1})$$

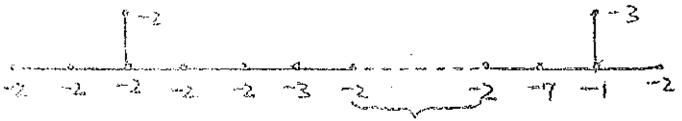
$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots 1\ 1\ 2$



$$z^2 = (x^2+y^3)(x+y^{5+2l_1})(x^2+y^{r+11+4l_1})$$

$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots\cdots 1\ 1\ \substack{1\\1}$$

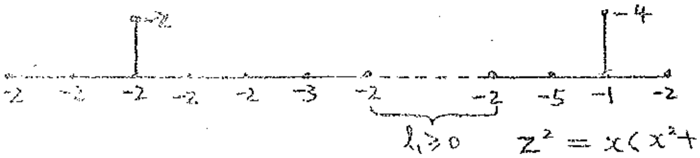
(34)



$$z^2 = (x^2+y^3)(x^3+y^{16+6l_1})$$

$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots 1\ 1\ \overset{2}{6}\ 3$$

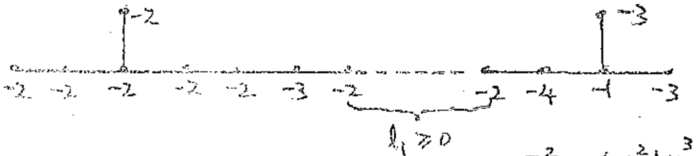
(35)



$$z^2 = x(x^2+y^3)(x^2+y^{11+4l_1})$$

$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots\cdots 1\ 1\ \overset{1}{4}\ 2$$

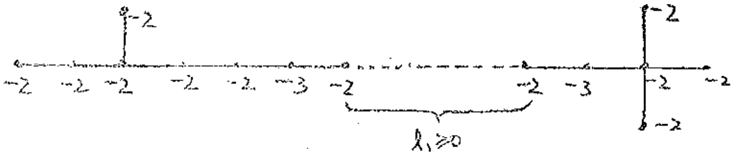
(36)



$$z^2 = (x^2+y^3)(x^3+y^{17+6l_1})$$

$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots\cdots 1\ 1\ \overset{1}{3}\ 1$$

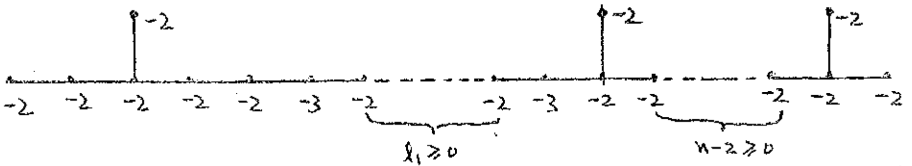
(37)



$$z^2 = (x^2+y^3)(x^3+y^{18+6l_1})$$

$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots\cdots 1\ 1\ \overset{1}{\underset{1}{2}}\ 1$$

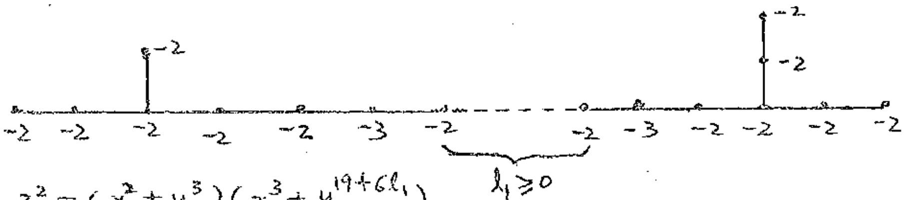
(38)



$$z^2 = (x^2+y^3)(x+y^{6+2l_1})(x^2+y^{n+11+4l_1})$$

$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots\cdots 1\ 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ \overset{1}{2}\ 1$$

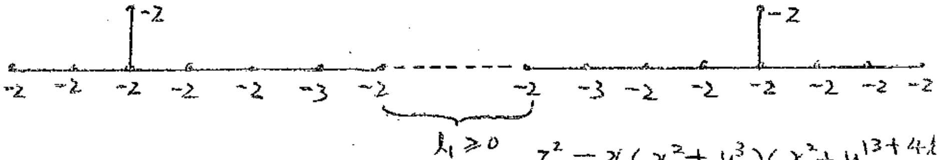
(39)



$$z^2 = (x^2+y^3)(x^3+y^{19+6l_1})$$

$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots\cdots 1\ 1\ 2\ \overset{1}{\overset{2}{3}}\ 2\ 1$$

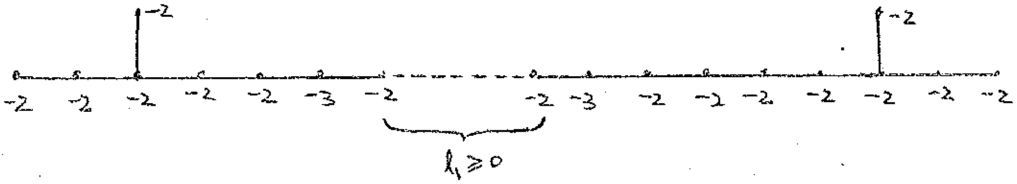
(40)



$$z^2 = x(x^2+y^3)(x^2+y^{13+4l_1})$$

$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots\cdots 1\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$$

(41)

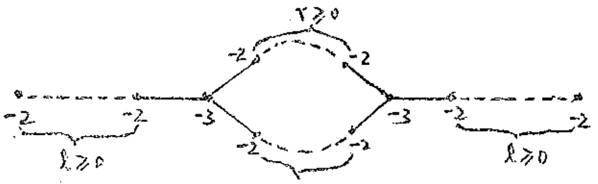


$$z^2 = (x^2+y^3)(x^3+y^{20+6l_1})$$

$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1 \cdots\cdots 1\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$$

Corollary 5.11: Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists and the set of self intersection numbers of elliptic sequence consists of −2. Then Γ must be one of the following forms.

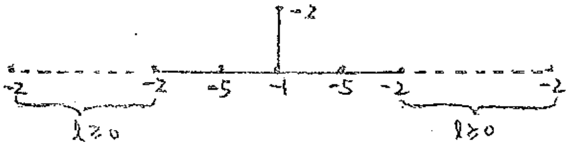
(1)

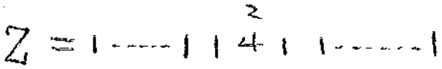


$$z^2 = (x^2 + y^{r+3+2\ell})((x+y^{1+\ell})^2 + y^{s+3+2\ell})$$

Z =

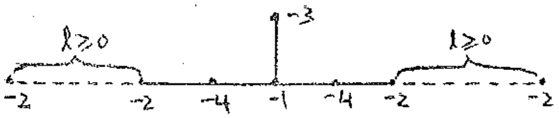
(2)

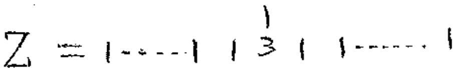




$$z^2 = y^4 + x^{5+4\ell}$$

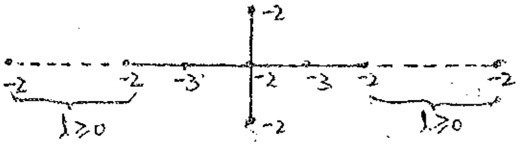
(3)

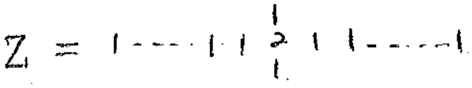




$$Z^2 = y^4 + x^{4+3\ell}y$$

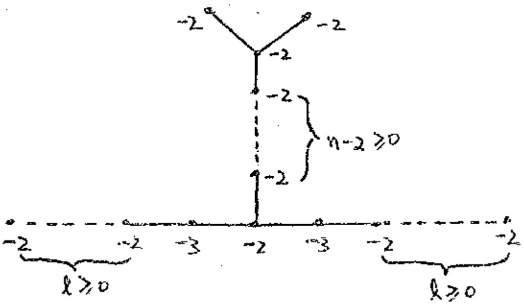
(4)





$$z^2 = y^4 + x^{6+4\ell}$$

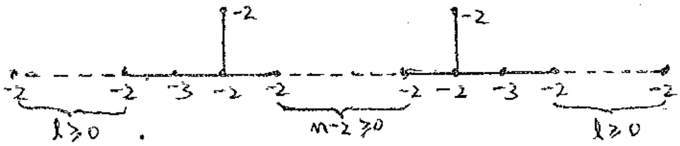
(5)



$$Z = 1 \cdots 1 \; 1 \; \overset{\substack{1 \quad 1 \\ 2 \\ 2 \\ \vdots \\ 2}}{2} \; 1 \; 1 \cdots 1$$

$$z^2 = (y^2 + x^{3+2l})(y^2 + x^{n+2l+2})$$

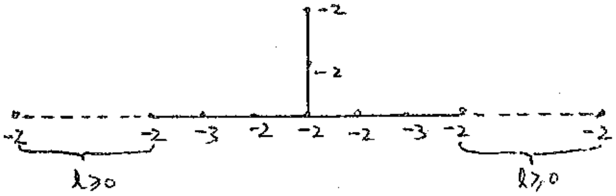
(6)



$$z^2 = (y^2 + x^{3+2l})^2 + x^a y^b \quad , \quad 2a + (3+2l)b = 11+n+8l$$

$$Z = 1 \cdots 1 \; 1 \; \overset{1}{2} \; 2 \cdots 2 \; \overset{1}{2} \; 1 \; 1 \cdots 1$$

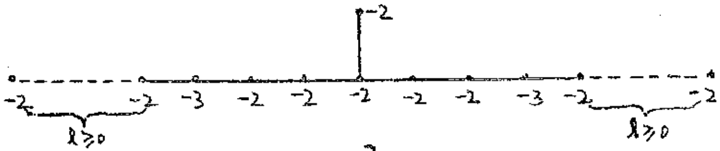
(7)



$$z^2 = y^4 + x^{5+3l} y$$

$$Z = 1 \cdots 1 \; 1 \; 2 \; \overset{\overset{1}{2}}{3} \; 2 \; 1 \; 1 \cdots 1$$

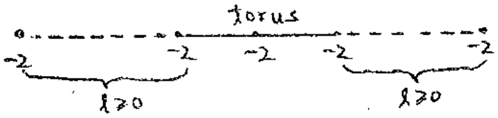
(8)



$$z^2 = y^4 + x^{7+4l}$$

$$Z = 1 \cdots 1 \; 1 \; 2 \; 3 \; \overset{2}{4} \; 3 \; 2 \; 1 \; 1 \cdots 1$$

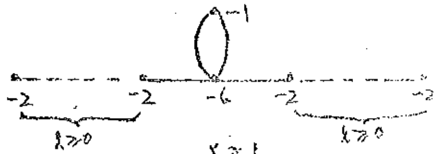
(9)

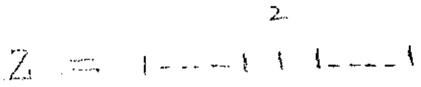


$$Z = 1 \cdots 1 \; 1 \; 1 \cdots 1$$

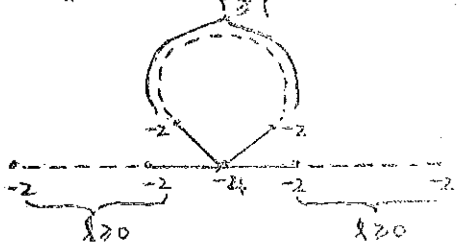
$$z^2 = y^4 + x^{4l+4}$$

(10)



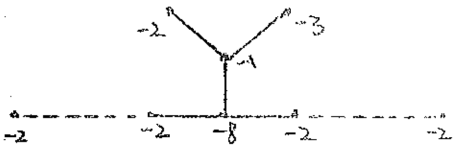


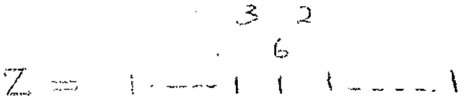
$$z^2 = (y+x^{l+1})(y+2x^{l+1})(y^2+x^{2l+3})$$



$$z^2 = (y+x^{l+1})(y+2x^{l+1})(y^2+x^{r+2l+3})$$

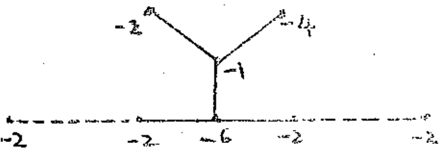
(11)

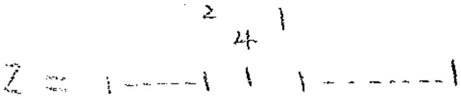




$$z^2 = (y+x^{l+1})(y^3+x^{4+3l})$$

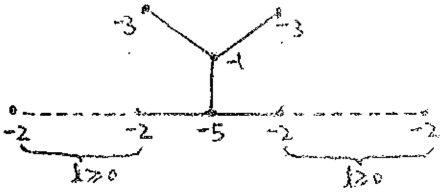
(12)

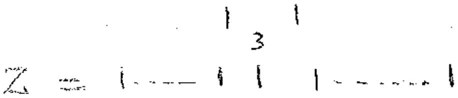




$$z^2 = x(x+y^{1+l})(x^2+y^{3+2l})$$

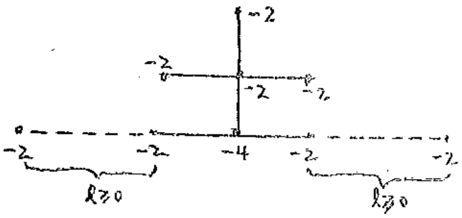
(13)

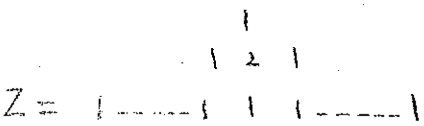




$$z^2 = (y+x^{l+1})(y^3+x^{5+3l})$$

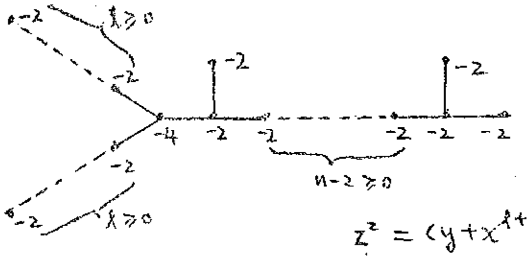
(14)



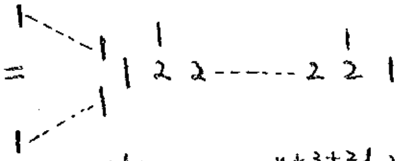


$$z^2 = (y+x^{l+1})(y^3+x^{6+3l})$$

(15)

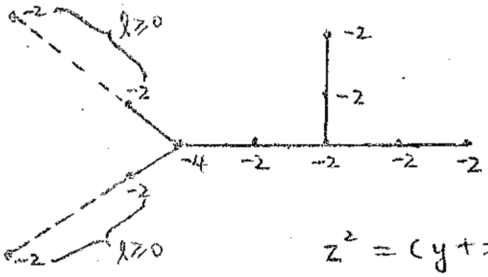


$Z=$

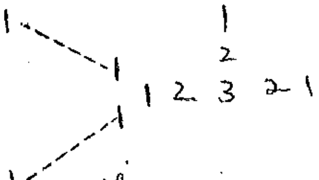


$$z^2=(y+x^{l+1})(y+x^{2+l})(y^2+x^{n+3+2l})$$

(16)

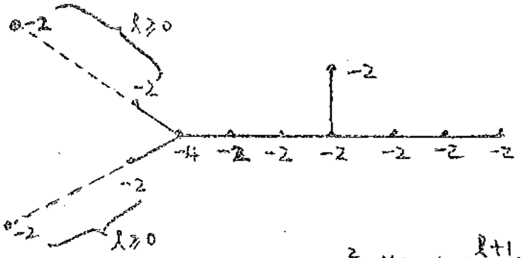


$Z=$

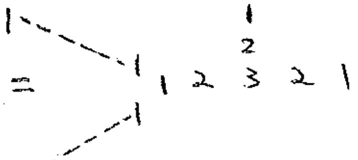


$$z^2=(y+x^{l+1})(y^3+x^{7+3l})$$

(17)

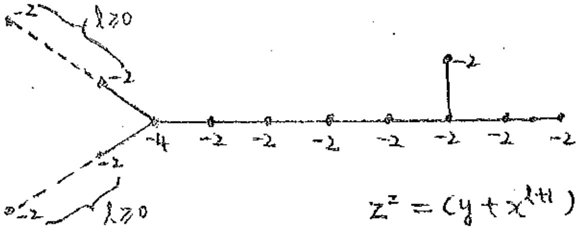


$Z=$

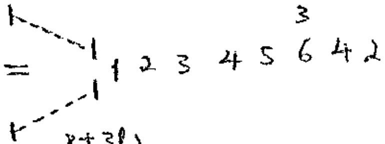


$$z^2=y(y+x^{l+1})(y^2+x^{5+2l})$$

(18)

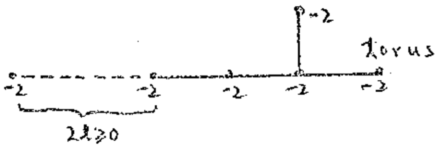


$Z=$



$$z^2=(y+x^{l+1})(y^3+x^{8+3l})$$

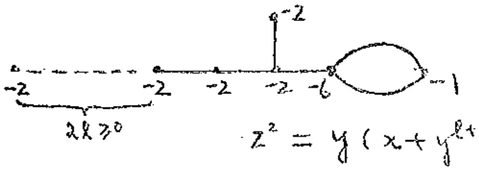
(19)



$Z=$

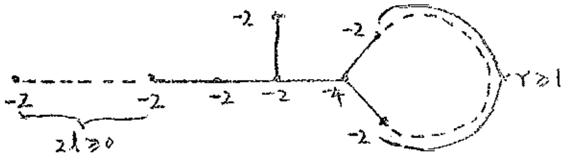
$$z^2=y(x+y^{l+1})(x^3+y^{6+3l})$$

(20)



$Z = 1\,2 \cdots 2\,2\,\overset{1}{2}\,1\,2$

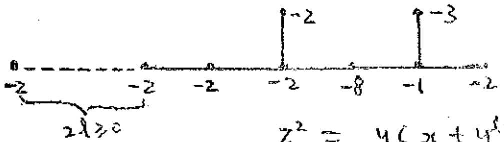
$z^2 = y(x+y^{\ell+1})(x+y^{\ell+2})(x^2+y^{5+2\ell})$



$Z = 1\,2 \cdots 2\,2\,\overset{1}{2}\,1\,\overset{1}{\underset{1}{\cdots}}$

$z^2 = y(x+y^{\ell+1})(x+y^{\ell+2})(x^2+y^{r+5+2\ell})$

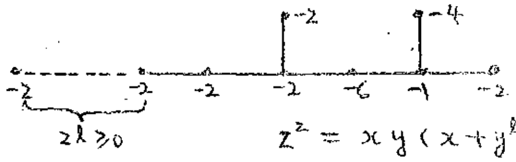
(21)



$Z = 1\,2 \cdots 2\,2\,\overset{1}{2}\,1\,\overset{3}{6}\,2$

$z^2 = y(x+y^{\ell+1})(x^3+y^{7+3\ell})$

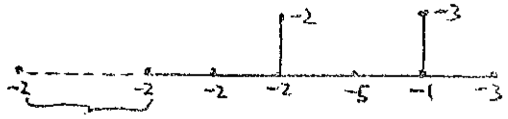
(22)



$Z = 1\,2 \cdots 2\,2\,\overset{1}{2}\,1\,\overset{1}{4}\,2$

$z^2 = xy(x+y^{\ell+1})(x^2+y^{5+2\ell})$

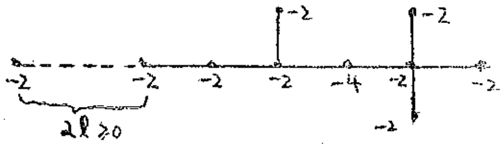
(23)



$Z = 1\,2 \cdots 2\,2\,\overset{1}{2}\,1\,\overset{1}{3}\,1$

$z^2 = y(x+y^{\ell+1})(x^3+y^{8+3\ell})$

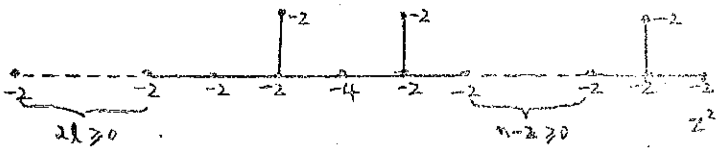
(24)



$Z = 1\,2 \cdots 2\,2\,\overset{1}{2}\,1\,\overset{1}{\underset{1}{2}}\,1$

$z^2 = y(x+y^{\ell+1})(x^3+y^{9+3\ell})$

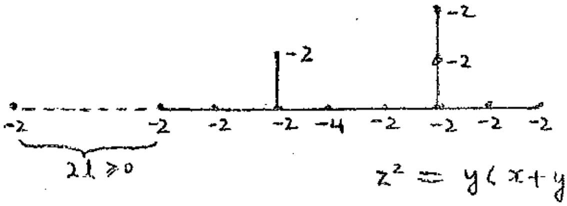
(25)



$z^2 = y(x+y^{\ell+1})(x+y^{\ell+3})(x^2+y^{n+5+2\ell})$

$Z = 1\,2 \cdots 2\,2\,\overset{1}{2}\,1\,\overset{1}{2}\,2 \cdots 2\,\overset{1}{2}\,1$

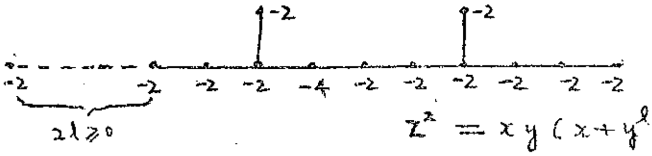
(26)



$$Z = 1\,2\cdots 2\,2\,\overset{1}{2}\,1\,2\,\overset{\overset{1}{2}}{3}\,2\,1$$

$$z^2 = y(x+y^{l+1})(x^3+y^{10+3l})$$

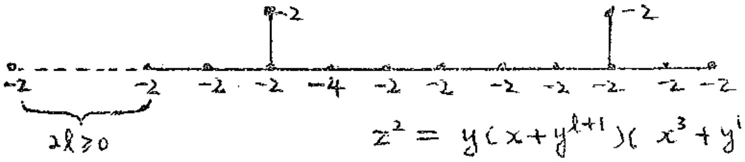
(27)



$$z^2 = xy(x+y^{l+1})(x^2+y^{7+2l})$$

$$Z = 1\,2\cdots 2\,2\,\overset{1}{2}\,1\,2\,3\,\overset{2}{4}\,3\,2\,1$$

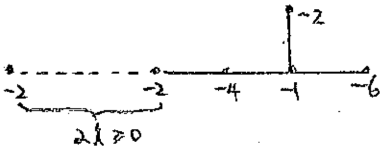
(28)



$$z^2 = y(x+y^{l+1})(x^3+y^{11+3l})$$

$$Z = 1\,2\cdots 2\,2\,\overset{1}{2}\,1\,2\,3\,4\,5\,\overset{3}{6}\,4\,2$$

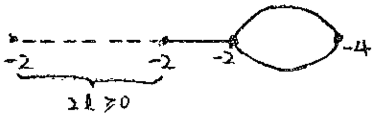
(29)



$$Z = 1\,2\cdots 2\,2\,\overset{3}{6}\,1$$

$$z^2 = xy(x^3+y^{2+3l})$$

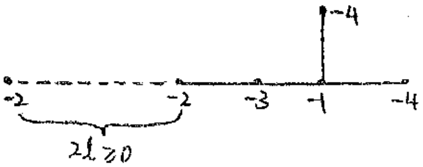
(30)



$$Z = 1\,2\cdots 2\;2\;1$$

$$z^2 = x(y^2+x^{1+2l})(y^2+x^{2+2l})$$

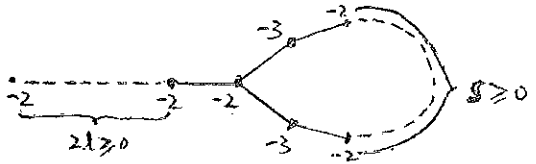
(31)



$$Z = 1\,2\cdots 2\,2\,\overset{1}{4}\,1$$

$$z^2 = y(x^4+y^{3+4l})$$

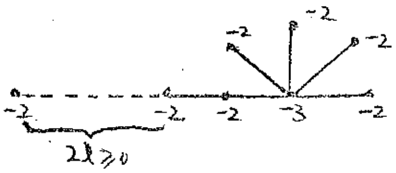
(32)



$$Z = \begin{matrix} & & & & 1 & 1 \\ 1 & 2 & & 2 \;\; 2 & & \\ & & & & 1 & 1 \end{matrix}$$

$$z^2 = x(y^2+x^{1+2l})(y^2+x^{3+2l+s})$$

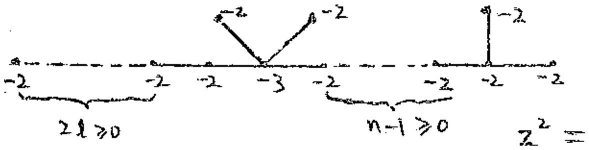
(33)



$$Z = 1\ 2 \cdots 2\ \ 2\ \ \overset{1\,1\,1}{2}\ \ 1$$

$$z^2 = y(x+y^{l+1})(x+2y^{l+1})(x+3y^{l+1})(x+4y^{l+1})$$

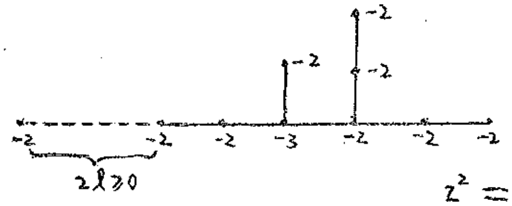
(34)



$$Z = 1\ 2 \cdots 2\ \ 2\ \ \overset{1\,1}{2}\ \ 2 \cdots 2\ \ \overset{1}{2}\ \ 1$$

$$z^2 = y(x^2+y^{2+2l})(x^2+y^{n+2+2l})$$

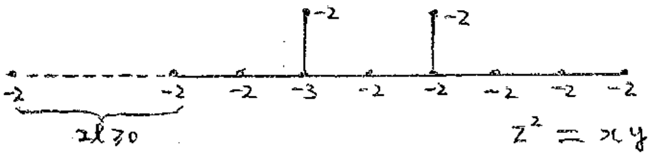
(35)



$$Z = 1\ 2 \cdots 2\ \ 2\ \ \overset{1}{2}\ \ \overset{\overset{1}{2}}{3}\ \ 2\ \ 1$$

$$z^2 = y(x+y^{l+1})(x^3+y^{4+3l})$$

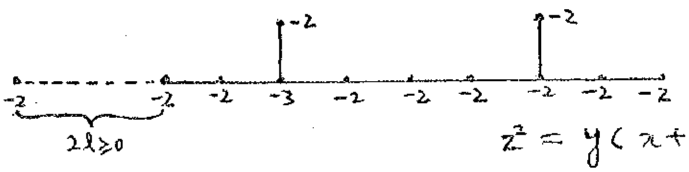
(36)



$$z^2 = xy(x+y^{l+1})(x^2+y^{3+2l})$$

$$Z = 1\ 2 \cdots 2\ \ 2\ \ \overset{1}{2}\ \ 3\ \ \overset{2}{4}\ \ 3\ \ 2\ \ 1$$

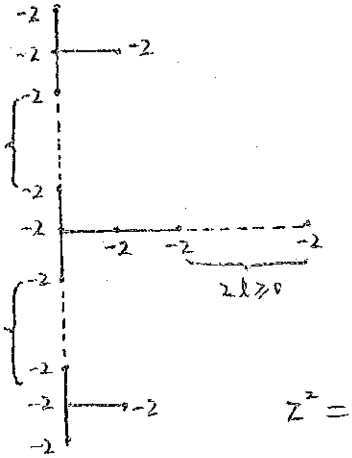
(37)



$$z^2 = y(x+y^{l+1})(x^3+y^{5+3l})$$

$$Z = 1\ 2 \cdots 2\ \ 2\ \ \overset{1}{2}\ \ 3\ \ 4\ \ 5\ \ \overset{3}{6}\ \ 4\ \ 2$$

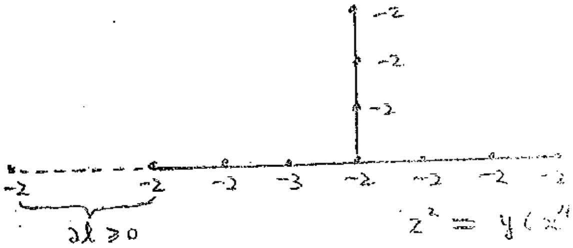
(38)



$$Z = \begin{array}{llll} 1 \\ 2 & 1 \\ 2 \\ \vdots \\ 2 \\ 2 & 2 & \cdots\cdots & 2\ \ 1 \\ 2 \\ \vdots \\ 2 \\ 2 & 1 \\ 1 \end{array}$$

$$z^2 = (x+y)(x^2+y^{2+n+2l})((x+y^{1+l})^2+y^{2+m+2l})$$

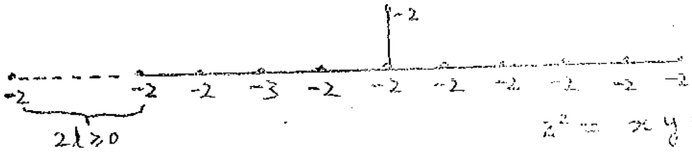
(39)



$$Z = 1\ 2 \cdots\cdots 2\ 2\ 2\ \overset{\overset{\overset{1}{2}}{3}}{4}\ 3\ 2\ 1$$

$$z^2 = y(x^4+y^{5+4l})$$

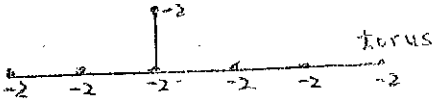
(40)



$$z^2 = xy(x^3+y^{4+3l})$$

$$Z = 1\ 2 \cdots\cdots 2\ 2\ 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2\ 1$$

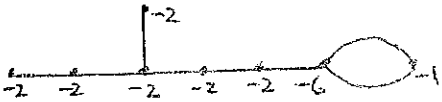
(41)



$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$$

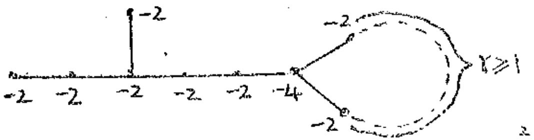
$$z^2 = (x^2+y^3)(x^3+y^9)$$

(42)



$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 2$$

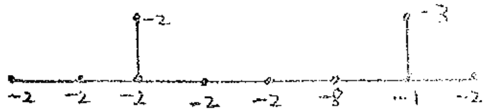
$$z^2 = (x^2+y^3)(x+y^3)(x^2+y^7)$$



$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ \begin{matrix}1\\ \vdots\\ 1\end{matrix}$$

$$z^2 = (x^2+y^3)(x+y^3)(x^2+y^{r+7})$$

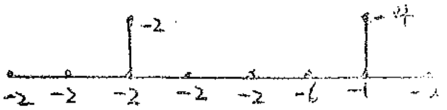
(43)



$Z = 2\,3\,\overset{2}{4}\,3\,2\,1\,\overset{2}{6}\,3$

$z^2 = (x^2+y^3)(x^3+y^{10})$

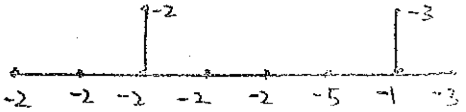
(44)



$Z = 2\,3\,\overset{2}{4}\,3\,2\,1\,\overset{1}{4}\,2$

$z^2 = x(x^2+y^3)(x^2+y^7)$

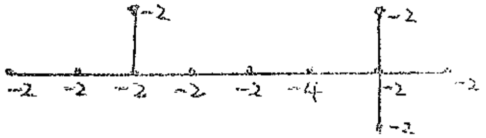
(45)



$Z = 2\,3\,\overset{2}{4}\,3\,2\,1\,\overset{1}{3}\,1$

$z^2 = (x^2+y^3)(x^3+y^{11})$

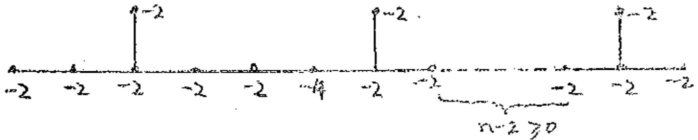
(46)



$Z = 2\,3\,\overset{2}{4}\,3\,2\,1\,\overset{1}{\underset{1}{2}}\,1$

$z^2 = (x^2+y^3)(x^3+y^{12})$

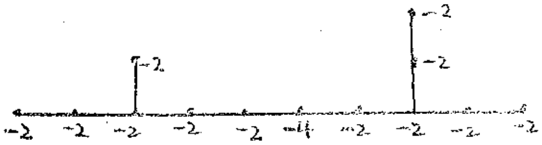
(47)



$z^2 = (x^2+y^3)(x+y^4)(x^2+y^{n+7})$

$Z = 2\,3\,\overset{2}{4}\,3\,2\,1\,\overset{1}{2}\,2 \quad 2\,\overset{1}{2}\,1$

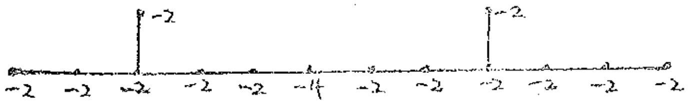
(48)



$Z = 2\,3\,\overset{2}{4}\,3\,2\,1\,2\,\overset{1}{\overset{2}{3}}\,2\,1$

$z^2 = (x^2+y^3)(x^3+y^{13})$

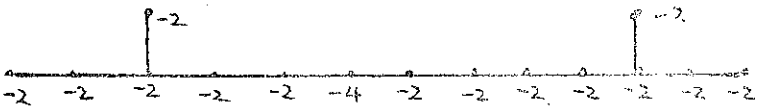
(49)



$z^2 = x(x^2+y^3)(x^2+y^9)$

$Z = 2\,3\,\overset{2}{4}\,3\,2\,1\,2\,3\,\overset{2}{4}\,3\,2\,1$

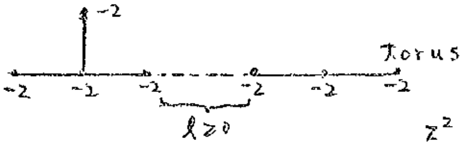
(50)



$z^2 = (x^2+y^3)(x^3+y^{14})$

$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

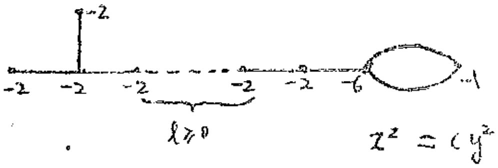
(51)



$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \ 2\ \ 1$

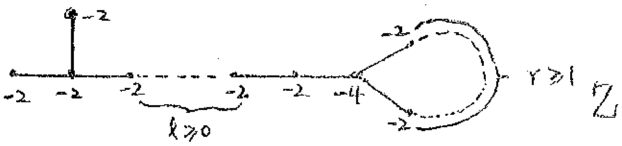
$z^2 = (y^2+x^{3+l})(x^3+y^6)$

(52)



$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \ 2\ 1\ 2$

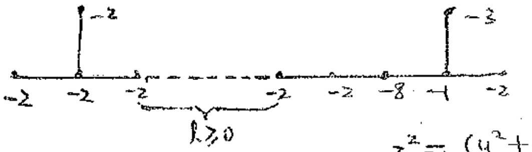
$z^2 = (y^2+x^{3+l})(x+y^2)(x^2+y^5)$



$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 1 \begin{matrix} 1 \cdots \\ 1 \cdots \end{matrix}\Big)$

$z^2 = (y^2+x^{3+l})(x+y^2)(x^2+y^{r+5})$

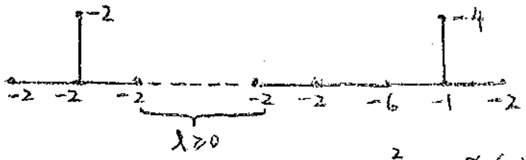
(53)



$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \ 2\ 1\ \overset{2}{6}\ 3$

$z^2 = (y^2+x^{3+l})(x^3+y^7)$

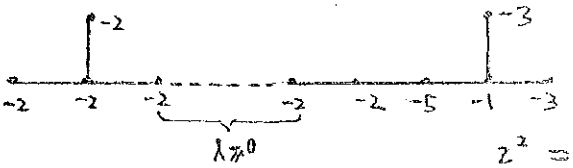
(54)



$Z = 1\ \overset{1}{2}\ 2 - 2\ \ 2\ 1\ \overset{1}{4}\ 2$

$z^2 = x(y^2+x^{3+l})(x^2+y^5)$

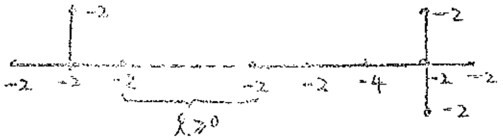
(55)



$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 1\ \overset{1}{3}\ 1$

$z^2 = (y^2+x^{3+l})(x^3+y^8)$

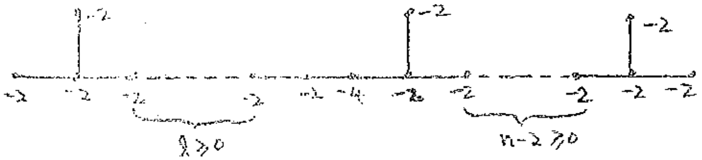
(56)



$$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 1\ \overset{1}{\underset{1}{2}}\ 1$$

$$z^2 = (y^2 + x^{3+l})(x^3 + y^9)$$

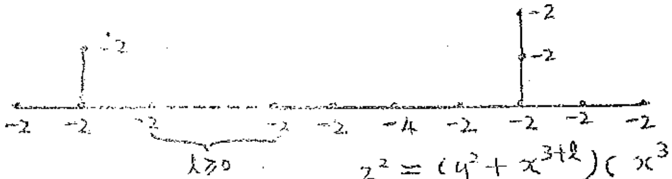
(57)



$$z^2 = (y^2 + x^{3+l})(x + y^3)(x^2 + y^{n+5})$$

$$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{2}\ 1$$

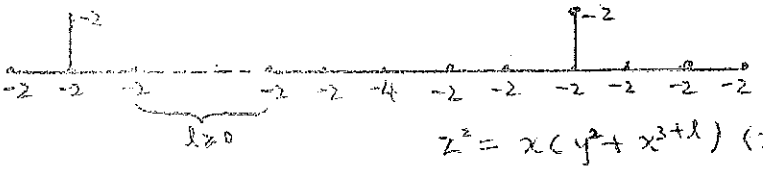
(58)



$$z^2 = (y^2 + x^{3+l})(x^3 + y^{10})$$

$$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 1\ 2\ \overset{\overset{1}{2}}{3}\ 2\ 1$$

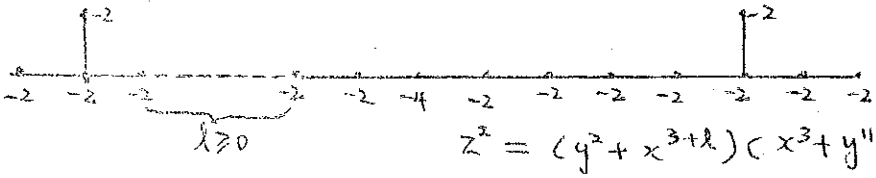
(59)



$$z^2 = x(y^2 + x^{3+l})(x^2 + y^7)$$

$$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$$

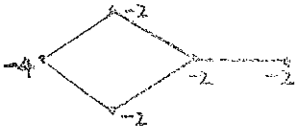
(60)



$$z^2 = (y^2 + x^{3+l})(x^3 + y^{11})$$

$$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$$

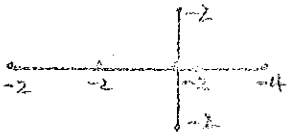
(61)



$Z = 1 \begin{smallmatrix}2\\2\end{smallmatrix} 3 \; 2$

$z^2 = (x^2+y^4)(x^3+y^4)$

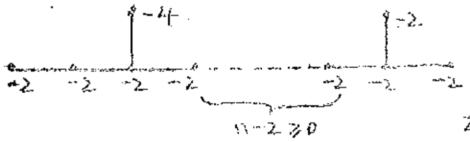
(62)



$Z = 2 \; 3 \; \overset{2}{\underset{2}{4}} \; 1$

$z^2 = x(x^4+y^6)$

(63)

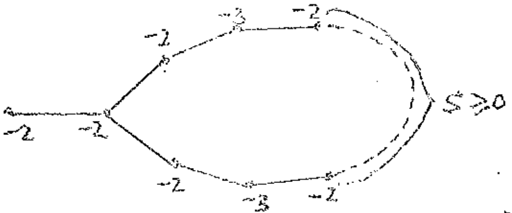


$Z = 2 \; 3 \; \overset{1}{4} \; 4 \cdots\cdots 4 \; \overset{2}{4} \; 2$

$z^2 = x(y^3+x^2)^2 + x^a y^b$

$3a+2b = n+14$

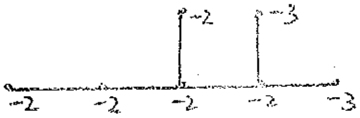
(64)



$Z = 2 \; 3 \begin{smallmatrix}2 & 1 & 1 \\ 2 & 1 & 1\end{smallmatrix}$

$z^2 = (x^2+y^{5+s})(x^3+y^4)$

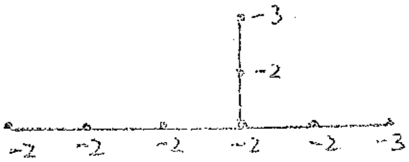
(65)



$Z = 2 \; 3 \; \overset{2}{4} \; \overset{1}{3} \; 1$

$z^2 = (x^2+y^3)(x^3+y^5)$

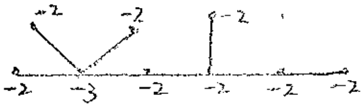
(66)



$Z = 2 \; 3 \; 4 \; \overset{\overset{1}{3}}{5} \; 3 \; 1.$

$z^2 = x^5+y^8$

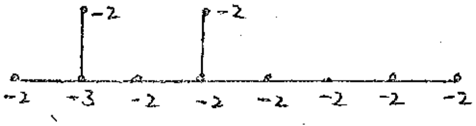
(67)



$Z = 1 \; \overset{1\;1}{2} \; 3 \; \overset{2}{4} \; 3 \; 2$

$z^2 = (x^3+y^6)(x^2+y^3)$

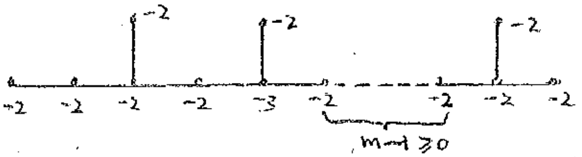
(68)



$Z = 1\ \overset{1}{2}\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2$

$z^2 = (x^2+y^4)(x^3+y^5)$

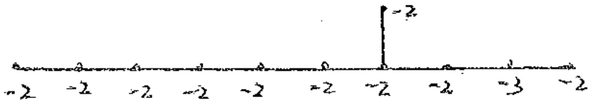
(69)



$Z = 2\ 3\ \overset{2}{4}\ 3\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{2}\ 1$

$z^2 = (x+y^2)(x^2+y^3)(x^2+y^{4+m})$

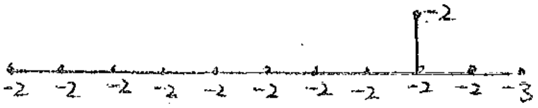
(70)



$Z = 2\ 3\ 4\ 5\ 6\ 7\ \overset{4}{8}\ 5\ 2\ 1$

$z^2 = x(x^4+y^7)$

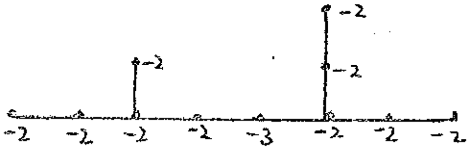
(71)



$Z = 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ \overset{5}{10}\ 6\ 2$

$z^2 = x^5+y^7$

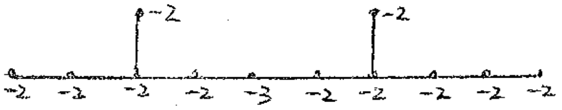
(72)



$Z = 2\ 3\ \overset{2}{4}\ 3\ \ 2\ \overset{\overset{1}{2}}{3}\ 2\ 1$

$z^2 = (x^2+y^3)(x^3+y^7)$

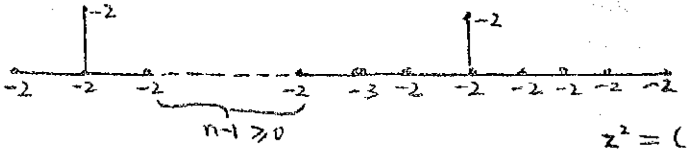
(73)



$Z = 2\ 3\ \overset{2}{4}\ \ 3\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

$z^2 = x(x^2+y^3)(x^2+y^5)$

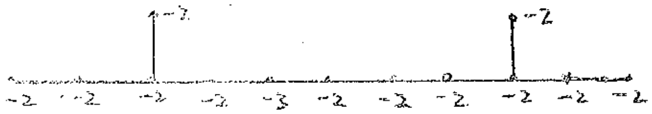
(74)



$z^2 = (x^2+y^{4+n})(x^3+y^5)$

$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2$

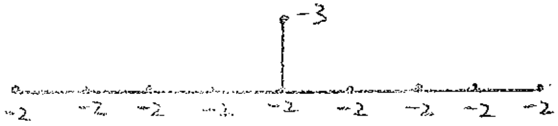
(75)



$z^2 = (x^2+y^3)(x^3+y^8)$

$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

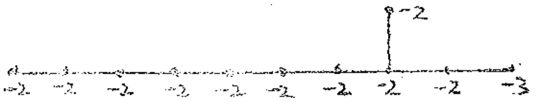
(76)



$Z = 1\ 2\ 3\ 4\ \overset{2}{5}\ 4\ 3\ 2\ 1$

$z^2 = x^5 + y^6$

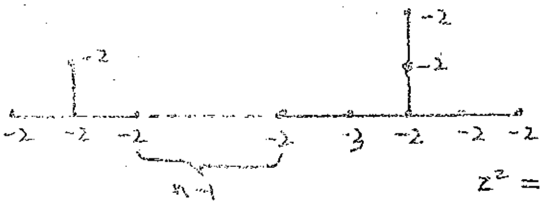
(77)



$Z = 1\ 2\ 3\ 4\ 5\ 6\ 7\ \overset{4}{8}\ 5\ 2$

$z^2 = x^5 + x y^5$

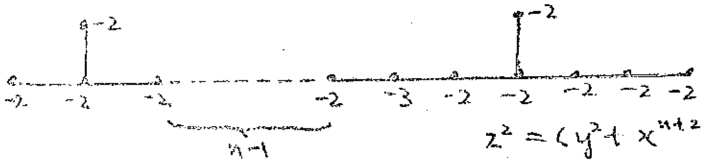
(78)



$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ \overset{1}{\overset{2}{3}}\ 2\ 1$

$z^2 = (y^2 + x^{n+2})(x^3 + y^4)$

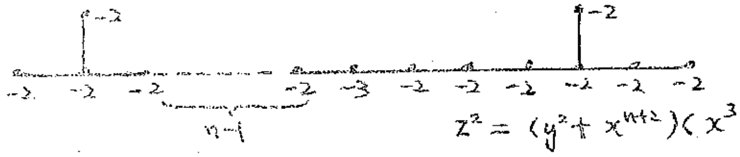
(79)



$z^2 = (y^2 + x^{n+2})(x^3 + x y^3)$

$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

(80)



$z^2 = (y^2 + x^{n+2})(x^3 + y^5)$

$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

Theorem 5.12: Let $\pi: M \to V$ be the minimal good resolution of a normal two dimensional Stein space with p as its only weakly elliptic double point. Then the associated weighted dual graph is one of the form shown in Corollary 5.6, Corollary 5.10 and Corollary 5.11. Moreover any such weighted dual graph has a weakly elliptic double point structure.

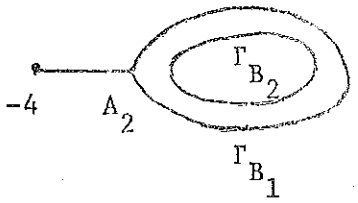
Theorem 5.12 give a complete topological classification of weakly elliptic double points because of the following fact. Suppose $p_o \in V_o$ and $p_1 \in V_1$ are isolated singularities of complex surfaces such that the graph of p_o is the same as the graph of p_1. Then there are open neighborhoods $U_o \ni p_o$ and $U_1 \ni p_1$ and a homeomorphism $h: U_o \to U_1$, such that $h(p_o) = p_1$. For the proof, see Remark 3.9 of [37].

§3 Topological Classification of Hypersurface Singularities with $h = \dim H^1(M, \mathcal{O}) = 2$.

Rational singularities have $H^1(M, \mathcal{O}) \cong 0$. The hypersurface rational singularities are actually double points. For $H^1(M, \mathcal{O}) \cong \mathbb{C}$, Laufer was able to list all weighted dual graphs of hypersurface singularities. In this section, we are going to list all possible weighted dual graphs of hypersurface singularities with $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$.

Proposition 5.13: Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z = Z_{B_o}, \ldots, Z_{B_\ell}, Z_E$ be the elliptic sequence. If $Z \cdot Z = -3$ and $Z_{B_1} \cdot Z_{B_1} = -1$, then Γ must be one of the following forms:

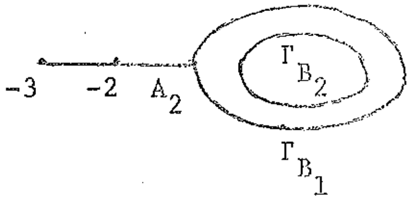
(1)



$Z = 1\ Z_{B_1}$ $\quad A_2 \subseteq B_1,\ A_2 \not\subseteq B_2$

$z_2 = 1,\ A_2 \cdot Z_{B_1} = -1$

(2)

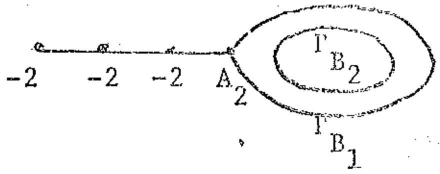


$Z = 1\ 2\ D$ $\quad |D| = B_1,$

$A_2 \subseteq B_1,\ A_2 \not\subseteq B_2$

$z_2 = 2,\ A_2 \cdot Z_{B_1} = 0$

(3)

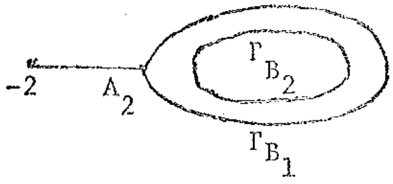


$Z = 1\ 2\ 3\ D$ $\quad |D| = B_1,$

$A_2 \subseteq B_1,\ A_2 \not\subseteq B_2$

$z_2 = 3,\ A_2 \cdot Z_{B_1} = 0$

(4)



$Z = 3\ D$ $\quad |D| = B_1$

$A_2 \subseteq B_1,\ A_2 \not\subseteq B_2$

$z_2 = 5,\ A_2 \cdot Z_{B_1} = 0.$

where Γ_{B_i} is the graph of B_i.

<u>Proof</u>: Since $Z \cdot Z = -3$, by the definition of the elliptic sequence, we have the following three cases:

(I) There are only three distinct $A_1, A_2, A_3 \nsubseteq B_1$ such that $A_i \cap B_1 \neq \phi$, $1 \leq i \leq 3$.

(II) There are only $A_1, A_2 \nsubseteq B_1$ such that $A_1 \cap B_1 \neq \phi \neq A_2 \cap B_1$ and $A_1 \neq A_2$.

(III) There exists unique $A_1 \nsubseteq B_1$ such that $A_1 \cap B_1 \neq \phi$.

In the first case, we have $A_1 \cdot Z = -1 = A_2 \cdot Z = A_3 \cdot Z$ and $z_1 = z_2 = z_3 = 1$. Since $0 \geq A_i \cdot (-K') = A_i \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_i \cdot (Z + Z_{B_1}) = 0$ for $i = 1, 2, 3$, we have $A_i \cap B_2 = \phi$ and $A_i \cdot A_i = -2$ for $i = 1, 2, 3$. Let $A_4, A_5, A_6 \subseteq B_1$ such that $A_1 \cdot A_4 = A_2 \cdot A_5 = A_3 \cdot A_6 = 1$. Then $z_4 = z_5 = z_6 = 1$. Hence $A_4 \cdot Z_{B_1} < 0$, $A_5 \cdot Z_{B_1} < 0$, $A_6 \cdot Z_{B_1} < 0$. Since $Z_{B_1} \cdot Z_{B_1} = -1$, we have $A_4 = A_5 = A_6$ and $A_4 \cdot Z_{B_1} = -1$. However, $z_4 = 1$ will imply that $Z = A_1 + A_2 + A_3 + Z_{B_1}$ and $A_4 \cdot Z = 2 > 0$. This is a contradiction.

In the second case, there are two subcases.

(IIA) $A_1 \cdot Z = -1 = A_2 \cdot Z$. Since $0 \geq A_i \cdot (-K') = A_i \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_i \cdot (Z + Z_{B_1}) = 0$, we have $A_i \cap B_2 = \phi$ and $A_i \cdot A_i = -2$ for $i = 1, 2$. We claim that there is no $A_3 \nsubseteq B_1$ such that $A_1 \neq A_3 \neq A_2$ and $A_3 \cdot Z < 0$. Otherwise $A_3 \cdot Z = -1$ and $z_1 = z_2 = z_3 = 1$. By our hypothesis, for any $A_i \nsubseteq B_1$, $A_1 \neq A_i \neq A_2$, we have $A_i \cap B_1 = \phi$. Since A is connected, there exists $A_j \nsubseteq B_1$, $A_1 \neq A_j \neq A_2$ such that $A_j \cdot A_1 = 1$ or $A_j \cdot A_2 = 1$.

It follows that either $A_1\cdot Z \geq 0$ or $A_2\cdot Z \geq 0$. This is a contradiction. Without lose of generality, we may assume that $z_1 = 1$, $z_2 = 2$. There is no $A_i \nsubseteq B_1$ such that $A_i\cdot A_1 = 1$. For $A_j \nsubseteq B_1$, $A_2 \neq A_j \neq A_1$, we have $A_j\cdot Z = 0 = A_j\cdot Z_{B_1}$. So $A_j\cdot(-K') = A_j\cdot(\sum_{i=0}^{\ell} Z_{B_i} + E) = 0$ and $A_j\cdot A_j = -2$. Let $A_3, A_4 \subseteq B_1$ such that $A_1\cdot A_3 = 1 = A_2\cdot A_4$. Then $z_3 = 1$ and $A_3\cdot Z_{B_1} < 0$. Since $Z_{B_1}\cdot Z_{B_1} = -1$, we have $A_3\cdot Z_{B_1} = -1$. If $A_3 = A_4$, then $Z = D + Z_{B_1}$ where $|D|$ consists of those A_i which are not in B_1. Hence $A_3\cdot Z_{B_1} = -3$. This is a contradiction.

We conclude that $A_3 \neq A_4$. z_4 cannot equal to one, otherwise $A_4\cdot Z_{B_1} = -2$, which is absurd. Therefore $z_4 \geq 2$. For any $A_i \subseteq B_1$, $A_i \neq A_3$, we have $A_i\cdot Z_{B_1} = 0$. Since B_1 is connected, there exists $A_5 \subseteq B_1$ such that $z_5 \geq {}_{B_1}z_5 + 1$ and $A_5\cdot A_3 = 1$. However, $z_3 = 1$ and $A_3\cdot Z_{B_1} = -1$ will imply that $A_3\cdot Z \geq 1$, which is absurd.

(IIB) $A_1\cdot Z = -1$ and $A_2\cdot Z = -2$. In this subcase, we have $z_1 = 1 = z_2$. Since $0 \geq A_1\cdot(-K') = A_1\cdot(\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_1\cdot(Z + Z_{B_1}) = 0$, $A_1\cdot A_1 = -2$ and $A_1 \cap B_2 = \phi$. Also $0 \geq A_2\cdot(-K') = A_2\cdot(\sum_{i=0}^{\ell} Z_{B_i} + E)$ $\geq A_2\cdot(Z + Z_{B_1}) = -2 + 1 = -1$. Either $A_2\cdot A_2 = -2$ and $A_2 \cap B_2 \neq \phi$ or $A_2\cdot A_2 = -3$, $A_2 \cap B_2 = \phi$. $A_2\cdot A_2 = -2$ and $A_2 \cap B_2 \neq \phi$ cannot occur otherwise $A_2\cdot Z \geq -1$ which contradicts to our assumption $A_2\cdot Z = -2$. Therefore $A_2\cdot A_2 = -3$ and $A_2 \cap B_2 = \phi$. Let $A_3, A_4 \subsetneq B_1$ such that $A_1\cdot A_3 = 1$, $A_2\cdot A_4 = 1$. Then $A = A_1 \cup A_2 \cup B_1$ and

$z_3 = 1 = z_4$. Moreover $Z = A_1+A_2+Z_{B_1}$ and $A_3 \cdot Z_{B_1} < 0$ and $A_4 \cdot Z_{B_1} < 0$. If $A_3 \neq A_4$ then $Z_{B_1} \cdot Z_{B_1} \lneq -2$, which is absurd. If $A_3 = A_4$, then $A_3 \cdot Z_{B_1} = -1$, since $Z_{B_1} \cdot Z_{B_1} = -1$. Hence $A_3 \cdot Z = A_3 \cdot (A_1+A_2+Z_{B_1}) = 2 - 1 = 1$. This is again a contradiction.

In the third case, there are three subcases.

(IIIA) $A_1 \cdot Z = -3$. In this case, $z_1 = 1$, $0 \geq A_1 \cdot (-K') = A_1 \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_1 \cdot (Z+Z_{B_1}) = -3+1 = -2$. Either (i) $A_1 \cdot A_1 = -3$, $A_1 \cap B_2 \neq \phi$ and $A_1 \cap B_3 = \phi$ or (ii) $A_1 \cdot A_1 = -4$ and $A_1 \cap B_2 = \phi$ or (iii) $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 \neq \phi \neq A_1 \cap B_3$. If (i) holds, then $A_1 \cdot Z \geq -2$ since $z_1 = 1$. This is a contradiction. If (iii) holds, then $A_1 \cdot Z \geq -1$. This is also impossible. Suppose $A_1 \cdot A_1 = -4$ and $A_1 \cap B_2 = \phi$. Let $A_2 \subsetneq B_1$ such that $A_1 \cdot A_2 = 1$. Since $A_1 \cdot Z = -3$, we have $z_2 = 1$ and $A = A_1 \cup B_1$. Moreover, $Z = A_1 + Z_{B_1}$ and hence $A_2 \cdot Z_{B_1} = 1$. So we are in (1).

(IIIB) $A_1 \cdot Z = -2$. In this case, we have $z_1 = 1$. Otherwise $z_1 \geq 2$ would imply that $Z \cdot Z \leq -4$ which is absurd. $0 \geq A_1 \cdot (-K') = A_1 \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_1 \cdot (Z + Z_{B_1}) = -2+1 = -1$. Either (i) $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 \neq \phi$ or (ii) $A_1 \cdot A_1 = -3$ and $A_1 \cap B_2 = \phi$. If (i) holds, then $A_1 \cdot Z \geq -1$ since $z_1 = 1$. This is a contradiction. Suppose $A_1 \cdot A_1 = -3$ and $A_1 \cap B_2 = \phi$. Since $A_1 \cdot Z = -2$, and $z_1 = 1$, there is no $A_i \not\subsetneq B_1$ such that $A_i \cdot A_1 = 1$. It follows that $A = A_1 \cup B_1$ and $Z = A_1 + Z_{B_1}$. But then $Z \cdot Z = (A_1+Z_{B_1}) \cdot Z = A_1 \cdot Z = -2$. This is a contradiction.

(IIIC) $A_1\cdot Z = -1$. Then $0 \geq A_1\cdot(-K') = A_1\cdot(\sum_{i=0}^{\ell} Z_{B_i}+E) \geq A_1\cdot(Z+Z_{B_1}) = 0$. So $A_1\cdot A_1 = -2$ and $A_1 \cap B_2 = \phi$. z_1 cannot equal to 1. Otherwise $A = A_1 \cup B_1$ and $Z = A_1 + Z_{B_1}$ which implies that $Z\cdot Z = (A_1+Z_{B_1})\cdot Z = A_1\cdot Z = -1$. This is a contradiction. Therefore either $z_1 = 2$ or $z_1 = 3$. Let $A_2 \subseteq B_1$ such that $A_2\cdot A_1 = 1$, $A_2 \not\subseteq B_2$.

(IIIC) $z_1 = 2$. Let $A_3 \not\subseteq B_1$, $A_3 \neq A_1$ such that $A_3\cdot Z < 0$. Then $A_3\cdot Z = -1$ and $z_3 = 1$. Since $2 + A_3\cdot A_3 = A_3\cdot(-K') = A_3\cdot(\sum_{i=0}^{\ell} Z_{B_i}+E) = A_3\cdot Z = -1$, $A_3\cdot A_3 = -3$. For any $A_i \not\subseteq B_1$, $A_3 \neq A_i \neq A_1$, we have $A_i\cdot Z = 0$ and $A_i \cap B_1 = \phi$. Hence, $A_i\cdot(-K') = A_i(\sum_{i=0}^{\ell} Z_{B_i}+E) = A_i\cdot Z = 0$ and $A_i\cdot A_i = -2$. There are four subcases.

(IIIC α i) $z_2 = 1$.

In this case $Z/B_1 = Z_{B_1}$. Therefore $A_2\cdot Z = 2A_2\cdot A_1+A_2\cdot Z_{B_1} = 2-1 = 1$. This is impossible.

(IIIC α ii) $z_2 = 2$

Since $z_1 = 2$ and $A_1\cdot A_1 = -2$, there exists $A_3 \not\subseteq B_1$, $A_3\cdot A_1 = 1$ and $z_3 = 1$. $A_3\cdot A_3$ is either -2 or -3. If $A_3\cdot A_3 = -2$, then $A = B_1 \cup A_1 \cup A_3$ and $Z = A_3 + 2A_1 + D$ where D is a positive cycle with $|D| = B_1$. Then $Z\cdot Z = (A_3+2A_1+D)\cdot Z = 2A_1\cdot Z = -2$. This is a contradiction. So $A_3\cdot A_3 = -3$ and we are in (2)

(IIIC α iii) $z_2 = 3$

Then $A = A_1 \cup B_1$ and $Z = 2A_1+D$ where D is a positive cycle with $|D| = B_1$. It follows that $Z \cdot Z = 2A_1 \cdot Z = -2$. This is a contradiction.

(IIIC α iv) $z_2 \geq 4$

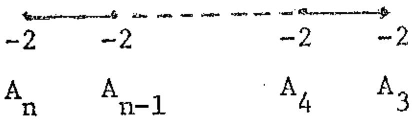
Then $A_1 \cdot Z \geq 0$. This is impossible by our hypothesis.

(IIICβ) $z_1 = 3$. Since $Z \cdot Z = -3$ and $A_1 \cdot Z = -1$, $A_i \cdot Z = 0$ for any $A_i \not\subseteq B_1$, $A_i \neq A_1$. Moreover $0 \geq A_i \cdot (-K') = A_i \cdot (\sum_{j=0}^{\ell} Z_{B_j} + E) = A_i \cdot Z = 0$. Hence $A_i \cdot A_i = -2$.

(IIICβi) · $z_2 = 1$

Then $Z/B_1 = Z_{B_1}$. Since $z_1 = 3$, we have $A_2 \cdot Z \geq 3A_1 \cdot A_2 + A_2 \cdot Z_{B_1} = 3-1 = 2$. This is a contradiction.

(IIICβii) $z_2 = 2$. Let Γ_1 be the subgraph of Γ consisting of those $A_i \not\subseteq B_1$, $A_i \neq A_1$. Since $z_1 = 3$, $z_2 = 2$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, we have $\deg A_1 = 2$. As $A_i \cdot A_i = -2$ for all A_i in Γ_1, Γ_1 is a graph of rational double point. There exists a unique $A_3 \subseteq \Gamma_1$ such that $z_3 = 3$. Because $A_3 \cdot Z = 0$ and $z_1 = 3$, $\deg A_3 = 2$. There exists a unique $A_4 \subseteq \Gamma_1$ such that $z_4 = 3$ and $A_4 \cdot A_3 = 1$. By induction Γ_1 is of the following form



$Z = 3A_n + 3A_{n-1} + \ldots + 3A_4 + 3A_3 + 3A_1 + D$ where D is a positive cycle with $|D| = B_1$. Then $A_n \cdot Z = -3$ and $Z \cdot Z < -3$. This is a contradiction.

(IIICβiii) $z_2 = 3$. It is easy to see that $\deg A_1 = 2$. Hence we are in (3).

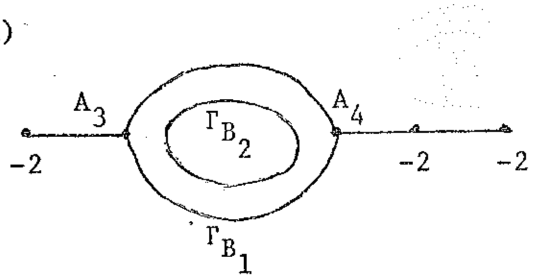
(IIICβiv) $z_2 = 4$. Since $z_1 = 3$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, there exists a unique $A_3 \not\subseteq B_1$ such that $A_3 \cdot A_1 = 1$ and $z_3 = 1$. Then $A_3 \cdot Z \geq 1$. This is a contradiction.

(IIICβv) $z_2 = 5$, we are in (5).

(IIICβvi) $z_2 \geq 6$. In this case, $A_1 \cdot Z \geq 0$. This is a contradiction.

<u>Proposition 5.14</u>: Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists and $Z = Z_{B_0}, \ldots, Z_{B_\ell}, Z_E$ be the elliptic sequence. If $Z \cdot Z = -3$ and $Z_{B_1} \cdot Z_{B_1} = -2$, then Γ must be one of the following forms:

(1)



Z = 1 D 2 1 $z_3 = 1$, $z_4 = 2$

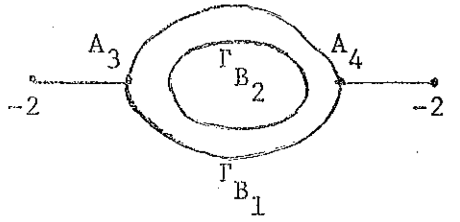
D is a positive cycle, $|D| = B_1$,

$A_3 \cdot Z_{B_1} < 0$

$A_4 \cdot Z_{B_1} = 0$, $A_3, A_4 \not\subseteq B_2$,

$A_3 \neq A_4$.

(2)



$Z = 1\ D\ 2 \qquad z_3 = 1,\ z_4 = 3$

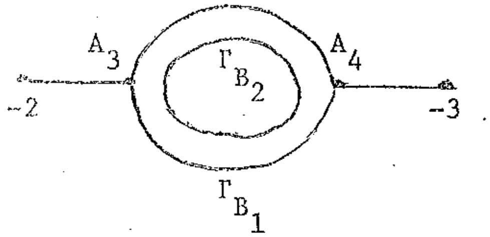
D is a positive cycle, $|D| = B_1$,

$A_3 \cdot Z_{B_1} < 0$

$A_4 \cdot Z_{B_1} = 0,\quad A_3, A_4 \subseteq B_1,$

$A_3, A_4 \nsubseteq B_2,\ A_3 \neq A_4$

(3)

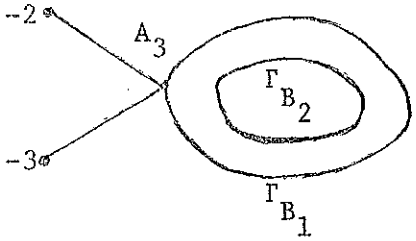


$Z = 1\ Z_{B_1}\ 1 \qquad z_3 = z_4 = 1,$

$A_3 \cdot Z_{B_1} = -1 = A_4 \cdot Z_{B_1}$

$A_3, A_4 \subseteq B_1,\quad A_3, A_4 \nsubseteq B_2,\quad A_3 \neq A_4$

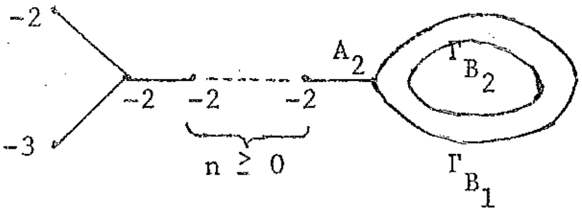
(4)



$Z = \begin{matrix} 1 \\ 1 \end{matrix} Z_{B_1} \qquad z_3 = 1,\quad A_3 \cdot Z_{B_1} = -2$

$A_3 \subseteq B_1,\quad A_3 \nsubseteq B_2.$

(5)

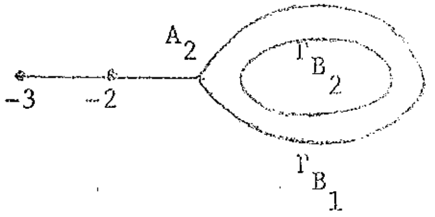


$Z = \begin{matrix} 1 \\ 1 \end{matrix} 2\quad 2 \ldots 2\ Z_{B_1} \qquad z_2 = 1$

$A_2 \cdot Z_{B_1} = -2$

$A_2 \subseteq B_1,\quad A_2 \nsubseteq B_2$

(6)

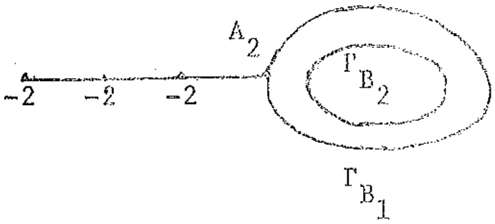


$Z = 1\ 2\ D$ D is a positive cycle $|D| = B_1$

$z_2 = 2$, $A_2 \subseteq B_1$, $A_2 \nsubseteq B_2$

$A_2 \cdot Z_{B_1} = 0$

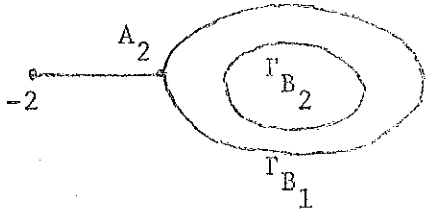
(7)



$Z = 1\ 2\ 3\ D$ D is a positive cycle, $|D| = B_1$

$z_2=3$, $A_2 \cdot Z_{B_1} = 0$, $A_2 \subseteq B_1$, $A_2 \nsubseteq B_2$

(8)



$Z = 3\ D$ D is a positive cycle $|D| = B_1$

$z_2=5$, $A_2 \cdot Z_{B_1} = 0$, $A_2 \subseteq B_1$, $A_2 \nsubseteq B_2$

where Γ_{B_i} is the graph of B_i.

<u>Proof</u>: Since $Z \cdot Z = -3$, by the definition of elliptic sequence, we have the following three cases:

(I) There are only three distinct $A_1, A_2, A_3 \nsubseteq B_1$ such that $A_i \cap B_1 \neq \phi$, $1 \leq i \leq 3$.

(II) There are only two distinct $A_1, A_2 \nsubseteq B_1$ such that $A_i \cap B_1 \neq \phi$ $1 \leq i \leq 2$.

(III) There exists a unique $A_1 \not\subseteq B_1$ such that $A_1 \cap B_1 \neq \phi$.

In the first case, we have $A_1 \cdot Z = -1 = A_2 \cdot Z = A_3 \cdot Z$ and $z_1 = z_2 = z_3 = 1$, $0 \geq A_i \cdot (-K') = A_i(\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_i(Z + Z_{B_1}) = 0$ for $i = 1,2,3$. So $A_i \cdot A_i = -2$ and $A_i \cap B_2 = \phi$, $1 \leq i \leq 3$. Let $A_4, A_5, A_6 \subseteq B_1$ such that $A_1 \cdot A_4 = 1 = A_2 \cdot A_5 = A_3 \cdot A_6$. Then $z_4 = z_5 = z_6 = 1$ and $Z = A_1 + A_2 + A_3 + Z_{B_1}$. Hence $A_4 \cdot Z_{B_1} < 0$, $A_5 \cdot Z_{B_1} < 0$, and $A_6 \cdot Z_{B_1} < 0$. If A_4, A_5, A_6 are distinct, then $Z_{B_1} \cdot Z_{B_1} \leq -3$. This is a contradiction. If $A_4 = A_5 \neq A_6$, then $A_4 \cdot Z_{B_1} = -2$ because $A_1 \cdot A_4 = A_2 \cdot A_4 = 1$ and $A_4 \cdot Z = 0$. Again we get $Z_{B_1} \cdot Z_{B_1} \leq -3$, which is absurd. If $A_4 = A_5 = A_6$, then $A_4 \cdot Z_{B_1} \leq -3$. In particular, $Z_{B_1} \cdot Z_{B_1} \leq -3$. This is absurd.

In the second case, there are two subcases.

(IIA) $A_1 \cdot Z = -1 = A_2 \cdot Z$. We claim that there are no $A_3 \not\subseteq B_1$ such that $A_1 \neq A_3 \neq A_2$ and $A_3 \cdot Z < 0$. Otherwise $A_1 \cdot Z = A_2 \cdot Z = A_3 \cdot Z = -1$ and $z_1 = z_2 = z_3 = 1$. By our hypothesis, for any $A_i \not\subseteq B_1$, $A_2 \neq A_i \neq A_1$, we have $A_i \cap B_1 = \phi$. Since A is connected, $\exists A_j \not\subseteq B_1$, $A_1 \neq A_j \neq A_2$ such that $A_j \cdot A_1 = 1$ or $A_j \cdot A_2 = 1$. As $0 \geq A_i \cdot (-K') = A_i(\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_i \cdot (Z + Z_{B_1}) = 0$ for $i = 1,2$, we have $A_1 \cdot A_1 = -2 = A_2 \cdot A_2$ and $A_1 \cap B_2 = \phi = A_2 \cap B_2$. It follows that either $A_1 \cdot Z \geq 0$ or $A_2 \cdot Z \geq 0$. This is a contradiction. Our claim is proved. Without lose of generality, we may assume that $z_1 = 1$, $z_2 = 2$. There is no $A_i \not\subseteq B_1$ such that $A_i \cdot A_1 = 1$. For any $A_j \not\subseteq B_1$, $A_2 \neq A_j \neq A_1$, we have $A_j\cdot Z = 0 = A_j\cdot Z_{B_1}$. So $A_j\cdot(-K') = A_j(\sum_{i=0}^{\ell} Z_{B_i}+E) = 0$ and $A_j\cdot A_j = -2$. Let $A_3, A_4 \subseteq B_1$ such that $A_4, A_3 \not\subseteq B_2$ and $A_1\cdot A_3 = 1 = A_2\cdot A_4$. Then $z_3 = 1$ and $A_3\cdot Z_{B_1} < 0$. If $A_3 = A_4$, then $Z/B_1 = Z_{B_1}$. So $A_3\cdot Z = 2A_1\cdot A_3 + A_2\cdot A_3 + A_3\cdot Z_{B_1} \geq 1$. This is a contradiction. We conclude that $A_3 \neq A_4$. z_4 cannot equal to 1. Otherwise $Z/B_1 = Z_{B_1}$ and $A_4\cdot Z_{B_1} = -2$. This would imply that $Z_{B_1}\cdot Z_{B_1} \leq -3$ which is absurd. Suppose $z_4 = 2$. Then there exists unique $A_5 \not\subseteq B_1$ such that $A_2 \neq A_5 \neq A_1$, $A_5\cdot A_2 = 1$ and $z_5 = 1$. It follows that $A = A_1 \cup A_2 \cup A_5 \cup B_1$. If $A_4\cdot Z_{B_1} < 0$, then $Z = A_1+A_2+A_5+Z_{B_1}$. This is a contradiction. So $A_4\cdot Z_{B_1} = 0$ and we are in (1). Suppose $z_4 = 3$ then $A = A_1 \cup A_2 \cup B_1$. Similar argument as above will show that $A_4\cdot Z_{B_1} = 0$ and we are in (2).

(IIB) $A_1\cdot Z = -1$ and $A_2\cdot Z = -2$. In this case, $z_1 = 1 = z_2$. $0 \geq A_1\cdot(-K') = A_1\cdot(\sum_{i=0}^{\ell} Z_{B_i}+E) \geq A_1\cdot(Z+Z_{B_1}) = 0$. Hence $A_1\cdot A_1 = -2$ and $A_1 \cap B_2 = \phi$. Since $0 \geq A_2\cdot(-K') = A_2\cdot(\sum_{i=0}^{\ell} Z_{B_i}+E) \geq A_2\cdot(Z+Z_{B_1}) = -1$, either $A_2\cdot A_2 = -2$ and $A_2 \cap B_2 \neq \phi$ or $A_2\cdot A_2 = -3$ and $A_2 \cap B_2 = \phi$. $A_2\cdot A_2$ cannot equal to -2 otherwise $A_2\cdot Z \geq -1$ which is a contradiction. Therefore $A_2\cdot A_2 = -3$ and $A_2 \cap B_2 = \phi$. Let $A_3, A_4 \subseteq B_1$, $A_3, A_4 \not\subseteq B_2$ such that $A_1\cdot A_3 = 1 = A_2\cdot A_4$. Then $A = A_1 \cup A_2 \cup B_1$ and $z_3 = z_4 = 1$. Moreover, $Z = A_1 + A_2 + Z_{B_1}$ and $A_3\cdot Z_{B_1} < 0$, $A_4\cdot Z_{B_1} < 0$. If $A_3 \neq A_4$, then $A_3\cdot Z_{B_1} = -1 = A_4\cdot Z_{B_1}$ and we are in (3). If $A_3 = A_4$, then

$A_3 \cdot Z_{B_1} = -2$ and we are in (4).

In the third case, there are three subcases.

(IIIA) $A_1 \cdot Z = -3$. In this case $z_1 = 1$. Since $0 \geq A_1 \cdot (-K') = A_1 (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_1 \cdot (Z + Z_{B_1}) = -3 + 1 = -2$, either $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 \neq \phi \neq A_1 \cap B_3$, or (ii) $A_1 \cdot A_1 = -3$, $A_1 \cap B_2 \neq \phi$ and $A_1 \cap B_3 = \phi$, or (iii) $A_1 \cdot A_1 = -4$ and $A_1 \cap B_2 = \phi$. In case (i) $A_1 \cdot Z \geq -1$, which is absurd. In case (ii), $A_1 \cdot Z \geq -2$, which is also absurd. In case (iii), it is easy to see that $A = A_1 \cup B_1$. Let $A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$ such that $A_1 \cdot A_2 = 1$. Then $z_2 = 1$ and hence $Z = A_1 + Z_{B_1}$. Since $A_2 \cdot Z = 0$, this implies that $A_2 \cdot Z_{B_1} = -1$. However, $Z_{B_1} \cdot Z_{B_1} = -2$, so there exists $A_3 \subseteq B_1$, $A_3 \neq A_2$ such that $A_3 \cdot Z_{B_1} = -1$. It follows that $A_3 \cdot Z = (A_1 + Z_{B_1}) \cdot A_3 = A_3 \cdot Z_{B_1} = -1 < 0$. This is a contradiction.

(IIIB) $A_1 \cdot Z = -2$. In this case, $z_1 = 1$. Otherwise $z_1 \geq 2$ and $Z \cdot Z \leq -4$. Since $0 \geq A_1 \cdot (-K') = A_1 \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_1 \cdot (Z + Z_{B_1}) = -2+1 = -1$, either (i) $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 \neq \phi$ or (ii) $A_1 \cdot A_1 = -3$ and $A_1 \cap B_2 = \phi$. If (i) holds, then $A_1 \cdot Z \geq 0$ which is absurd. Suppose (ii) holds. It follows easily that $A = A_1 \cup B_1$ and $Z = A_1 + Z_{B_1}$. Then $Z \cdot Z = (A_1 + Z_{B_1}) \cdot Z = A_1 \cdot Z = -2$. This is a contradiction.

(IIIC) $A_1 \cdot Z = -1$. Since $0 \geq A_1 \cdot (-K') = A_1 (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_1 \cdot (Z + Z_{B_1}) = 0$, we have $A_1 \cdot A_1 = -2$ and $A_1 \cap B_2 = \phi$. There are three subcases.

(IIICα) $z_1 = 1$. In this case, $A = A_1 \cup B_1$ and $Z = A_1 + Z_{B_1}$. Hence $Z \cdot Z = (A_1 + Z_{B_1}) \cdot Z = A_1 \cdot Z = -1$. This is a contradiction.

(IIICβ) $z_1 = 2$. Let $A_3 \nsubseteq B_1$, $A_3 \neq A_1$ such that $A_3 \cdot Z < 0$. Then $z_3 = 1$, $A_3 \cdot Z = -1$ and $A_3 \cap B_1 = \phi$. Since $A_3 \cdot (-K') = A_3 \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) = A_3 \cdot Z = -1$, we have $A_3 \cdot A_3 = -3$. For $A_i \nsubseteq B_1$, $A_3 \neq A_i \neq A_1$, we have $A_i \cdot Z = 0$ and $A_i \cap B_1 = \phi$. Because $A_i \cdot (-K') = A_i \cdot (\sum_{i=0}^{\ell} Z_{B_o} + E) = A_i \cdot Z = 0$, $A_i \cdot A_i = -2$. Let $A_2 \subseteq B_1$, $A_2 \nsubseteq B_2$ such that $A_1 \cdot A_2 = 1$.

(IIICβi) $z_2 = 1$. It is easy to see that we are in (5).

(IIICβii) $z_2 = 2$. It is easy to see that we are in (6).

(IIICβiii) $z_2 = 3$. In this case, $A = A_1 \cup B_1$ and $Z = 2A_1 + D$ where D is a positive cycle with $|D| = B_1$. It follows that $Z \cdot Z = (2A_1 + D) \cdot Z = 2A_1 \cdot Z = -2$. This is a contradiction.

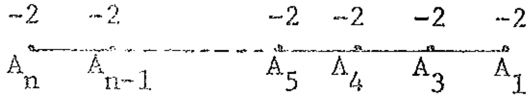
(IIICβiv) $z_2 \geq 4$. In this case, $A_1 \cdot Z \geq 0$, which is absurd.

(IIICγ) $z_1 = 3$. Since $Z \cdot Z = -3$ and $A_1 \cdot Z = -1$, $A_i \cdot Z = 0$ for any $A_i \nsubseteq B_1$, $A_i \neq A_1$. Moreover, $0 \geq A_i \cdot (-K') = A_i \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) \geq A_i \cdot (Z + Z_{B_1}) = 0$. Hence $A_i \cdot A_i = -2$ for all $A_i \neq A_1$

and $A_i \nsubseteq B_1$. Let $A_2 \subseteq B_1$ such that $A_2 \nsubseteq B_2$ and $A_1 \cdot A_2 = 1$.

(IIICγi) $z_2 = 1$. In this case $Z/B_1 = Z_{B_1}$. So $A_2 \cdot Z = 3A_1 \cdot A_2 + A_2 \cdot Z_{B_1} \geq 3-2 = 1$. This is a contradiction.

(IIICγii) $z_2 = 2$. Let Γ_1 be the subgraph of Γ consisting of those $A_i \nsubseteq B_1$. Since $A_i \cdot A_i = -2$ for all A_i in Γ_1, Γ_1 is a graph of rational double point. Since $z_1 = 3$, $z_2 = 2$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, it is easy to see that $\deg A_1 = 2$. Hence there exists a unique $A_3 \subseteq \Gamma_1$ such that $z_3 = 3$. Since $A_3 \cdot Z = 0$ and $z_1 = 3$, $\deg A_3 = 2$. There exists a unique $A_4 \subseteq \Gamma_1$ such that $z_4 = 3$, $A_4 \neq A_1$ and $A_4 \cdot A_3 = 1$. By induction Γ_1 is of the following form.



$Z = 3A_4 + \ldots + 3A_3 + 3A_1 + D$, where D is a positive cycle with $|D| = B_1$. Then $A_n \cdot Z = -3$ and $Z \cdot Z < -3$. This is a contradiction.

(IIICγiii) $z_2 = 3$. Since $z_1 = 3$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, we have $2 \leq \deg A_1 \leq 3$. It is not hard to see that $\deg A_1 = 3$ cannot occur. Therefore $\deg A_1 = 2$. It follows that we are in (7).

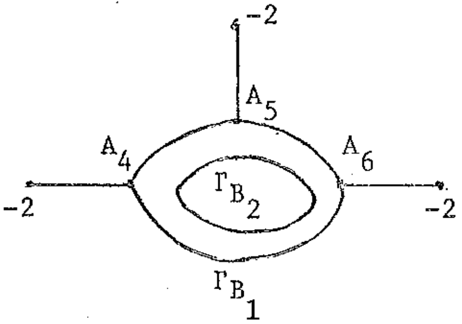
(IIICγiv) $z_2 = 4$. There exists a unique $A_3 \nsubseteq B_1$ such that $A_1 \cdot A_3 = 1$ and $z_3 = 1$. Therefore $A_3 \cdot Z \geq 3-2 = 1$. This is a contradiction.

(IIICγv) $z_2 = 5$. Since $z_1 = 3$, $z_2 = 5$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, we have $A = A_1 \cup B_1$. So we are in (8).

(IIICγvi) $z_2 \geq 6$. Then $A_1 \cdot Z \geq 0$, which is a contradiction.

Proposition 5.15: Let Γ be a weighted dual graph including genera for the vertices associated to the minimal good resolution of weakly elliptic singularity. Suppose K' exists. Let $Z = Z_{B_0}, \ldots, Z_{B_\ell}, Z_E$ be the elliptic sequence. If $Z \cdot Z \leq -3$ and $Z_{B_1} \cdot Z_{B_1} = -3$, then Γ must be one of the following forms.

(1)



$Z = 1 \overset{1}{Z_{B_1}} 1$

$z_4 = z_5 = z_6 = 1$

$A_4 \cdot Z_{B_1} = -1 = A_5 \cdot Z_{B_1}$

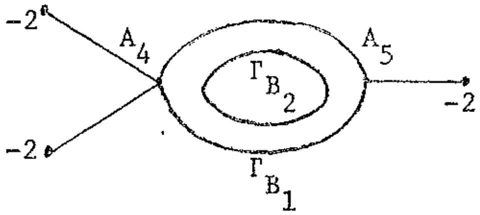
$= A_6 \cdot Z_{B_1}$

$A_4, A_5, A_6 \subseteq B_1$,

$A_4, A_5, A_6 \nsubseteq B_2$,

$A_4 \neq A_5 \neq A_6 \neq A_4$

(2)



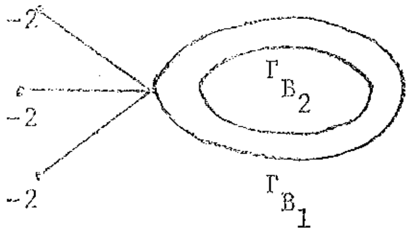
$Z = \begin{smallmatrix}1\\1\end{smallmatrix} Z_{B_1}\ 1 \qquad z_4 = z_5 = 1$

$A_4 \cdot Z_{B_1} = -2, \quad A_5 \cdot Z_{B_1} = -1$

$A_4, A_5 \subseteq B_1, \quad A_4, A_5 \nsubseteq B_2$,

$A_4 \neq A_5$

(3)

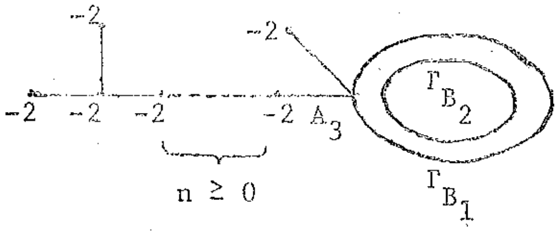


$$Z = \begin{matrix}1\\1\\1\end{matrix}\ Z_{B_1}$$

$z_4 = 1 \qquad A_4\cdot Z_{B_1} = -3$

$A_4 \subseteq B_1, \quad A_4 \not\subseteq B_2$

(4)

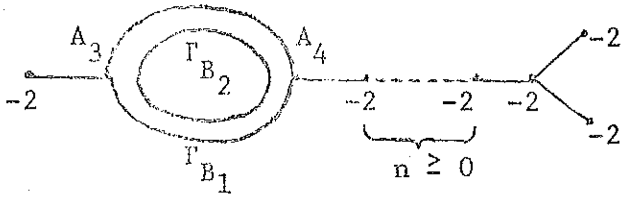


$$Z = 1\ \overset{1}{2}\ 2\ldots2\ {}^{1}\ Z_{B_1}$$

$z_3 = 1, \qquad A_3\cdot Z_{B_1} = -3$

$A_3 \subseteq B_1, \qquad A_3 \not\subseteq B_2$

(5)



$$Z = 1\ Z_{B_1}\ 2\ldots\ldots2\ 2\ \begin{matrix}1\\1\end{matrix}$$

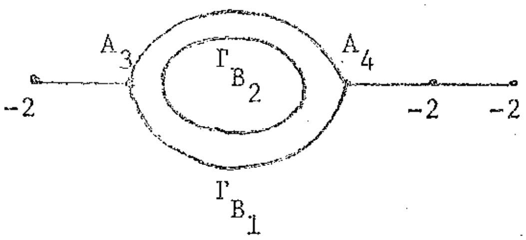
$z_3 = z_4 = 1, \quad A_3\cdot Z_{B_1} = -1$

$A_4\cdot Z_{B_1} = -2$

$A_3,\ A_4 \subseteq B_1, \quad A_3, A_4 \not\subseteq B_2,$

$A_3 \neq A_4$

(6)

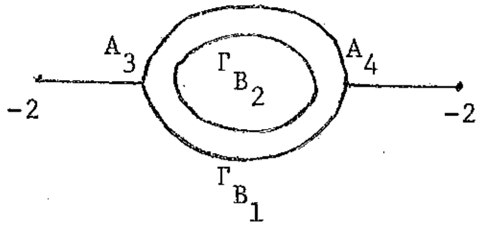


$Z = 1\ \ D\ \ 2\ \ 1$. D is a positive cycle, $|D| = B_1$

$z_3=1, \quad z_4=2, \quad A_3\cdot Z_{B_1}<0, \ A_4\cdot Z_{B_1}=0$

$A_3,A_4 \subseteq B_1, \quad A_3,A_4 \not\subseteq B_2, \quad A_3 \neq A_4$

(7)



$Z = 1 \;\; D \;\; 2$ D is a positive cycle $|D| = B_1$

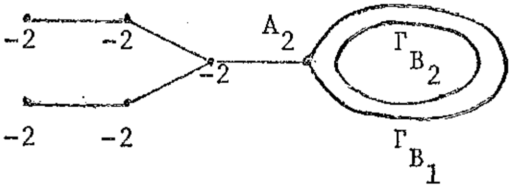
$z_3=1$, $z_4=3$, $A_3 \cdot Z_{B_1} < 0$,

$A_4 \cdot Z_{B_1} = 0$

$A_3, A_4 \subseteq B_1$, $A_3, A_4 \not\subseteq B_2$,

$A_3 \neq A_4$

(8)



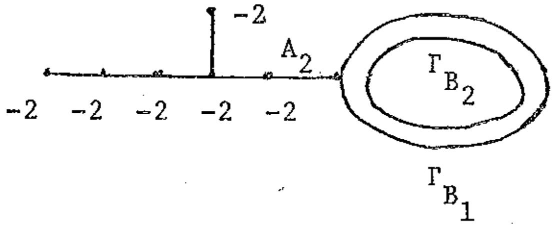
$Z = \begin{matrix}1\\1\end{matrix} \;\; \begin{matrix}2\\2\end{matrix} \;\; 3 \;\; Z_{B_1}$

$z_3 = 1.$

$A_2 \cdot Z_{B_1} = -3$, $A_2 \subseteq B_1$

$A_2 \not\subseteq B_2$

(9)

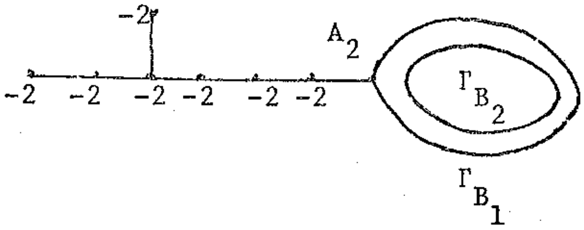


$Z = 1 \;\; 2 \;\; 3 \;\; \overset{2}{4} \;\; 3 \;\; Z_{B_1}$

$z_2 = 1$, $A_2 \cdot Z_{B_1} = -3$

$A_2 \subseteq B_1$, $A_2 \not\subseteq B_2$

(10)



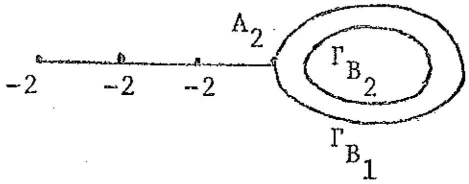
$Z = 2 \;\; 4 \;\; \overset{3}{6} \;\; 5 \;\; 4 \;\; 3 \;\; Z_{B_1}$

$z_2 = 1$, $A_2 \cdot Z_{B_1} = -3$,

$A_2 \subseteq B_1$

$A_2 \not\subseteq B_2$

(11)

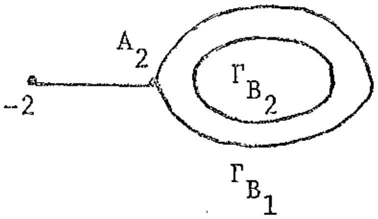


$Z = 1\ \ 2\ \ 3\ \ D$, D is a positive cycle $|D| = B_1$

$z_2 = 3$, $A_2\cdot Z_{B_1} = 0$,

$A_2 \subsetneq B_1$, $A_2 \not\subseteq B_2$

(12)



$Z = 3\ \ D$, D is a positive cycle $|D| = B_1$

$z_2 = 5$, $A_2\cdot Z_{B_1} = 0$,

$A_2 \subsetneq B_1$, $A_2 \not\subseteq B_2$

where Γ_{B_i} is the graph of B_i.

<u>Proof</u>: Since $Z\cdot Z = Z_{B_1}\cdot Z_{B_1}$, $(Z-Z_{B_1})\cdot(Z+Z_{B_1}) = 0$. For all $A_i \not\subseteq B_1$, we have $0 \geq A_i\cdot(-K') = A_i\cdot(\sum_{i=0}^{\ell} Z_{B_i}+E) \geq A_i\cdot(Z+Z_{B_1}) = 0$. Therefore, $A_i\cdot A_i = -2$ and $A_i \cap B_2 = \phi$ for all $A_i \not\subseteq B_1$. As $Z\cdot Z = -3$, by the definition of elliptic sequence, we have the following three cases.

(I) There exist $A_1, A_2, A_3 \not\subseteq B_1$ such that $A_i \cap B_1 \neq \phi$, $1 \leq i \leq 3$ and A_1, A_2, A_3 are distinct.

(II) There are only $A_1, A_2 \not\subseteq B_1$, $A_1 \neq A_2$ such that $A_1 \cap B_1 \neq \phi \neq A_2 \cap B_1$.

(III) There exists unique $A_1 \not\subseteq B_1$ such that $A_1 \cap B_1 \neq \phi$.

In the first case, we have $A_1\cdot Z = -1 = A_2\cdot Z = A_3\cdot Z$ and $z_1 = z_2 = z_3 = 1$. It follows that there is no $A_i \not\subseteq B_1$, $A_i \neq A_j$, $1 \leq j \leq 3$ such that $A_i\cdot A_1 > 0$ or $A_i\cdot A_2 > 0$ or $A_i\cdot A_3 > 0$, i.e., $A = A_1 \cup A_2 \cup A_3 \cup B_1$. Let A_4, A_5, $A_6 \subseteq B_1$ such that A_4, A_5, $A_6 \not\subseteq B_2$ and $A_1\cdot A_4 = 1 = A_2\cdot A_5 = A_3\cdot A_6$. Then $z_4 = z_5 = z_6$ and $A_4\cdot Z < 0$, $A_5\cdot Z < 0$, $A_6\cdot Z < 0$. If A_4, A_5, A_6 are distinct, then $Z = A_1 + A_2 + A_3 + Z_{B_1}$ and we are in (1). If $A_4 = A_5 \neq A_6$, then $A_4\cdot Z_{B_1} = -2$ because $A_4\cdot Z = 0$ and $A_1\cdot A_4 = A_2\cdot A_4 = 1$. Hence $A_6\cdot Z_{B_1} = -1$. We are in (2). Suppose $A_4 = A_5 = A_6$. Since $A_4\cdot Z = 0$, we have $A_4\cdot Z_{B_1} = -3$. We are in (3).

In the second case, we claim that there is no $A_3 \not\subseteq B_1$ such that $A_1 \neq A_3 \neq A_2$ and $A_3\cdot Z < 0$. Otherwise $A_1\cdot Z = -1 = A_2\cdot Z = A_3\cdot Z$ and $z_1 = z_2 = z_3 = 1$. By our hypothesis, $A_i \cap B_1 = \phi$ for any $A_i \not\subseteq B_1$, $A_1 \neq A_i \neq A_2$. Since A is connected, there exists $A_j \not\subseteq B_1$, $A_2 \neq A_j \neq A_1$ such that $A_j\cdot A_1 = 1$ or $A_j\cdot A_2 = 1$. Consequently, either $A_1\cdot Z \geq 0$, or $A_2\cdot Z>0$. This is a contradiction. Our claim is proved. There are two subcases:

(IIA) $A_1\cdot Z = -1$, $A_2\cdot Z = -2$. Since $Z\cdot Z = -3$, we conclude that $z_1 = 1$, $z_2 = 1$, However, $A_2\cdot A_2 = -2$, so $A_2\cdot Z \geq -1$. This is a contradiction.

(IIB) $A_1\cdot Z = -1 = A_2\cdot Z$. Without lose of generality, we may assume that $z_1 = 1$, $z_2 = 2$. Let A_3, $A_4 \subseteq B_1$, A_3, $A_4 \not\subseteq B_2$ such that $A_1\cdot A_3 = 1 = A_2\cdot A_4$. Since $z_1 = 1$ and $A_1\cdot A_1 = -2$, we have $z_3 = 1$

and $A_3 \cdot Z_{B_1} < 0$. If $A_3 = A_4$, then $A_3 \cdot Z_{B_1} = -3$ because $A_3 \cdot Z = A_3(2A_2 + A_1 + Z_{B_1}) = 0$. As $A_2 \cdot A_2 = -2$ and $A_2 \cdot Z = -1$, we have $2 \leq \deg A_2 \leq 3$. We are in (4). Suppose $A_3 \neq A_4$. Because $A_2 \cdot Z = -1$, the proof breaks up into four subcases.

(IIBi) There exist $A_5, A_6 \not\subseteq B_1$, $A_1 \neq A_5 \neq A_6 \neq A_1$ such that $A_5 \cdot A_2 = A_6 \cdot A_2 = 1$. In this case, we have $z_4 = z_5 = z_6 = 1$. Hence $Z = A_1 + A_5 + A_6 + 2A_2 + Z_{B_1}$. $A_4 \cdot Z = 0$, $A_3 \cdot Z = 0$ imply that $A_4 \cdot Z_{B_1} = -2$. $A_3 \cdot Z_{B_1} = -1$. Then we are in (5).

(IIBii) There exists a unique $A_5 \not\subseteq B_1$, $A_1 \neq A_5 \neq A_2$ such that $A_5 \cdot A_2 = 1$, $z_4 = 1$ and $z_5 = 2$. In this case, $Z/B_1 = Z_{B_1}$. So $A_4 \cdot Z_{B_1} = -2$ and $A_3 \cdot Z_{B_1} = -1$. Since $A_5 \cdot A_5 = -2$, $A_5 \cdot Z = 0$ and $z_5 = z_2 = 2$, we have $2 \leq \deg A_5 \leq 3$. It follows easily that we are in (5).

(IIBiii) There exists a unique $A_5 \not\subseteq B_1$, $A_1 \neq A_5 \neq A_2$ such that $A_5 \cdot A_2 = 1$, $z_5 = 1$ and $z_4 = 2$. In this case $A = A_1 \cup A_2 \cup A_5 \cup B_1$. Since $z_4 > 1$, it is easy to see that $A_4 \cdot Z_{B_1} = 0$. We are in (6).

(IIBiv) $z_4 = 3$. In this case, $A = A_1 \cup A_2 \cup B_1$. Hence $A_4 \cdot Z_{B_1} = 0$. We are in (7).

In the third case, there are three subcases.

(IIIA) There exist $A_2, A_3 \not\subseteq B_1$ such that A_1, A_2, A_3 are distinct and $A_2 \cdot Z < 0$, $A_3 \cdot Z < 0$. Because $Z \cdot Z = -3$, we have $A_1 \cdot Z = A_2 \cdot Z = A_3 \cdot Z = -1$ and $z_1 = z_2 = z_3 = 1$. There exists

$A_i \nsubseteq B_1$ such that $A_i \cdot A_1 = 1$. Since $z_1 = 1$ and $A_1 \cdot A_1 = -2$, we have $A_1 \cdot Z \geq 0$. This is a contradiction.

(IIIB) There exists a unique $A_2 \nsubseteq B_1$ such that $A_2 \neq A_1$ and $A_2 \cdot Z < 0$. Since $Z \cdot Z = -3$, we have three subcases.

(IIIBα) $z_1 = 2$, $z_2 = 1$. In this case, we have $A_1 \cdot Z = -1 = A_2 \cdot Z$. Let $A_3 \subseteq B_1$, $A_3 \nsubseteq B_2$ such that $A_1 \cdot A_3 = 1$. If $z_3 = 2$, then there exists a unique $A_4 \nsubseteq B_1$, such that $A_4 \cdot A_1 = 1$ and $z_4 = 1$. It follows easily that $A = A_1 \cup A_4 \cup B_1$. Since $z_4 = 1$, $z_1 = 2$ and $A_4 \cdot A_4 = -2$, we have $A_4 \cdot Z = 0$. This implies that $A_4 \neq A_2$ which is absurd. It $z_3 = 1$, then $Z/B_1 = Z_{B_1}$. Since $0 = A_3 \cdot Z = A_3 \cdot (2A_1 + Z_{B_1}) = 2 + A_3 \cdot Z_{B_1}$, we have $A_3 \cdot Z_{B_1} = -2$. As $Z_{B_1} \cdot Z_{B_1} = -3$, there exists $A_i \nsubseteq B_1$ such that $A_i \cdot Z_{B_1} = -1$. It follows that $A_i \cdot Z = A_i \cdot Z_{B_1} = -1 < 0$. This is a contradiction.

(IIIBβ) $z_1 = 1$, $z_2 = 2$. In this case, there exists $A_i \nsubseteq B_1$, $A_i \cdot A_1 = 1$. Since $z_1 = 1$, $A_1 \cdot A_1 = -2$, we have $A_1 \cdot Z \geq 0$. This is a contradiction.

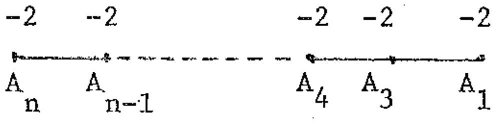
(IIIBγ) $z_1 = 1 = z_2$. The similar argument as (IIIBβ) shows that this case cannot occur.

(IIIC) There is no $A_i \nsubseteq B_1$, $A_i \neq A_1$ such that $A_i \cdot Z < 0$. In this case, $A_1 \cdot Z = -1$ and $z_1 = 3$. Otherwise, $A_1 \cdot Z \leq -2$ and $A_1 \cdot (-K') = A_1 \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) = A_1 \cdot (Z + Z_{B_1}) \leq -1$.

This would imply that $A_1 \cdot A_1 \leq -3$ which is a contradiction. Let $A_2 \subseteq B_1$ such that $A_1 \cdot A_2 = 1$ and $A_2 \nsubseteq B_2$. There are five subcases.

(IICi) $z_2 = 1$. Let Γ_1 be the subgraph of Γ consisting of those $A_i \nsubseteq B_1$. Since $A_i \cdot A_i = -2$ for all A_i in Γ_1, Γ_1 is a graph of rational double point. Since $z_1 = 3$, $z_2 = 1$ and $A_1 \cdot Z = -1$, it is not hard to check that we are in (8), (9) or (10).

(IICii) $z_2 = 2$. Let Γ_1 be the subgraph of Γ consisting of those $A_i \nsubseteq B_1$. Since $A_i \cdot A_i = -2$ for all A_i in Γ_1, Γ_1 is a graph of rational double point. As $z_1 = 3$, $z_2 = 2$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, we have $\deg A_1 = 2$. There exists a unique $A_3 \subseteq \Gamma_1$ such that $z_3 = 3$. Since $A_3 \cdot Z = 0$ and $z_1 = 3$, we have $\deg A_3 = 2$. There exists a unique $A_4 \subseteq \Gamma_1$ such that $z_4 = 3$, $A_4 \neq A_1$ and $A_4 \cdot A_3 = 1$. By induction, Γ_1 is of the following form



$Z = 3A_n + \ldots + 3A_3 + 3A_1 + D$, where D is a positive cycle with $|D| = B_1$. Then $A_n \cdot Z = -3$ and $Z \cdot Z < -3$. This is a contradiction.

(IIICiii) $z_2 = 3$. Then we are in (11).

(IIICiv) $z_2 = 4$. Since $z_1 = 3$, $A_1 \cdot A_1 = -2$ and $A_1 \cdot Z = -1$, there exists a unique $A_3 \nleq B_1$ such that $A_3 \cdot A_1 = 1$ and $z_3 = 1$. Then $A_3 \cdot Z \geq 1 > 0$, which is absurd.

(IIICv) $z_2 = 5$. Then we are in (12).

Theorem 5.16: Let $\pi: M \to V$ be a resolution of normal two dimensional Stein space with p as its only singular point. If $\dim H^1(M, \mathcal{O}) \leq 2$ and p is a hypersurface singularity, then the multiplicity ${}_V\mathcal{O}_p$ is less than or equal to 3.

Proof: This is a trivial consequence of Theorem 5.3.

Corollary 5.17: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only singular point. Suppose $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$. If p is a hypersurface singularity, then the elliptic sequence is one of the following forms:

(I) elliptic sequence is $\{Z, Z_E\}$

(a) $Z \cdot Z = -3, \quad Z_E \cdot Z_E = -1$

(b) $Z \cdot Z = -3, \quad Z_E \cdot Z_E = -2$

(c) $Z \cdot Z = -3 = Z_E \cdot Z_E$

(d) $Z \cdot Z = -1, \quad Z_E \cdot Z_E = -1$

(e) $Z \cdot Z = -2, \quad Z_E \cdot Z_E = -1$

(f) $Z \cdot Z = -2, \quad Z_E \cdot Z_E = -2$

(II) elliptic sequence is $\{Z, Z_{B_1}, Z_E\}$

(g) $Z \cdot Z = -2, \quad Z_{B_1} \cdot Z_{B_1} = -1 = Z_E \cdot Z_E$

(h) $Z \cdot Z = -1 = Z_{B_1} \cdot Z_{B_1} = Z_E \cdot Z_E$

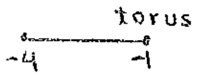
(III) elliptic sequence is $\{Z, Z_{B_1}, Z_{B_2}, Z_E\}$

(i) $Z\cdot Z = -1 = Z_{B_1}\cdot Z_{B_1} = Z_{B_2}\cdot Z_{B_2} = Z_E\cdot Z_E$

Proof: This is an easy consequence of Corollary 4.11, Proposition 5.4, and Theorem 5.16.

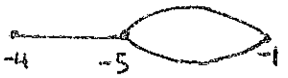
Theorem 5.18: Let π: $M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only singular point. Suppose $H^1(M,\mathcal{O}) \cong \mathbb{C}^2$ and p is a hypersurface singularity. Then the associated weighted dual graph is one of the following forms:

(1)

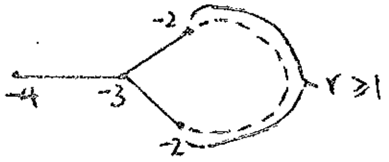


$Z = 1 \quad 1$

(2)

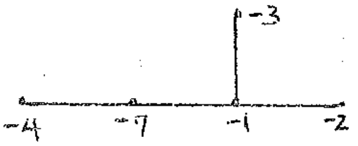


$Z = 1 \quad 1 \quad 2$



$Z = 1 \quad 1 \begin{matrix}1\\1\end{matrix}$

(3)

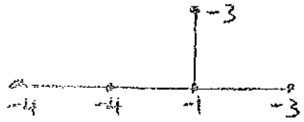


$$Z = \begin{matrix} & & 2 & \\ 1 & 1 & 6 & 3\end{matrix}$$

(4)

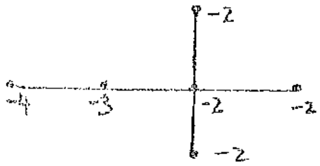
$$Z = \begin{matrix} & & 1 & \\ 1 & 1 & 4 & 2\end{matrix}$$

(5)



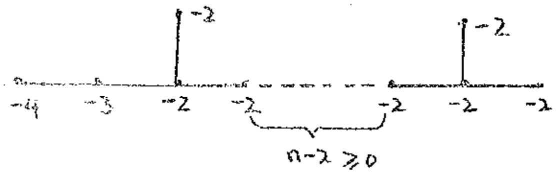
$Z = 1\ 1\ \overset{1}{3}\ 1$

(6)



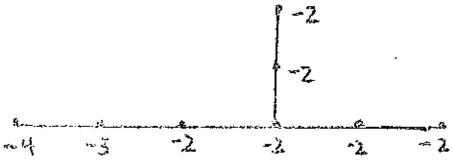
$Z = 1\ 1\ \overset{1}{\underset{1}{2}}\ 1$

(7)



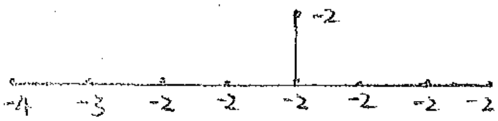
$Z = 1\ 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{2}\ 1$

(8)



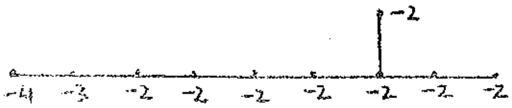
$Z = 1\ 1\ 2\ \overset{1}{\overset{2}{3}}\ 2\ 1$

(9)



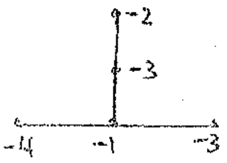
$Z = 1\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

(10)



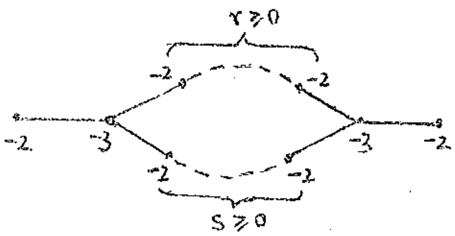
$Z = 1\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

(11)

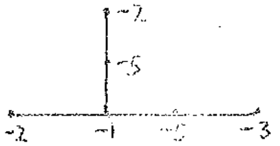


$Z = 3\ \overset{3}{\overset{5}{12}}\ 4$

(12)

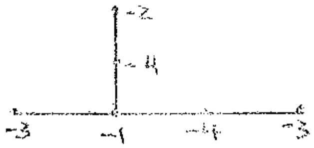


(13)



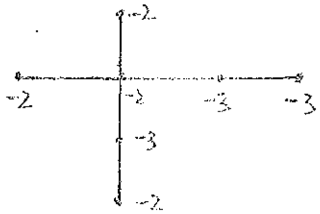
$$Z = \begin{array}{cccc} & 1 & & \\ & 1 & & \\ 2 & 4 & 1 & 1 \end{array}$$

(14)



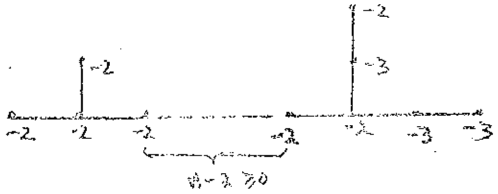
$$Z = \begin{array}{cccc} & 1 & & \\ & 1 & & \\ 1 & 3 & 1 & 1 \end{array}$$

(15)



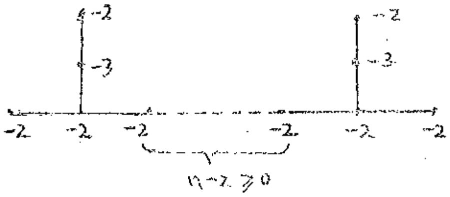
$$Z = \begin{array}{cccc} & 1 & & \\ 1 & 2 & 1 & 1 \\ & 1 & & \\ & 1 & & \end{array}$$

(16)



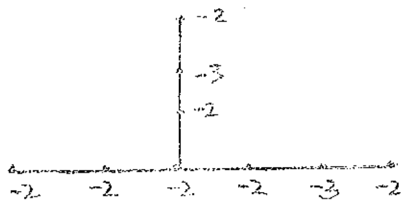
$$Z = \begin{array}{cccccccc} & & & & & 1 & & \\ & 1 & & & & 1 & & \\ 1 & 2 & 2 & \cdots & 2 & 2 & 1 & 1 \end{array}$$

(17)



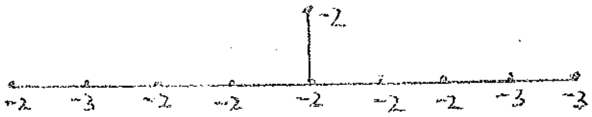
$$Z = \begin{array}{ccccccc} & 1 & & & & 1 & \\ & 1 & & & & 1 & \\ 1 & 2 & 2 & \cdots & 2 & 2 & 1 \end{array}$$

(18)



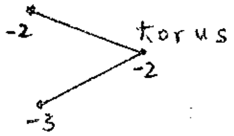
$$Z = \begin{array}{cccccc} & & 1 & & & \\ & & 1 & & & \\ & & 2 & & & \\ 1 & 2 & 3 & 2 & 1 & 1 \end{array}$$

(19)



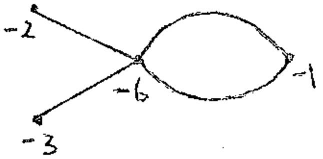
$$Z = \begin{array}{ccccccccc} & & & & 2 & & & & \\ 1 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 1 \end{array}$$

(20)

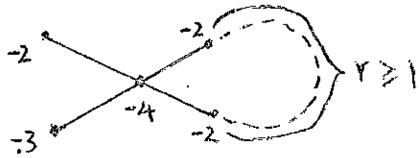


$Z = \begin{array}{cc} 1 & \\ & 1 \\ 1 & \end{array}$

(21)

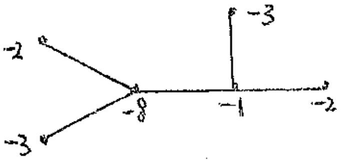


$Z = \begin{array}{ccc} 1 & & \\ & 1 & 2 \\ 1 & & \end{array}$



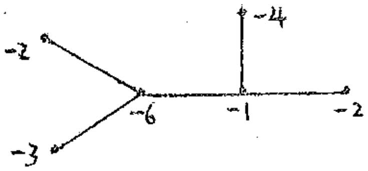
$Z = \begin{array}{cccc} 1 & & 1 & \cdots \\ & 1 & & \vdots \\ 1 & & 1 & \cdots \end{array}$

(22)



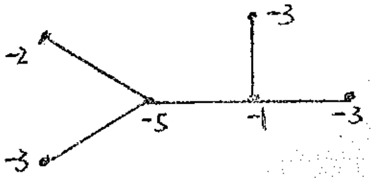
$Z = \begin{array}{cccc} 1 & & 2 & \\ & 1 & 6 & 3 \\ 1 & & & \end{array}$

(23)



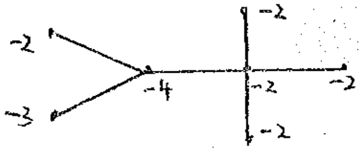
$Z = \begin{array}{cccc} 1 & & 1 & \\ & 1 & 4 & 2 \\ 1 & & & \end{array}$

(24)



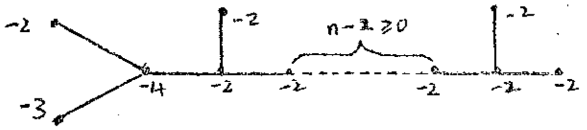
$Z = \begin{array}{cccc} 1 & & 1 & \\ & 1 & 3 & 1 \\ 1 & & & \end{array}$

(25)



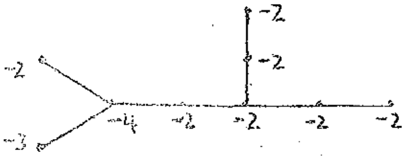
$Z = \begin{array}{cccc} 1 & & 1 & \\ & 1 & 2 & 1 \\ 1 & & 1 & \end{array}$

(26)



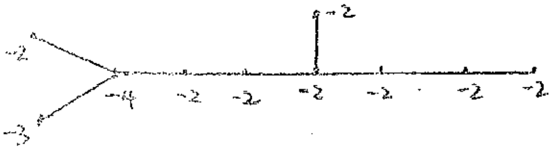
$Z = \begin{array}{cccccccc} 1 & & 1 & & & & 1 & \\ & 1 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 1 & & & & & & & \end{array}$

(27)



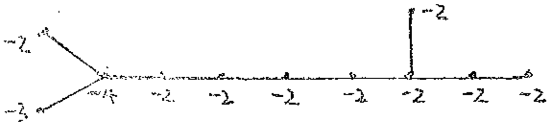
$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 1\ \ 2\ \ \overset{\overset{1}{2}}{3}\ \ 2\ \ 1$

(28)



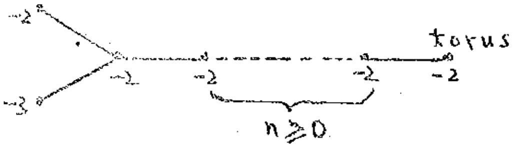
$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

(29)



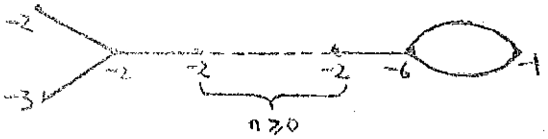
$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

(30)

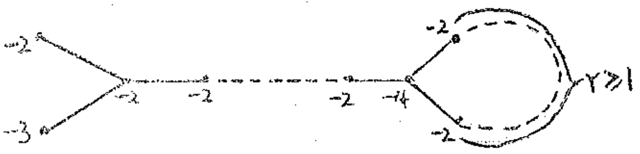


$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 2\ 2\ \cdots\ 2\ 1$

(31)

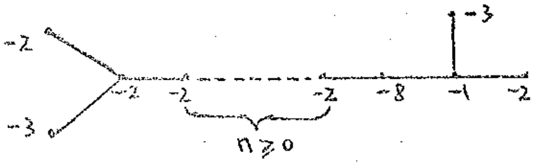


$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 2\ 2\ \cdots\ 2\ 1\ 2$



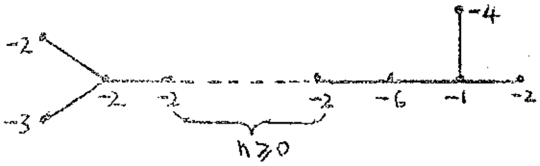
$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 2\ 2\ \cdots\ 2\ \begin{smallmatrix}1\\1\end{smallmatrix}$

(32)



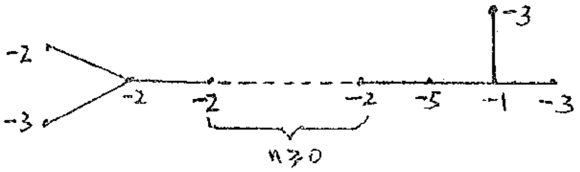
$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 2\ 2\ \cdots\ 2\ 1\ \overset{2}{6}\ 3$

(33)



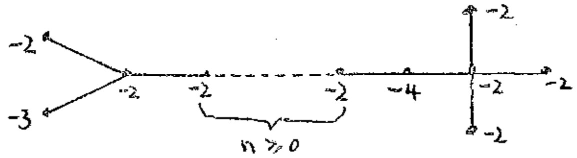
$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 2\ 2\ \cdots\ 2\ 1\ \overset{1}{4}\ 2$

(34)



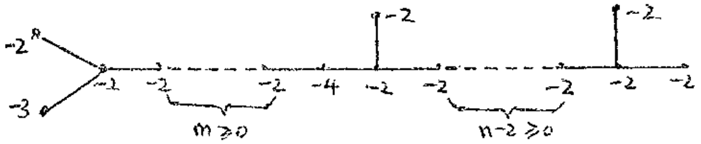
$Z = {}^{1}_{1}\ 2\ 2 \cdots 2\ 1\ \overset{1}{3}\ 1$

(35)



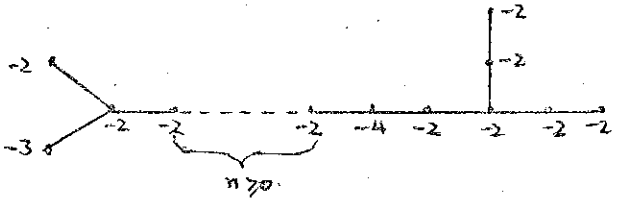
$Z = {}^{1}_{1}\ 2\ 2 \cdots 2\ 1\ \overset{1}{\underset{1}{2}}\ 1$

(36)



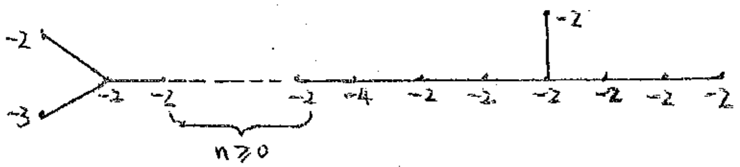
$Z = {}^{1}_{1}\ 2\ 2 \cdots\cdots 2\ 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ \overset{1}{2}\ 1$

(37)



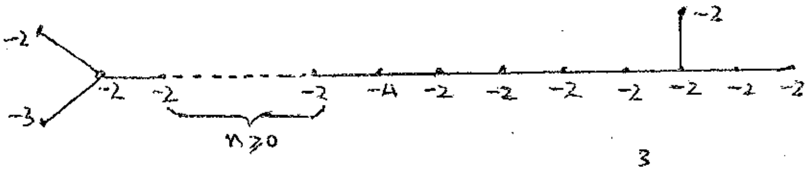
$Z = {}^{1}_{1}\ 2\ 2 \cdots 2\ 1\ 2\ \overset{1}{\overset{2}{3}}\ 2\ 1$

(38)



$Z = {}^{1}_{1}\ 2\ 2 \cdots\cdots 2\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

(39)

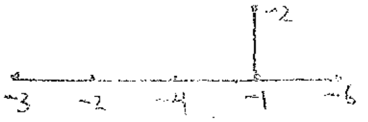


$Z = {}^{1}_{1}\ 2\ 2 \cdots\cdots 2\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

(40)

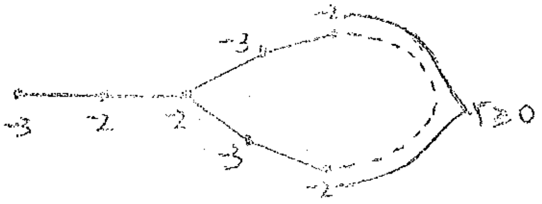
$Z = 1\ 2\ 2\ 1$

(41)



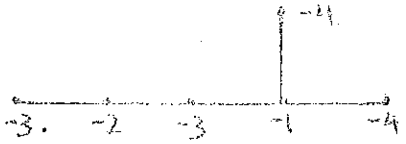
$Z = 1\ 2\ 2\ \overset{3}{6}\ 1$

(42)



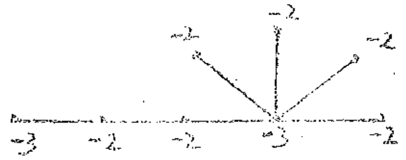
$Z = 1\ 2\ 2\ \begin{smallmatrix}1 & 1 & \cdots \\ 1 & 1 & \cdots\end{smallmatrix}$

(43)



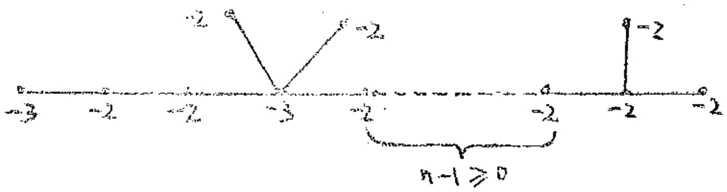
$Z = 1\ 2\ 2\ \overset{1}{4}\ 1$

(44)



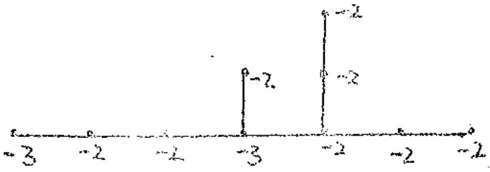
$Z = 1\ 2\ 2\ \overset{1\ 1\ 1}{2}\ 1$

(45)



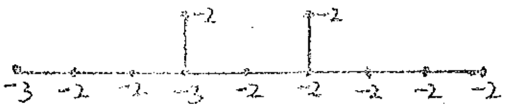
$Z = 1\ 2\ 2\ \overset{1\ 1}{2}\ 2 \cdots\cdots 2\ \overset{1}{2}\ 1$

(46)



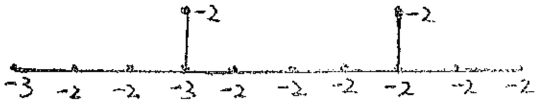
$Z = 1\ 2\ 2\ \overset{1}{2}\ \overset{\overset{1}{2}}{3}\ 2\ 1$

(47)



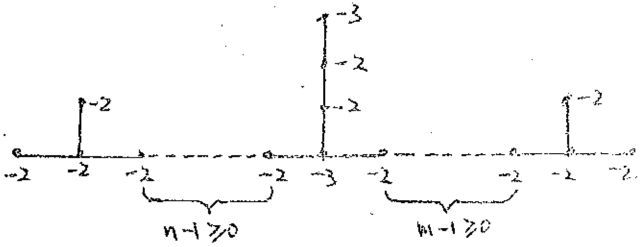
$Z = 1\ 2\ 2\ \overset{1}{2}\ 3\ \overset{2}{4}\ 3\ 2\ 1$

(48)



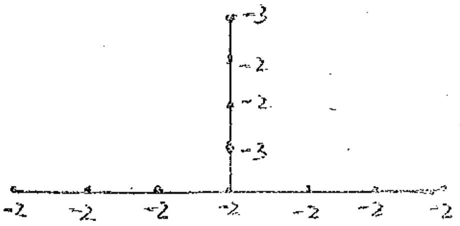
$$Z = 1\,2\,2\,\overset{1}{2}\,3\,4\,5\,\overset{3}{6}\,4\,2$$

(49)



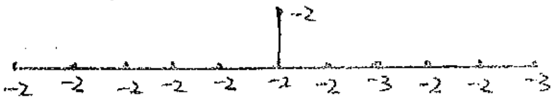
$$Z = 1\,\overset{1}{2}\,2\cdots 2\,\overset{\begin{smallmatrix}1\\2\\2\end{smallmatrix}}{2}\,2\cdots 2\,\overset{1}{2}\,1$$

(50)



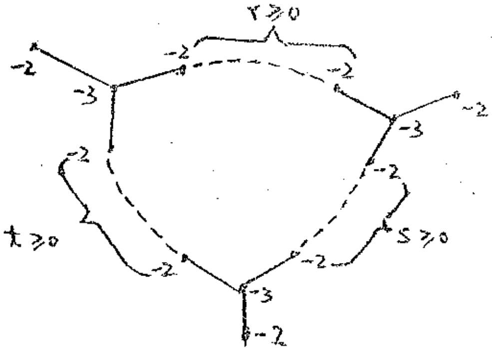
$$Z = 1\,2\,3\,\overset{\begin{smallmatrix}1\\2\\2\\2\end{smallmatrix}}{4}\,3\,2\,1$$

(51)

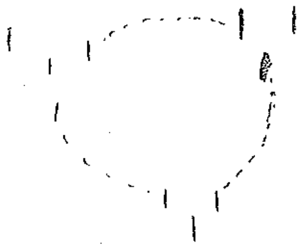


$$Z = 1\,2\,3\,4\,5\,\overset{3}{6}\,4\,2\,2\,2\,1$$

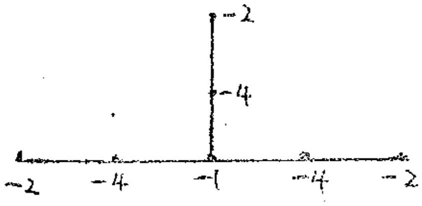
(52)



Z =

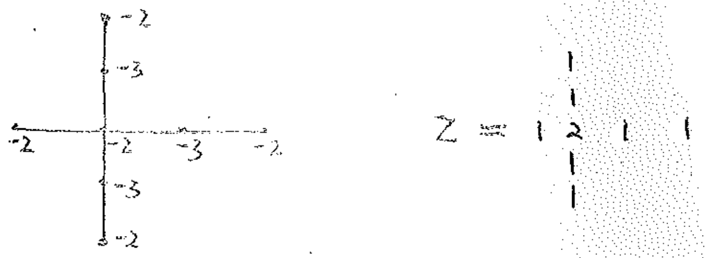


(53)

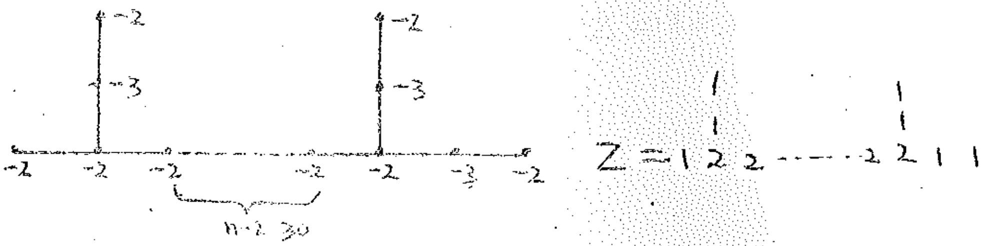


$$Z = 1\,1\,\overset{\begin{smallmatrix}1\\1\end{smallmatrix}}{3}\,1\,1$$

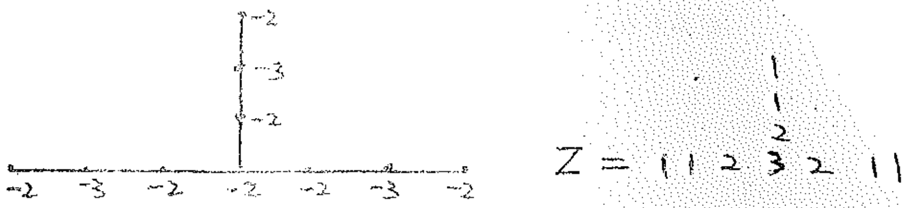
(54)



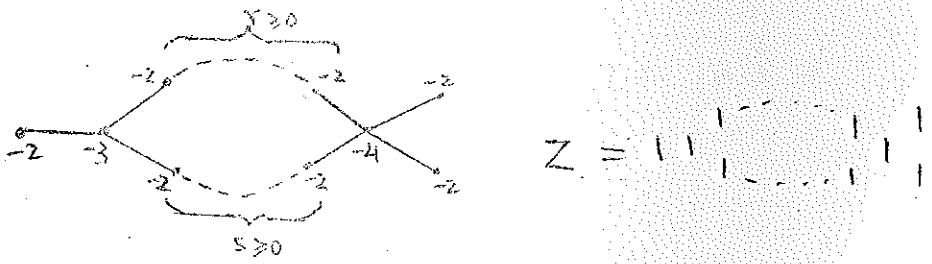
(55)



(56)

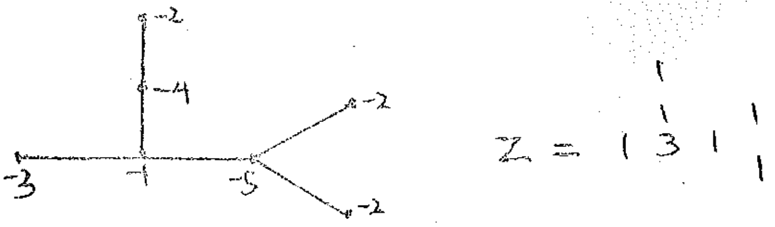


(57)

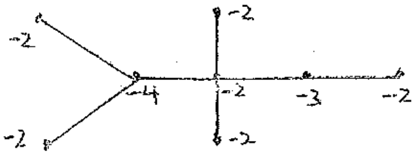


(58)

(59)

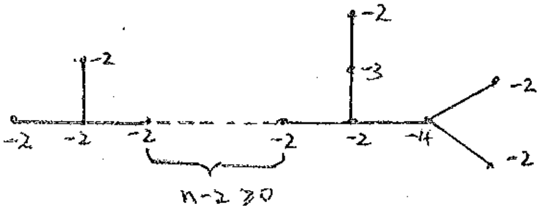


(60)



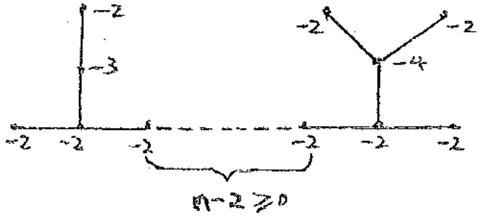
$Z = {}^{1}_{1}\ 1\ \overset{1}{\underset{1}{2}}\ 1\ 1$

(61)



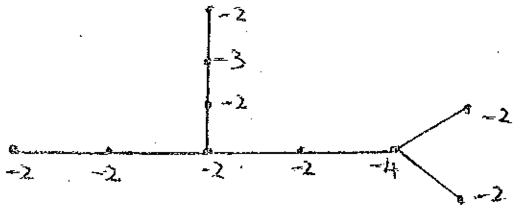
$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{\overset{1}{2}}\ 1\ {}^{1}_{1}$

(62)



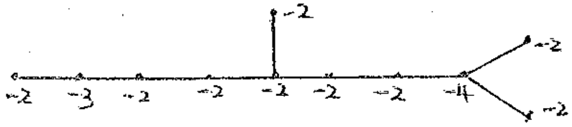
$Z = 1\ \overset{1}{\overset{1}{2}}\ 2 \cdots 2\ \overset{1\ \ 1}{\overset{1}{2}}\ 1$

(63)



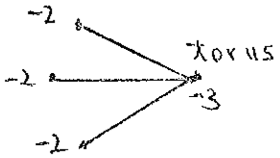
$Z = 1\ 2\ \overset{1}{\overset{1}{\overset{2}{3}}}\ 2\ 1\ {}^{1}_{1}$

(64)



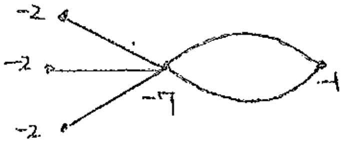
$Z = 1\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ {}^{1}_{1}$

(65)

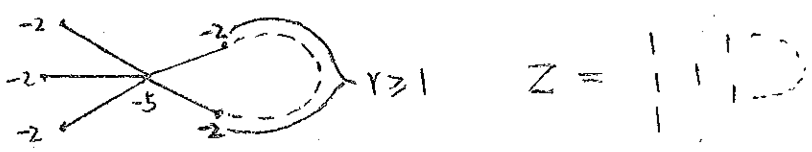


$Z = 1\ {}^{1}_{1}\ 1$

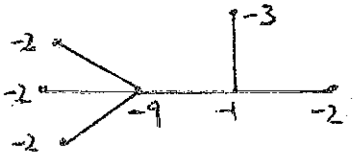
(66)



$Z = \begin{smallmatrix}1\\1\\1\end{smallmatrix}\ 1\ 2$

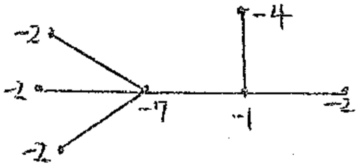


(67)



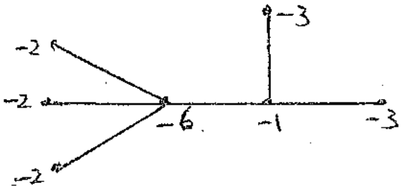
$$Z = \begin{array}{cccc} 1 & & 2 & \\ 1 & 1 & 6 & 3 \\ 1 & & & \end{array}$$

(68)



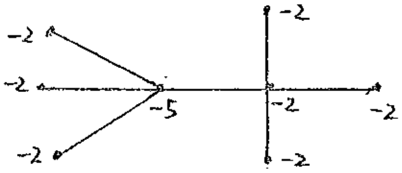
$$Z = \begin{array}{cccc} 1 & & 1 & \\ 1 & 1 & 4 & 2 \\ 1 & & & \end{array}$$

(69)



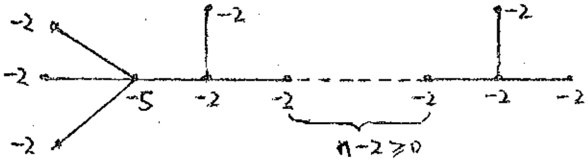
$$Z = \begin{array}{cccc} 1 & & 1 & \\ 1 & 1 & 3 & 1 \\ 1 & & & \end{array}$$

(70)



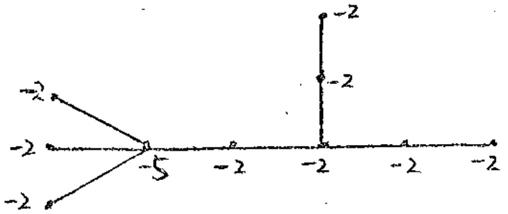
$$Z = \begin{array}{cccc} 1 & & 1 & \\ 1 & 1 & 2 & 1 \\ 1 & & 1 & \end{array}$$

(71)



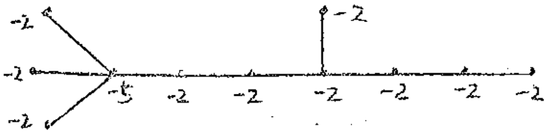
$$Z = \begin{array}{ccccccc} 1 & & 1 & & & 1 & \\ 1 & 1 & 2 & 2 & \cdots 2 & 2 & 1 \\ 1 & & & & & & \end{array}$$

(72)



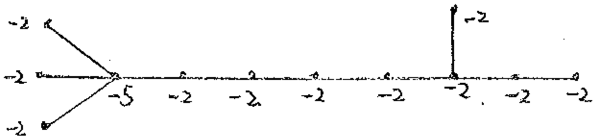
$$Z = \begin{array}{cccccc} & & & 1 & & \\ 1 & & & 2 & & \\ 1 & 1 & 2 & 3 & 2 & 1 \\ 1 & & & & & \end{array}$$

(73)



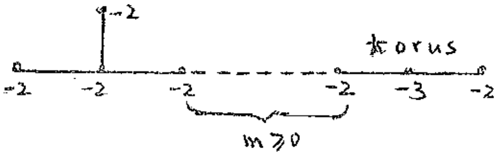
$Z = \begin{smallmatrix}1\\1\\1\end{smallmatrix}\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

(74)



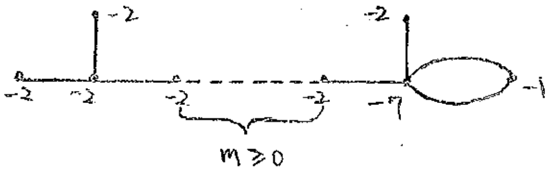
$Z = \begin{smallmatrix}1\\1\\1\end{smallmatrix}\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

(75)

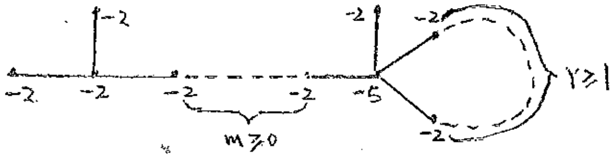


$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 1\ 1$

(76)

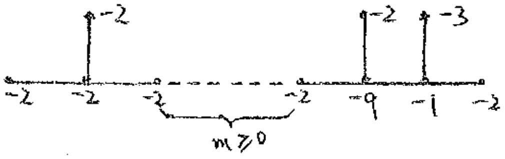


$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{1}\ 2$



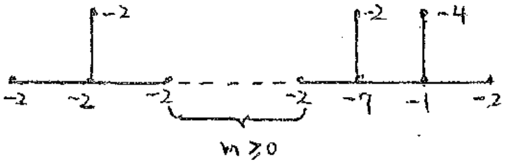
$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{1}\ 1\ \begin{smallmatrix}1\\1\end{smallmatrix}$

(77)



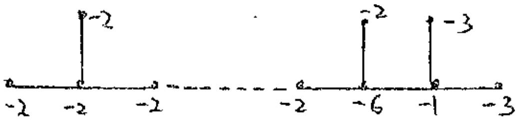
$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{1}\ \overset{2}{6}\ 3$

(78)



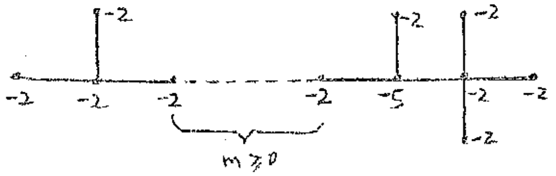
$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{1}\ \overset{1}{4}\ 2$

(79)



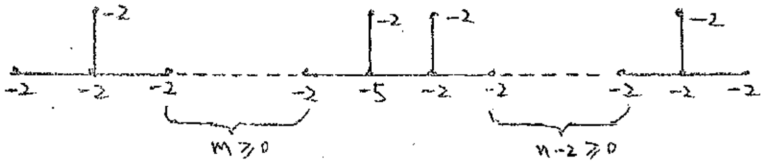
$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{1}\ \overset{1}{3}\ 1$

(80)



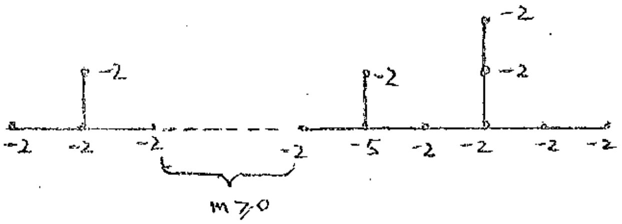
$Z = 1\overset{1}{2}\,2 \cdots 2\,\overset{1}{1}\,\underset{1}{\overset{1}{2}}\,1$

(81)



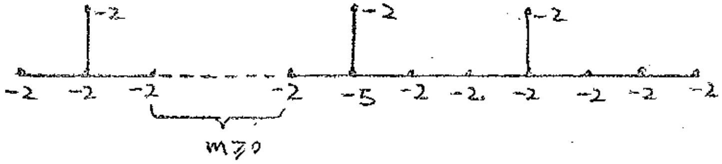
$Z = 1\overset{1}{2}\,2 \cdots 2\,\overset{1}{1}\,\overset{1}{2}\,2 \cdots 2\,\overset{1}{2}\,1$

(82)



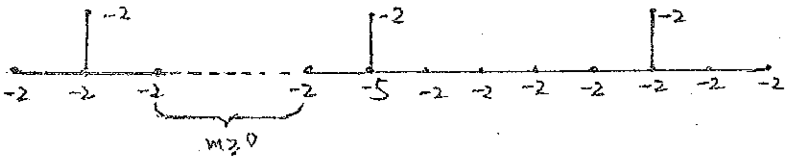
$Z = 1\overset{1}{2}\,2 \cdots 2\,\overset{1}{1}\,2\,\overset{\overset{1}{2}}{3}\,2\,1$

(83)



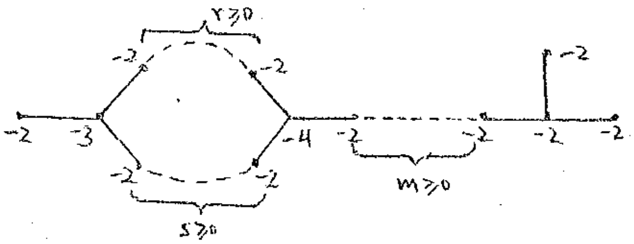
$Z = 1\overset{1}{2}\,2 \cdots 2\,\overset{1}{1}\,2\;3\;\overset{2}{4}\;3\;2\;1$

(84)



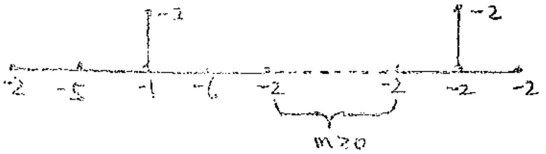
$Z = 1\overset{1}{2}\,2 \cdots 2\,\overset{1}{1}\,2\;3\;4\;5\;\overset{3}{6}\;4\;2$

(85)



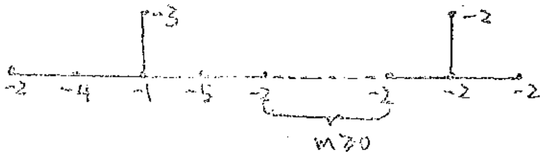
$Z = 1\;1\begin{matrix}1 \cdots 1\\ 1 \cdots 1\end{matrix}1\;2 \cdots 2\,\overset{1}{2}\,1$

(86)



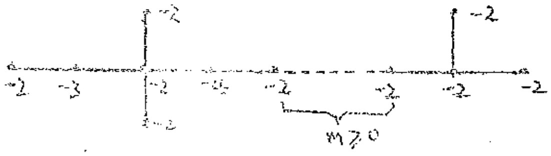
$$Z = 1\ 1\ \overset{2}{4}\ 1\ 2 \cdots 2\ \overset{1}{2}\ 1$$

(87)



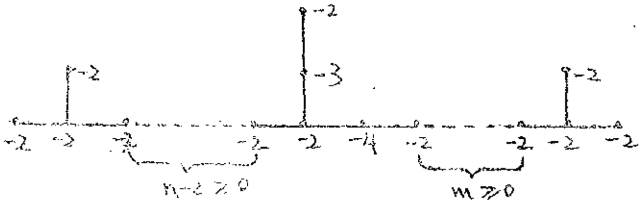
$$Z = 1\ 1\ \overset{1}{3}\ 1\ 2 \cdots 2\ \overset{1}{2}\ 1$$

(88)



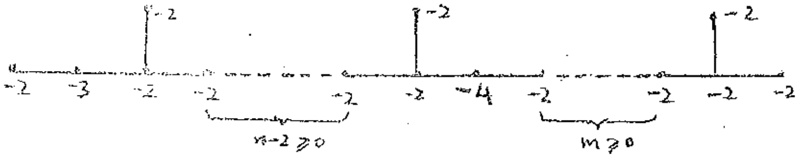
$$Z = 1\ 1\ \overset{1}{\underset{1}{2}}\ 1\ 2 \cdots 2\ \overset{1}{2}\ 1$$

(89)



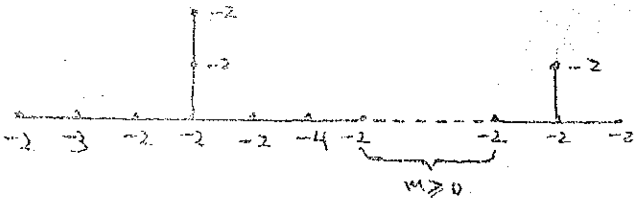
$$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{\overset{1}{2}}\ 1\ 2 \cdots 2\ \overset{1}{2}\ 1$$

(90)



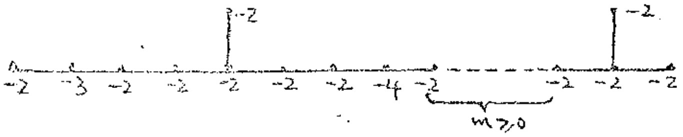
$$Z = 1\ 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{2}\ 1\ 2 \cdots 2\ \overset{1}{2}\ 1$$

(91)



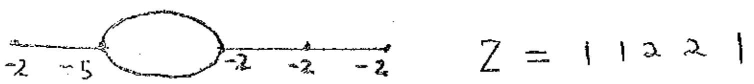
$$Z = 1\ 1\ 2\ \overset{1}{\overset{2}{3}}\ 2\ 1\ 2 \cdots 2\ \overset{1}{2}\ 1$$

(92)

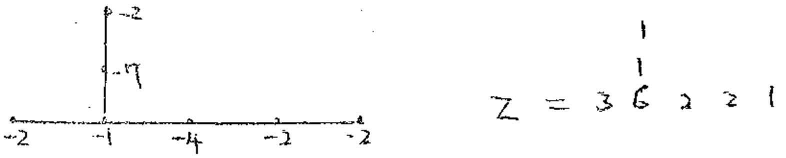


$$Z = 1\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 2 \cdots 2\ \overset{1}{2}\ 1$$

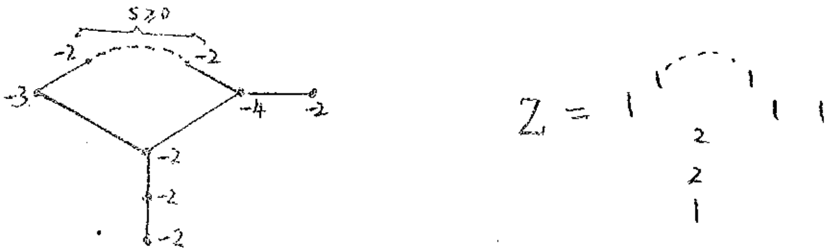
(93)



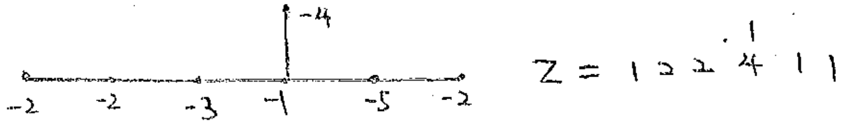
(94)



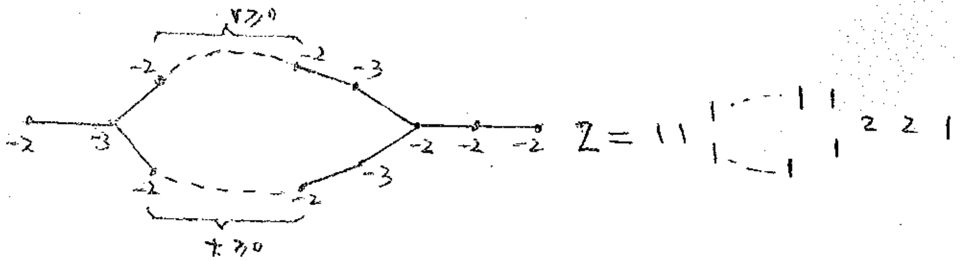
(95)



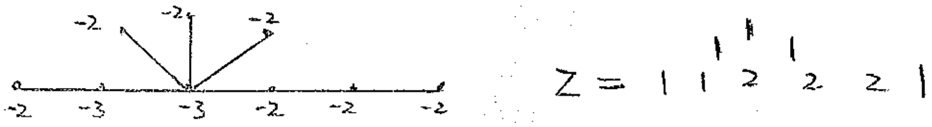
(96)



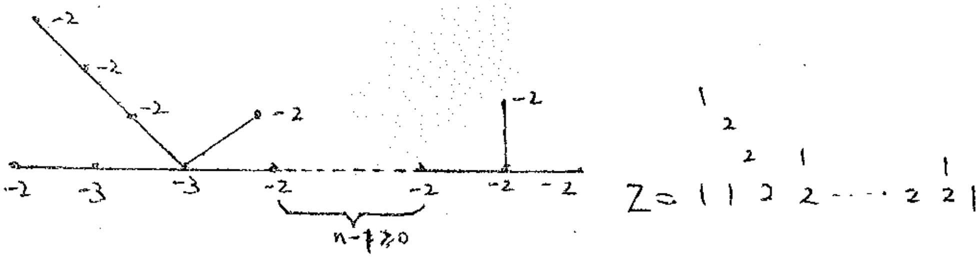
(97)



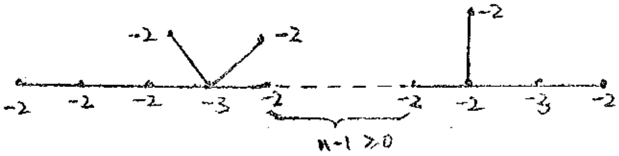
(98)



(99)

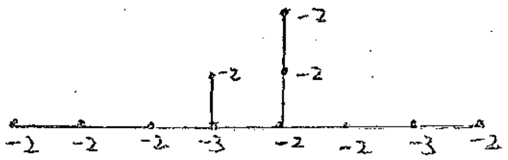


(100)



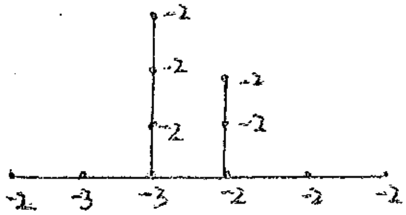
$$Z=\begin{array}{cccccccccc} & & & 1\ \ 1 & & & & 1 & & \\ 1 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & 1 & 1 \end{array}$$

(101)



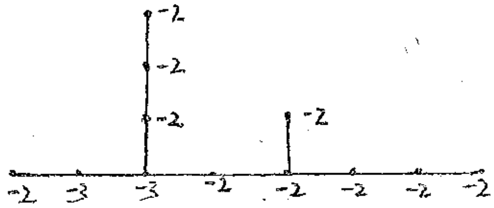
$$Z=\begin{array}{ccccccc} & & & & 1 & & \\ & & & 1 & 2 & & \\ 1 & 2 & 2 & 2 & 3 & 2 & 1 \end{array}$$

(102)



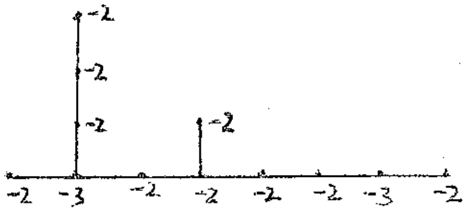
$$Z=\begin{array}{cccccc} & & 1 & & & \\ & & 2 & 1 & & \\ & & 2 & 2 & & \\ 1 & 1 & 2 & 3 & 2 & 1 \end{array}$$

(103)



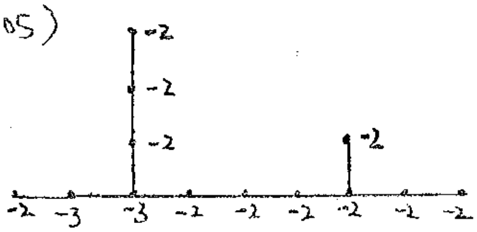
$$Z=\begin{array}{cccccccc} & & 1 & & & & & \\ & & 2 & & & & & \\ & & 2 & & 2 & & & \\ 1 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{array}$$

(104)



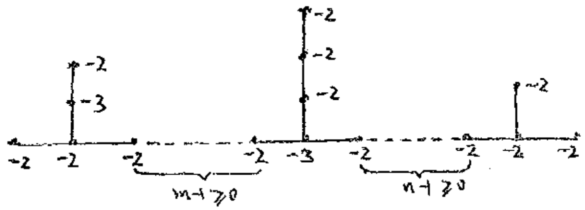
$$Z=\begin{array}{cccccccc} & 1 & & & & & & \\ & 2 & & & & & & \\ & 2 & & 2 & & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 & 1 \end{array}$$

(105)



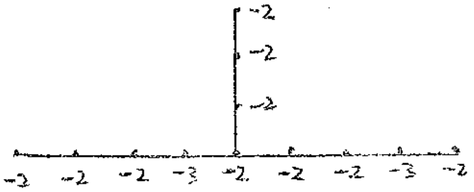
$$Z=\begin{array}{ccccccccc} & & 1 & & & & & & \\ & & 2 & & & & & & \\ & & 2 & & & & 3 & & \\ 1 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{array}$$

(106)



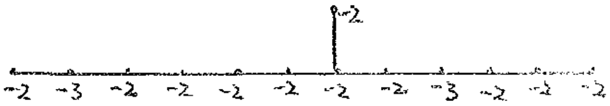
$$Z=\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & 1 & & & & 2 & & & & & \\ & 1 & & & & 2 & & & & 1 & \\ 1 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 & 2 & 1 \end{array}$$

(107)



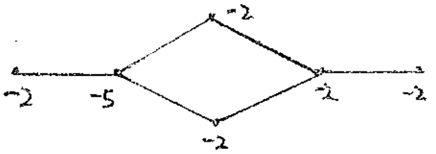
$$Z = \begin{matrix} & & & & 1 & & & & \\ & & & & 2 & & & & \\ & & & & 3 & & & & \\ 1 & 2 & 2 & 2 & 4 & 3 & 2 & 1 & 1 \end{matrix}$$

(108)



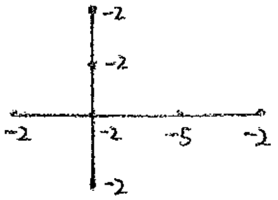
$$Z = \begin{matrix} & & & & & & 3 & & & & & \\ 1 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 & 2 & 2 & 1 \end{matrix}$$

(109)



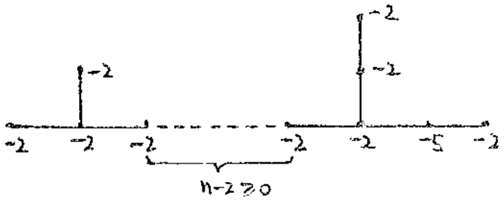
$$Z = \begin{matrix} & & 2 & & \\ 1 & 1 & & 3 & 2 \\ & & 2 & & \end{matrix}$$

(110)



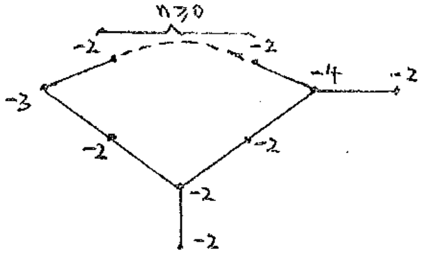
$$Z = \begin{matrix} & 2 & & \\ & 3 & & \\ 2 & 4 & 1 & 1 \\ & 2 & & \end{matrix}$$

(111)



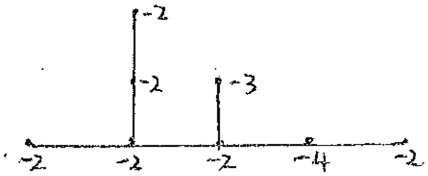
$$Z = \begin{matrix} & & & & & 2 & & \\ & 2 & & & & 3 & & \\ 2 & 4 & 4 & \cdots & 4 & 4 & 1 & 1 \end{matrix}$$

(112)



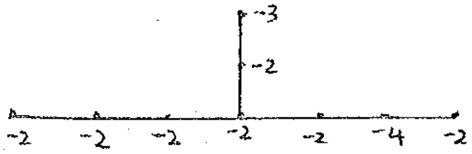
$$Z = \begin{matrix} & 1 & \cdots & 1 & & \\ 1 & & & & 1 & 1 \\ & 2 & & 2 & & \\ & & 3 & & & \\ & & 2 & & & \end{matrix}$$

(113)



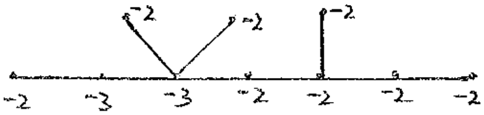
$$Z = \begin{matrix} & 2 & & & \\ & 3 & 1 & & \\ 2 & 4 & 3 & 1 & 1 \end{matrix}$$

(114)



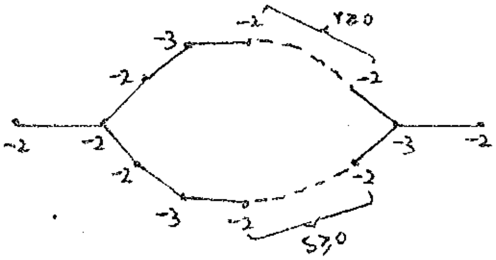
$$Z = 2\,3\,4\,\overset{\overset{1}{3}}{5}\,3\,1\,1$$

(115)



$$Z = 1\,1\,\overset{1\;1}{2}\;3\;\overset{2}{4}\,3\,2$$

(116)

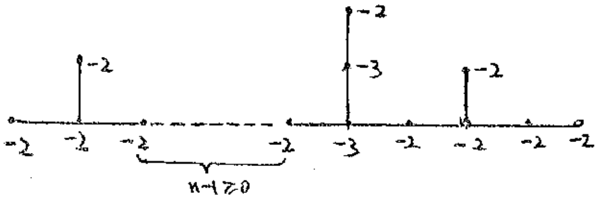


$$Z = 2\,3\begin{matrix}2 & 1 & 1 & \cdots & 1 & 1\,1\\ 2 & 1 & 1 & \cdots & 1 & \end{matrix}$$

(117)

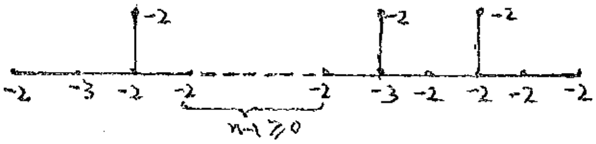
$$Z = 1\,1\,\overset{1}{2}\,4\,\overset{3}{6}\,5\,4\,3\,2$$

(118)

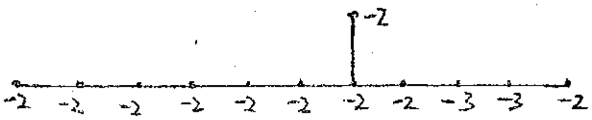


$$Z = 1\,\overset{1}{2}\,2\cdots 2\,\overset{\overset{1}{1}}{2}\,3\,\overset{2}{4}\,3\,2$$

(119)



(120)

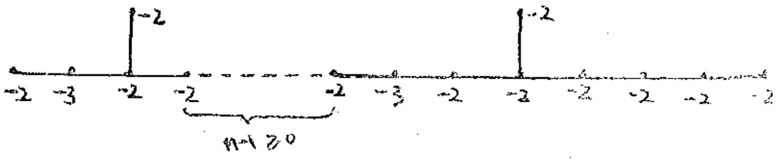


$$Z = 2\,3\,4\,5\,6\,7\,\overset{4}{8}\,5\,2\,1\,1$$

(121)

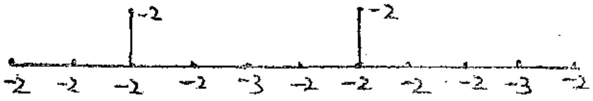
$$Z = 2\,3\,\overset{2}{4}\,3\;\;2\,\overset{\overset{1}{2}}{3}\,2\,1\,1$$

(122)



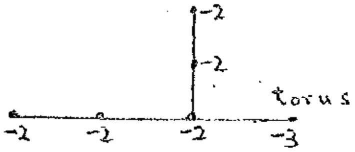
$$Z = 1\ 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2$$

(123)



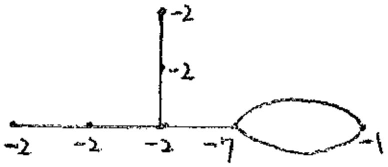
$$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1$$

(124)

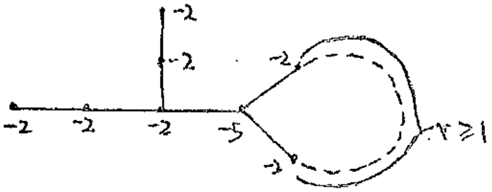


$$Z = 1\ 2\ \overset{\overset{1}{2}}{3}\ 1$$

(125)

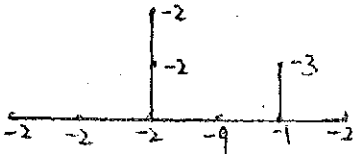


$$Z = 1\ 2\ \overset{\overset{1}{2}}{3}\ 1\ \ 2$$



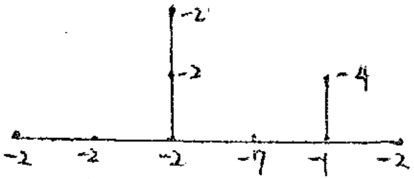
$$Z = 1\ 2\ \overset{\overset{1}{2}}{3}\ 1\ \begin{smallmatrix}1 \cdots \\ 1 \cdots\end{smallmatrix}$$

(126)



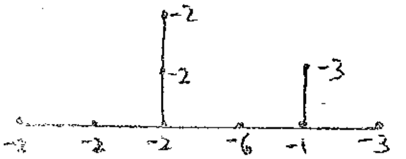
$$Z = 1\ 2\ \overset{\overset{1}{2}}{3}\ 1\ \overset{2}{6}\ 3$$

(127)



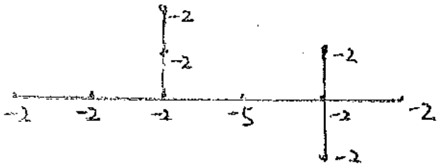
$$Z = 1\ 2\ \overset{\overset{1}{2}}{3}\ 1\ \overset{1}{4}\ 2$$

(128)



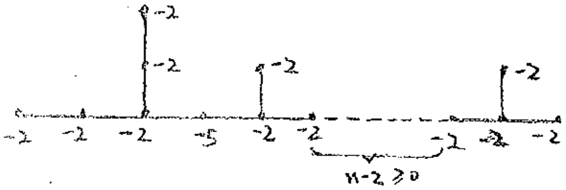
$Z = 12\overset{\overset{1}{2}}{3}\,1\overset{1}{3}1$

(129)



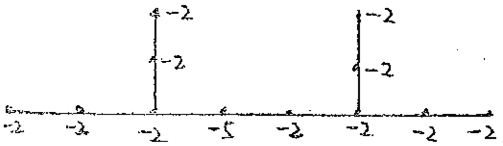
$Z = 12\overset{\overset{1}{2}}{3}\,1\underset{1}{\overset{1}{2}}1$

(130)



$Z = 12\overset{\overset{1}{2}}{3}\,1\overset{1}{2}2\cdots 2\overset{1}{2}1$

(131)



$Z = 12\overset{\overset{1}{2}}{3}\,1\,2\,\overset{\overset{1}{2}}{3}\,2\,1$

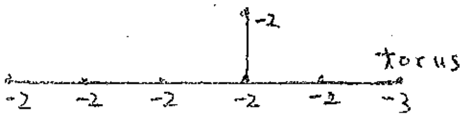
(132)

$Z = 12\overset{\overset{1}{2}}{3}\,1\,2\,3\,\overset{2}{4}\,3\,2\,1$

(133)

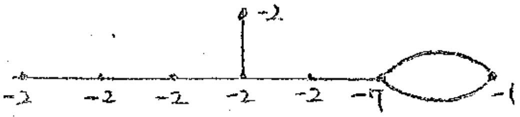
$Z = 12\overset{\overset{1}{2}}{3}\,1\,2\,3\,4\,5\,\overset{3}{6}\,4\,2$

(134)

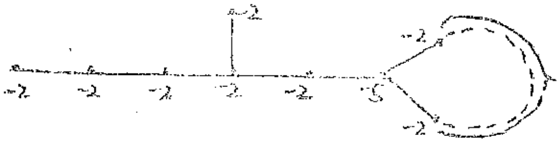


$Z = 1\,2\,3\,\overset{2}{4}\,3\,1$

(135)

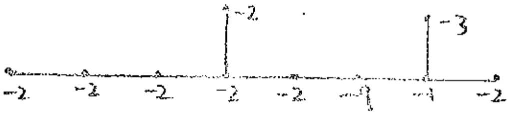


$Z = 1\,2\,3\,\overset{2}{4}\,3\,1\,2$



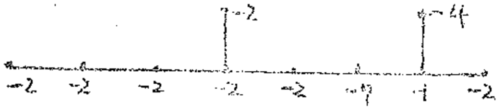
$r \geq 1$ $Z = 1\,2\,3\,\overset{2}{4}\,3\,1\,\overset{1\,-}{\underset{1\,-}{}}$

(136)



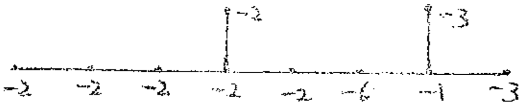
$Z = 1\,2\,3\,\overset{2}{4}\,3\,1\,\overset{2}{6}\,3$

(137)



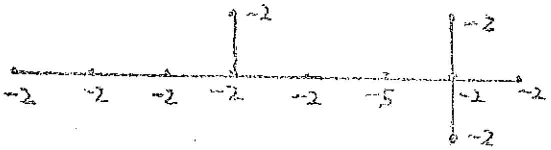
$Z = 1\,2\,3\,\overset{2}{4}\,3\,1\,\overset{1}{4}\,2$

(138)



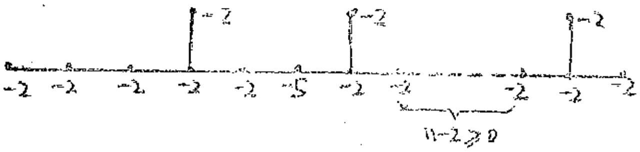
$Z = 1\,2\,3\,\overset{2}{4}\,3\,1\,\overset{1}{3}\,1$

(139)



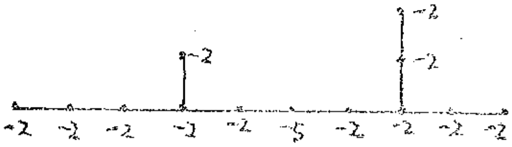
$Z = 1\,2\,3\,\overset{2}{4}\,3\,1\,\underset{1}{\overset{1}{2}}\,1$

(140)



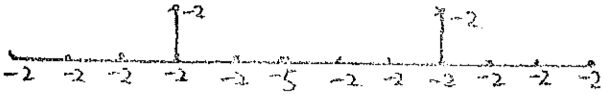
$Z = 1\,2\,3\,\overset{2}{4}\,3\,1\,\overset{1}{2}\,2\cdots 2\,\overset{1}{2}\,1$

(141)



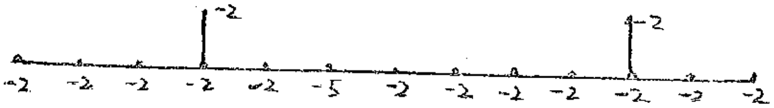
$Z = 1\,2\,3\,\overset{2}{4}\,3\,1\,2\,\overset{1}{\overset{2}{3}}\,2\,1$

(142)



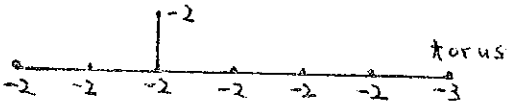
$Z = 1\,2\,3\,\overset{2}{4}\,3\,1\,2\,3\,\overset{2}{4}\,3\,2\,1$

(143)



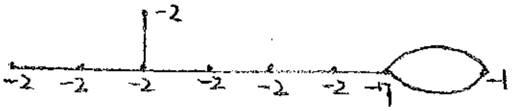
$Z = 1\ 2\ 3\ \overset{2}{4}\ 3\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

(144)

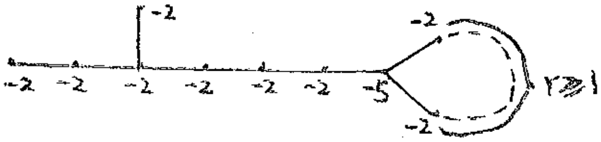


$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1$

(145)

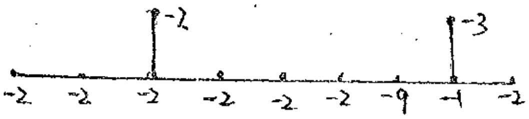


$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1\ 2$



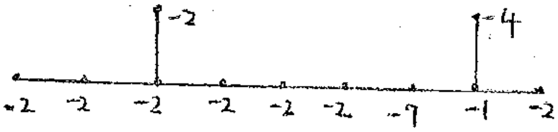
$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1\ \begin{smallmatrix}1\\1\end{smallmatrix}$

(146)



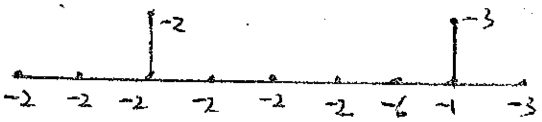
$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1\ \overset{2}{6}\ 3$

(147)



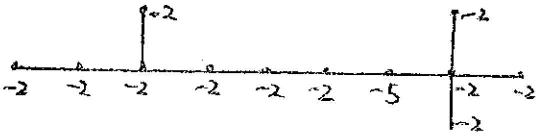
$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1\ \overset{1}{4}\ 2$

(148)



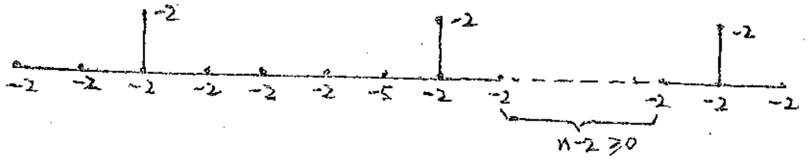
$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1\ \overset{1}{3}\ 1$

(149)



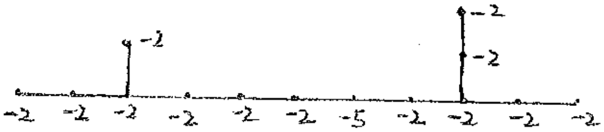
$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1\ \underset{1}{\overset{1}{2}}\ 1$

(150)



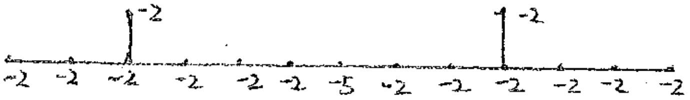
$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1\ \overset{1}{2}\ 2 - - - - 2\ \overset{1}{2}\ 1$

(151)



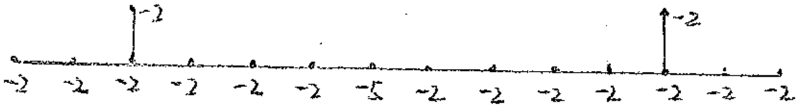
$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1\ 2\ \overset{1}{\overset{2}{3}}\ 2\ 1$

(152)



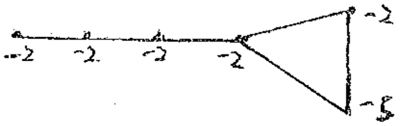
$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

(153)



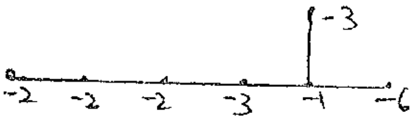
$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

(154)



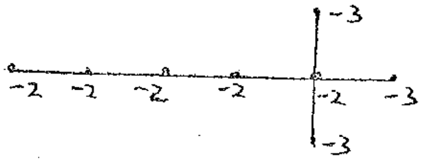
$Z = 1\ 2\ 3\ 3\ \begin{matrix}2\\1\end{matrix}$

(155)



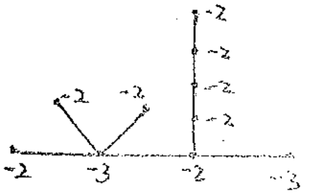
$Z = 1\ 2\ 3\ 3\ \overset{2}{6}\ 1$

(156)



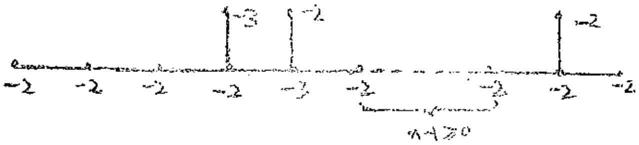
$Z = 1\ 2\ 3\ 3\ \overset{1}{\underset{1}{3}}\ 1$

(157)



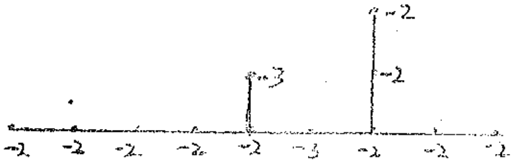
$$Z = 1\,\overset{1\;1}{2}\;\overset{\begin{smallmatrix}1\\2\\3\\3\end{smallmatrix}}{3}\;1$$

(158)



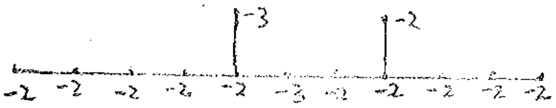
$$Z = 1\,2\,3\,\overset{1}{3}\,\overset{1}{2}\,2 - - - 2\,\overset{1}{2}\,1$$

(159)



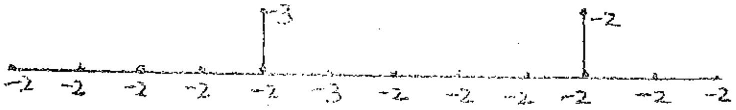
$$Z = 1\,2\,3\,3\,\overset{1}{3}\,2\,\overset{\begin{smallmatrix}1\\2\end{smallmatrix}}{3}\,2\,1$$

(160)



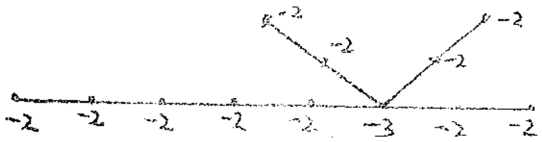
$$Z = 1\,2\,3\,3\,\overset{1}{3}\,2\,3\,\overset{2}{4}\,3\,2\,1$$

(161)



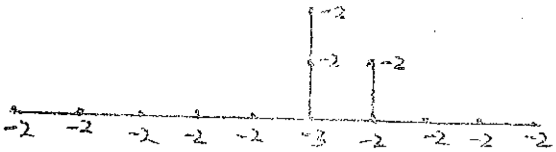
$$Z = 1\,2\,3\,3\,\overset{1}{3}\,2\,3\,4\,5\,\overset{3}{6}\,4\,2$$

(162)



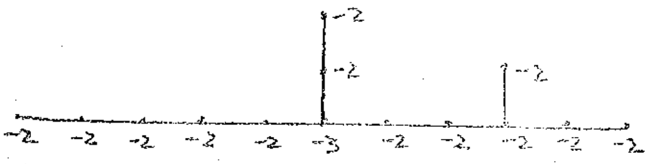
$$Z = 1\,2\,3\,3\,\overset{\begin{smallmatrix}1&&1\\2&&2\end{smallmatrix}}{3}\,3\;2\;1$$

(163)



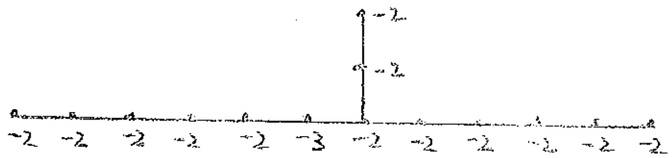
$$Z = 1\,2\,3\,3\,3\,\overset{\begin{smallmatrix}1\\2\end{smallmatrix}}{3}\,\overset{2}{4}\;3\,2\,1$$

(164)



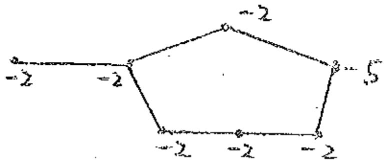
$$Z = 1\ 2\ 3\ 3\ 3\ \overset{\overset{1}{2}}{3}\ 4\ 5\ \overset{3}{6}\ 4\ 2$$

(165)



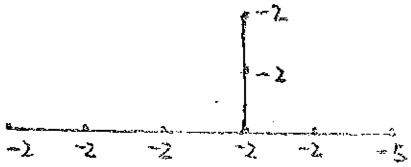
$$Z = 1\ 2\ 3\ 3\ 3\ 3\ \overset{\overset{2}{4}}{6}\ 5\ 4\ 3\ 2\ 1$$

(166)



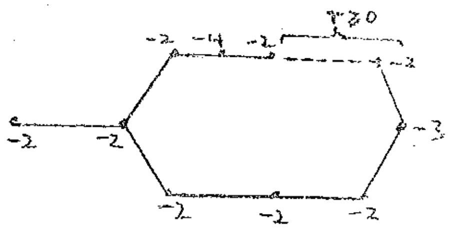
$$Z = \begin{array}{ccccc} & & 3 & & \\ 3 & 5 & & & 1 \\ & 4 & & 3 & 2 \end{array}$$

(167)



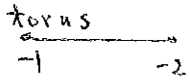
$$Z = 3\ 5\ 7\ \overset{\overset{3}{6}}{9}\ 5\ 1$$

(168)



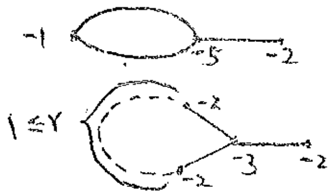
$$Z = \begin{array}{cccccc} & & 3 & 1 & 1 \cdots 1 & \\ 3 & 5 & & & & 1 \\ & & 4 & 3 & 2 & \end{array}$$

(169)



$$Z = 1\ \ 1$$

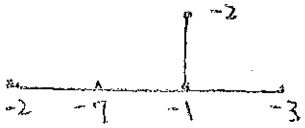
(170)



$$Z = 2\ 1\ 1$$

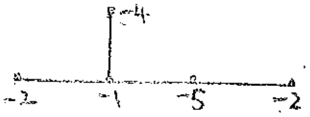
$$Z = \begin{array}{c} 1 \\ \\ 1 \end{array}\ 1\ 1$$

(171)



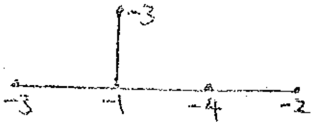
$Z = 1\ 1\ \overset{3}{6}\ 2$

(172)



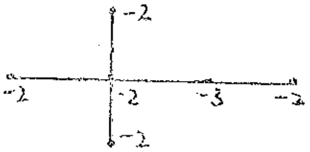
$Z = 2\ \overset{1}{4}\ 1\ 1$

(173)



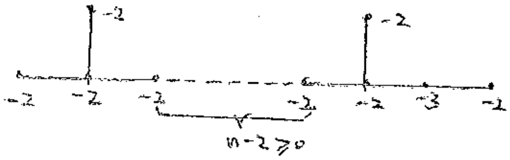
$Z = 1\ \overset{1}{3}\ 1\ 1$

(174)



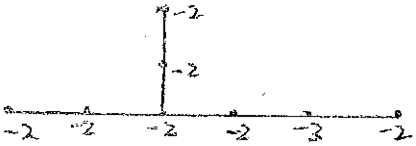
$Z = 1\ \overset{1}{\underset{1}{2}}\ 1\ 1$

(175)



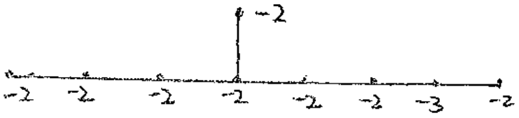
$Z = 1\ \overset{1}{2}\ 2\ \cdots\ 2\ \overset{1}{2}\ 1$

(176)



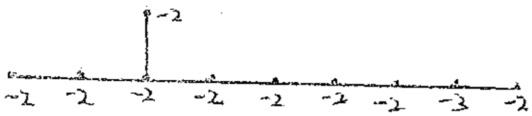
$Z = 1\ 2\ \overset{1}{\overset{2}{3}}\ 2\ 1\ 1$

(177)



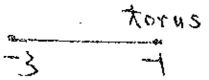
$Z = 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1$

(178)



$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2\ 1$

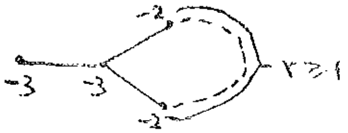
(179)



$Z = 1\ 1$

(180)

$Z = 1\ 1\ 2$

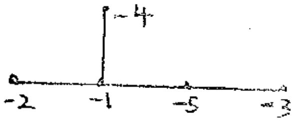


$Z = 1\ 1\ \begin{matrix}1\\ 1\end{matrix}$

(181)

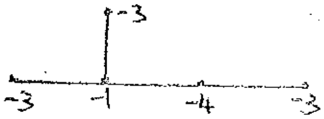
$Z = 1\ 1\ \overset{3}{6}\ 2$

(182)



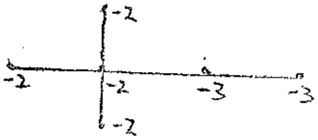
$Z = 2\ \overset{1}{4}\ 1\ 1$

(183)



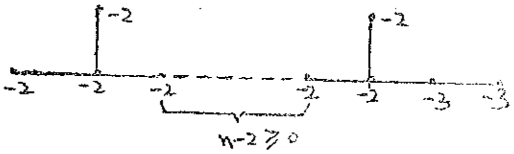
$Z = 1\ \overset{1}{3}\ 1\ 1$

(184)



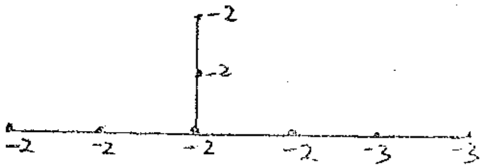
$Z = 1\ \underset{1}{\overset{1}{2}}\ 1\ 1$

(185)



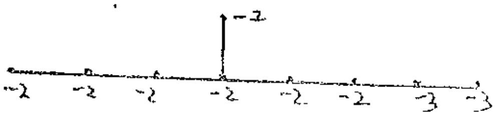
$Z = 1\ \overset{1}{2}\ 2 - - -\ 2\ \overset{1}{2}\ 1\ 1$

(186)



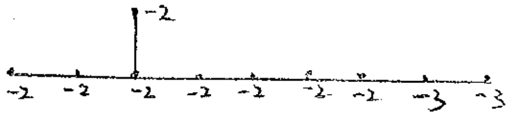
$Z = 1\ 2\ \overset{\overset{1}{2}}{3}\ 2\ 1\ 1$

(187)



$Z = 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

(188)

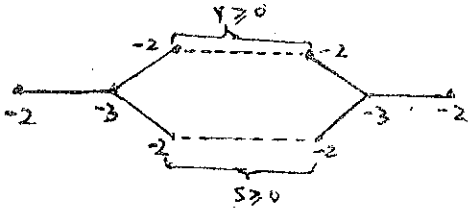


$Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2\ 1\ 1$

(189)

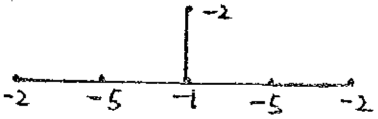
$Z = 2\ 3\ \overset{5}{10}\ 2$

(190)



$Z = 1\ 1\ \begin{matrix}1 \cdots\cdots 1\\ 1 \cdots\cdots 1\end{matrix}\ 1\ 1$

(191)

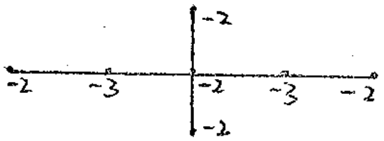


$Z = 1\ 1\ \overset{2}{4}\ 1\ 1$

(192)

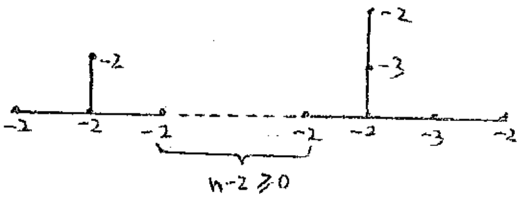
$Z = 1\ 1\ \overset{1}{3}\ 1\ 1$

(193)



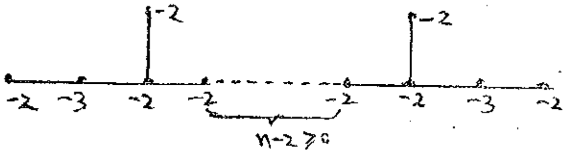
$Z = 1\ 1\ \overset{1}{\underset{1}{2}}\ 1\ 1$

(194)



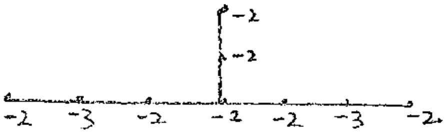
$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{\overset{1}{2}}\ 1\ 1$

(195)



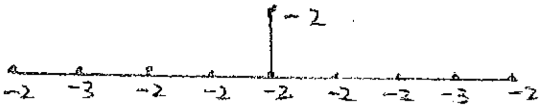
$Z = 1\ 1\ \overset{1}{2}\ 2 \cdots\cdots 2\ \overset{1}{2}\ 1\ 1$

(196)



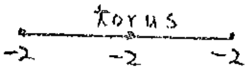
$$Z = 1\ 1\ 2\ \overset{\overset{1}{2}}{3}\ 2\ 1\ 1$$

(197)



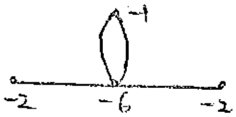
$$Z = 1\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1\ 1$$

(198)

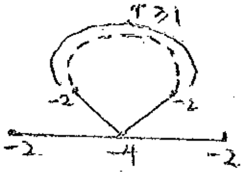


$$Z = 1\ 1\ 1$$

(199)

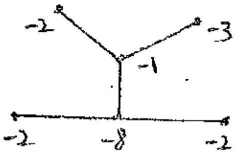


$$Z = \begin{matrix} & 1 & \\ 1 & 1 & 1 \end{matrix}$$



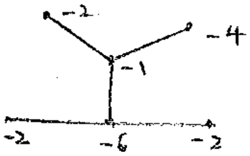
$$Z = \begin{matrix} & 1 & & 1 & \\ 1 & & 1 & & 1 \end{matrix}$$

(200)



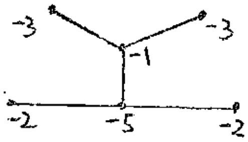
$$Z = \begin{matrix} 3 & & 2 \\ & 6 & \\ 1 & 1 & 1 \end{matrix}$$

(201)



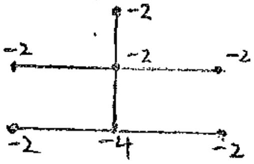
$$Z = \begin{matrix} 2 & & 1 \\ & 4 & \\ 1 & 1 & 1 \end{matrix}$$

(202)



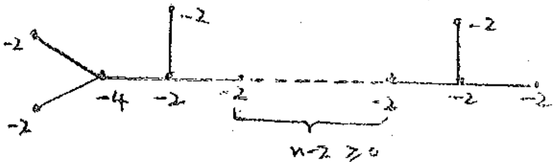
$$Z = \begin{matrix} 1 & & 1 \\ & 3 & \\ 1 & 1 & 1 \end{matrix}$$

(203)



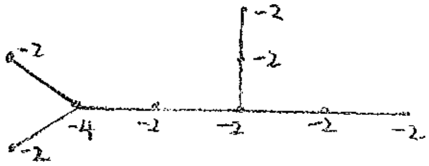
$$Z = \begin{matrix} & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{matrix}$$

(204)



$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{1}{2}\ 1$

(205)

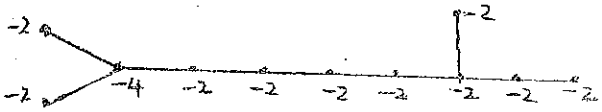


$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 1\ 2\ \overset{1}{\overset{2}{3}}\ 2\ 1$

(206)

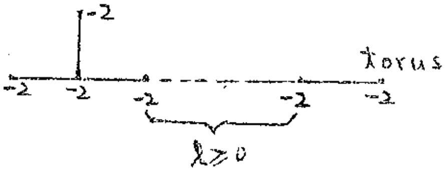
$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

(207)



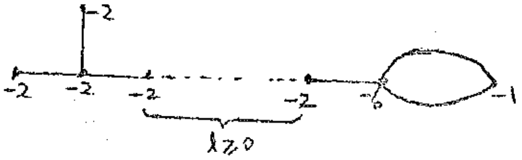
$Z = \begin{smallmatrix}1\\1\end{smallmatrix}\ 1\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

(208)

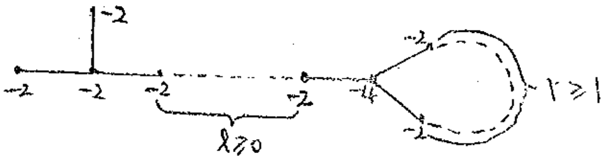


$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 1$

(209)

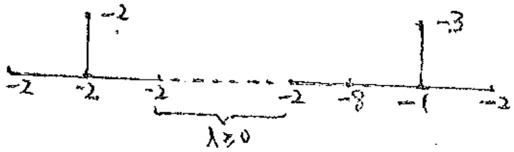


$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 1\ 2$



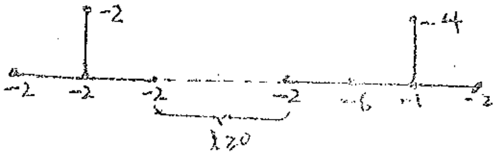
$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 1\ \begin{smallmatrix}1 \cdots\\ \vdots\\ 1 \cdots\end{smallmatrix}$

(210)



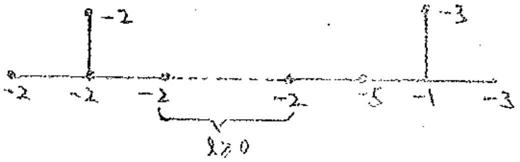
$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 1\ \overset{3}{6}\ 2$

(211)



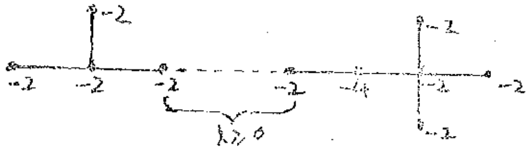
$$Z = 1\,\overset{1}{2}\,2\cdots 2\,1\,\overset{1}{4}\,2$$

(212)



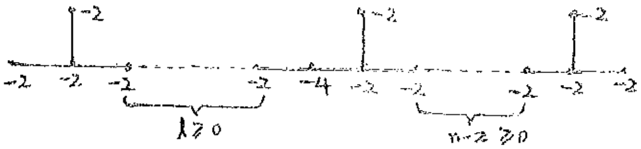
$$Z = 1\,\overset{1}{2}\,2\cdots 2\,1\,\overset{1}{3}\,1$$

(213)



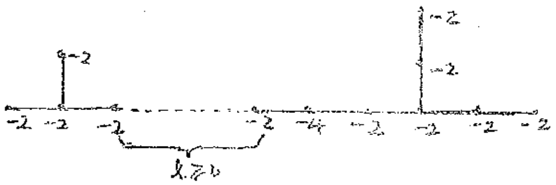
$$Z = 1\,\overset{1}{2}\,2\cdots 2\,1\,\underset{1}{\overset{1}{2}}\,1$$

(214)



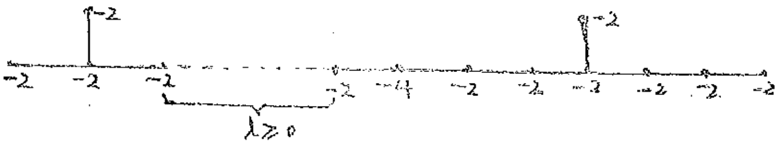
$$Z = 1\,\overset{1}{2}\,2\cdots 2\,1\,\overset{1}{2}\,2\cdots 2\,\overset{1}{2}\,1$$

(215)



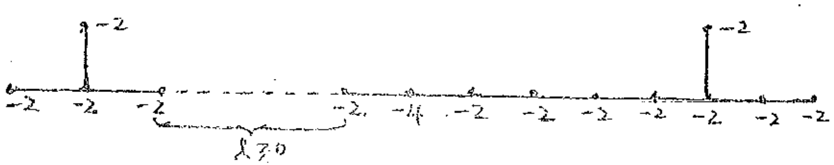
$$Z = 1\,\overset{1}{2}\,2\cdots 2\,1\,2\,\overset{\overset{1}{2}}{3}\,2\,1$$

(216)



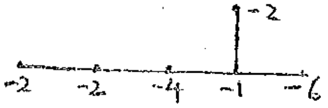
$$Z = 1\,\overset{1}{2}\,2\cdots 2\,1\,2\,3\,\overset{2}{4}\,3\,2\,1$$

(217)



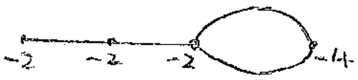
$$Z = 1\,\overset{1}{2}\,2\cdots 2\,1\,2\,3\,4\,5\,\overset{3}{6}\,4\,2$$

(218)



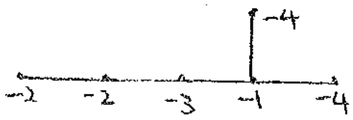
$Z = 1\,2\,2\,\overset{3}{6}\,1$

(219)



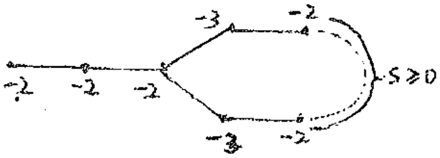
$Z = 1\,2\,2\,1$

(220)



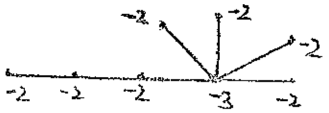
$Z = 1\,2\,2\,\overset{1}{4}\,1$

(221)



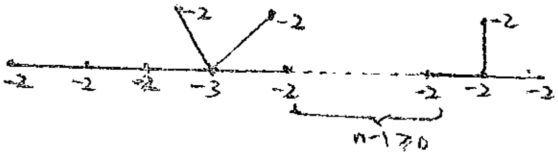
$Z = 1\,2\,2\,\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\,\cdots$

(222)



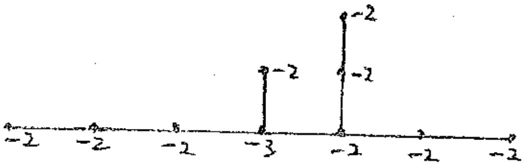
$Z = 1\,2\,2\,\overset{1\;1\;1}{2}\,1$

(223)



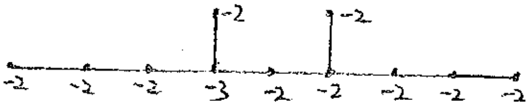
$Z = 1\,2\,2\,\overset{1\;1}{2}\,2\cdots2\,\overset{1}{2}\,1$

(224)



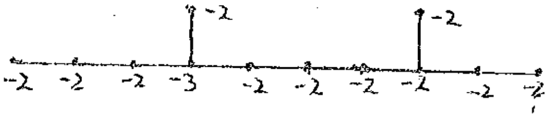
$Z = 1\,2\,2\,\overset{1}{2}\,\overset{\overset{1}{2}}{3}\,2\,1$

(225)



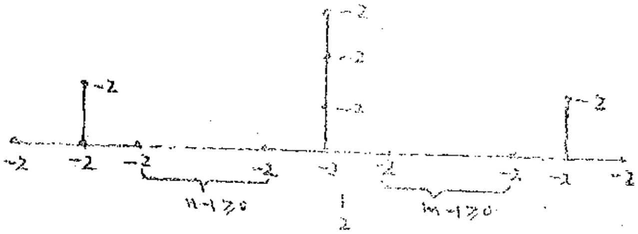
$Z = 1\,2\,2\,\overset{1}{2}\,3\,\overset{2}{4}\,3\,2\,1$

(226)



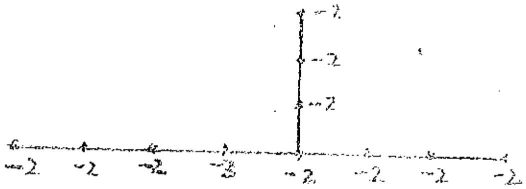
$Z = 1\,2\,2\,\overset{1}{2}\,3\,4\,5\,\overset{3}{6}\,4\,2$

(227)



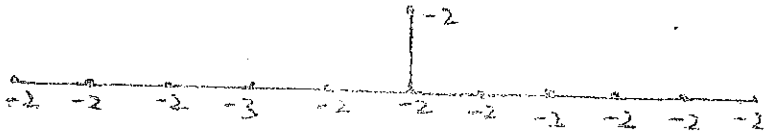
$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ \overset{\substack{1\\2\\2}}{2}\ 2 \cdots 2\ \overset{1}{2}\ 1$

(228)



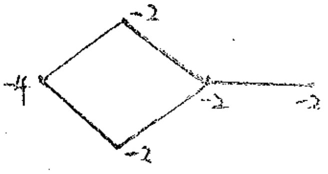
$Z = 1\ 2\ 2\ 2\ \overset{\substack{1\\2\\3}}{4}\ 3\ 2\ 1$

(229)



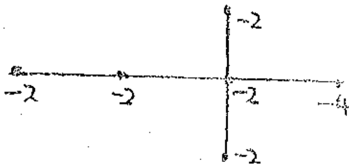
$Z = 1\ 2\ 2\ 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2\ 1$

(230)



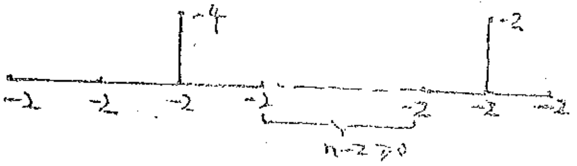
$Z = 1\ \begin{matrix}2\\2\end{matrix}\ 3\ 2$

(231)



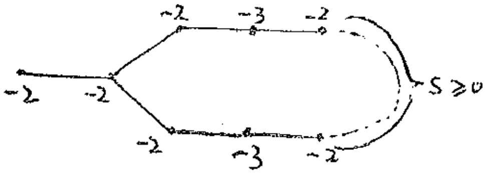
$Z = 2\ 3\ \overset{2}{\underset{2}{4}}\ 1$

(232)



$Z = 2\ 3\ \overset{1}{4}\ 4 \cdots 4\ \overset{2}{4}\ 2$

(233)

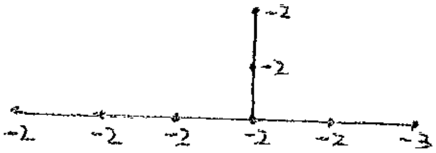


$Z = 2\,3\begin{smallmatrix}2 & 1 & 1 & \cdots \\ 2 & 1 & 1 & \cdots\end{smallmatrix}$

(234)

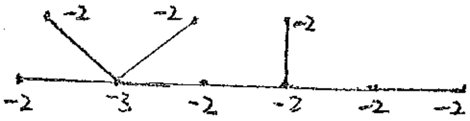
$Z = 2\;3\;\overset{2}{4}\;\overset{1}{3}\;1$

(235)



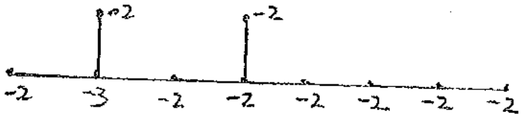
$Z = 2\;3\;4\;\overset{\overset{1}{3}}{5}\;3\;1$

(236)



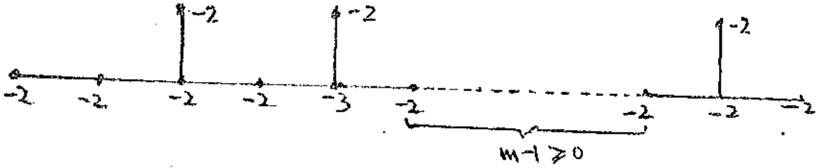
$Z = 1\;\overset{1\,1}{2}\;3\;\overset{2}{4}\;3\;2$

(237)



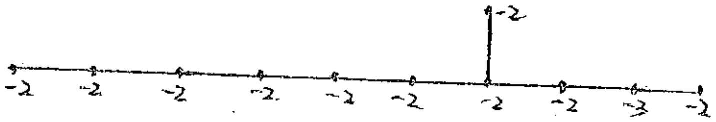
$Z = 1\;\overset{1}{2}\;4\;\overset{3}{6}\;5\;4\;3\;2$

(238)



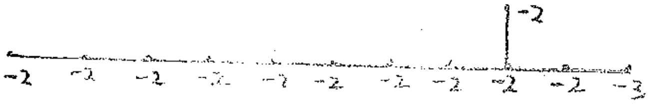
$Z = 2\;3\;\overset{2}{4}\;3\;\overset{1}{2}\;2\;\cdots\cdots\;2\;\overset{1}{2}\;1$

(239)



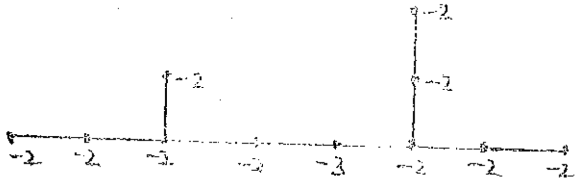
$Z = 2\;3\;4\;5\;6\;7\;\overset{4}{8}\;5\;2\;1$

(240)



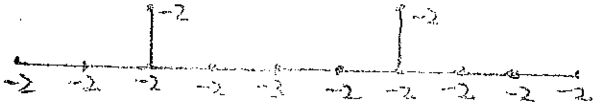
$Z = 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ \overset{5}{10}\ 6\ 2$

(241)



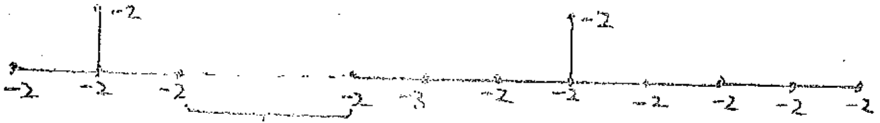
$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ \overset{1}{\overset{2}{3}}\ 2\ 1$

(242)



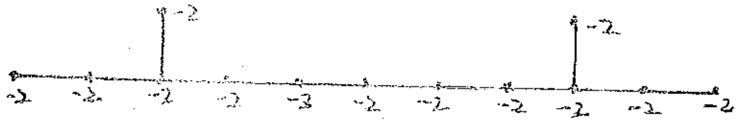
$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$

(243)



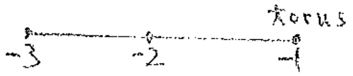
$Z = 1\ \overset{1}{2}\ 2 \cdots 2\ 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2$

(244)



$Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ 2$

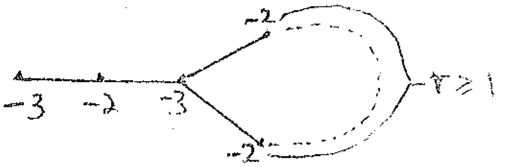
(245)



$Z = 1\ 1\ 1$

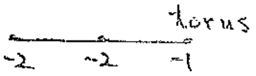
(246)

$Z = 1\ 1\ 1\ 2$



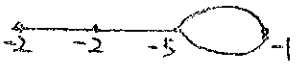
$Z = 1\ 1\ 1\ \begin{matrix}1\\1\end{matrix}$

(247)



Z = 1 1 1

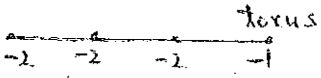
(248)



Z = 1 1 1 2

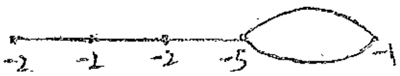
$Z = 1\ 1\ 1\ \begin{matrix}1\\1\end{matrix}$

(249)

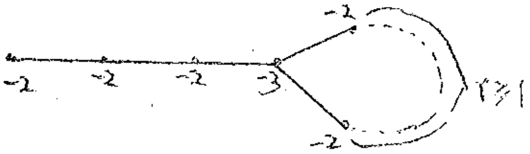


Z = 1 1 1 1

(250)



Z = 1 1 1 1 2



$Z = 1\ 1\ 1\ 1\ \begin{matrix}1\\1\end{matrix}$

Proof of Theorem 5.18: This is a consequence of Proposition 5.13, Proposition 5.14, Proposition 5.15 and Corollary 5.17.

By the virtue of Theorem 5.18 we have the following Theorem.

Theorem 5.19: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only singular point. Suppose $H^1(M,\mathcal{O}) \simeq \mathbb{C}^2$ and p is a hypersurface singularity. Let A be the exceptional set. If $H^1(|A|,Z) = 0$, then p is an almost minimally elliptic singularity.

Proof: The condition $H^1(|A|,Z) = 0$ rules out case (245), (246), (247), (248), (249) and (250) in Theorem 5.18. All the remaining cases in Theorem 5.18 are almost minimally elliptic. Q.E.D.

Remark 5.20: Using proposition 5.13, 5.14, and 5.15, we can list all possible weighted dual graphs of weakly elliptic singularities such that K' exists and $Z \cdot Z \leq -3$. By Corollary 5.2, we know that all hypersurface maximally elliptic singularities must be one of these forms. However, the list is too long to be included here. We remark only that the condition on the elliptic sequence of Theorem 4.7 is automatically satisfied if $Z \cdot Z \leq -3$ and K' exists.

CHAPTER VI

EXISTENCE PROBLEM

It is known that if ${}_V\mathcal{O}_p$ is Gorenstein, then the cycle K' exists. One can ask the following converse question. Given a weighted dual graph such that K' exists. Is there a singularity corresponding to the given weighted dual graph and which has Gorenstein structure? In section 1, we give a necessary and sufficient condition for the existence of Gorenstein structures for weakly elliptic singularities. In section 2, we give a positive answer to the above question for a very special kind of singularity.

§1 Necessary and Sufficient Condition for the Existence of Gorenstein Structure for Weakly Elliptic Singularities

Theorem 6.1: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only weakly elliptic singular point. Suppose K' exists. Let $Z_{B_o} = Z, \ldots, Z_{B_\ell}$, $Z_E = Z_{B_{\ell+1}}$ be the elliptic sequence. Then ${}_V\mathcal{O}_p$ is Gorenstein if and only if $H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} +E}) \to H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}})$ is surjective and $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=2}^{\ell} Z_{B_i} -E)$ is the sheaf of germs of sections of a trivial line bundle over $(|E|, \mathcal{O}_E)$.

Proof: "$\Longrightarrow$" Choose a computation sequence for Z as follows: $Z_o = 0, Z_1, \dots, Z_k = E, \dots$. By Theorem 3.7, $-K' = \sum_{i=0}^{\ell} Z_{B_i} + E$. So $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) = 0$. The exact sequence

$$H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$$
$$\to 0$$

shows that $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$ is an isomorphism. If $E = A_1$ is an elliptic curve, then either $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$ $\simeq 0 \simeq H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$ or $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$ $\simeq \mathbb{C} \simeq H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$ by Serre duality and Riemann Roch Theorem. If $|E|$ has at least two irreducible components, we consider the following sheaf exact sequence

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_1)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)$$
$$\to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(\sum_{i=0}^{\ell} Z_{B_i} - Z_1) \to 0$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_2)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_1)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)$$
$$\to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_1)/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_2) \to 0$$

$$\vdots \qquad\qquad\qquad\qquad\qquad \vdots$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E) \to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-2})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)$$

$$\to \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-2})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1}) \to 0.$$

The usual long cohomology exact sequences will show that either

$$H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \simeq 0 \simeq H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$$

or

$$H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \simeq \mathbb{C} \simeq H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)).$$

Thus either $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \simeq \mathbb{C}$ or $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) = 0$. We claim that $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \simeq \mathbb{C}$ and $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to H^1(M, \mathcal{O})$ is injective. Otherwise $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to H^1(M, \mathcal{O})$ is a zero map. As ${}_V\mathcal{O}_p$ is Gorenstein, there exists $\omega \epsilon H^1(M-A, \Omega)$ having no zeros near A and the image of ω in $H^o(M-A,\Omega)/H^o(M,\Omega)$ is nonzero. Let w_i be the order of the pole of ω on A_i. Consider a cover as in Lemma 3.8 of [24]. On p_1 where $A_1 \subsetneqq |E|$

$$\omega = \frac{\omega_1(x_1,y_1)}{y_1^{w_1}} dx_1 \wedge dy_1$$

where $\omega_1(x_1,y_1)$ is a holomorphic function, $\omega_1(x_1,0) \not\equiv 0$. There is a holomorphic function $f(x_1)$ $r \le |x_1| \le R$ such that

$$\int_{\substack{|x_1|=R\\ |y_1|=R}} y_1^{w_1-1} f(x_1) \frac{\omega_1(x_1,y_1)}{y_1^{w_1}} dx_1 \wedge dy_1 \neq 0$$

Let $\lambda_{o_1} = y_1^{w_1-1} f(x_1)$ and $\lambda_{o_j} = 0$ for $2 \le j \le t$. Then by Lemma 3.8 of [24], $\mathrm{cls}[\lambda] \neq 0$ in $H^1(M^*, \mathcal{O})$. Let $E = \Sigma e_i A_i$, $Z_{B_i} = \Sigma_{j}{}_{B_i} z_j A_j$. Then $w_1 = \sum_{i=0\,B_i}^{\ell} z_1 + e_1$ and $w_1 - 1 \geq \sum_{i=0\,B_i}^{\ell} z_1$. Hence λ may be thought of as also a cocycle in $H^1(N(\mathfrak{U}), \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}))$. It follows that $\mathrm{cls}[\lambda] = 0$ in $H^1(M', \mathcal{O})$. This leads to a contradiction. Our claim is proved. Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \arrow[r] \arrow[d] & H^0(M, \mathcal{O}) \arrow[r] \arrow[d] & H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) \arrow[d] \\
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \arrow[r] & H^0(M, \mathcal{O}) \arrow[r] & H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}})
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \arrow[r] \arrow[d] & H^1(M, \mathcal{O}) \arrow[r] \arrow[d] & H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \arrow[r] & H^1(M, \mathcal{O}) \arrow[r] & H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \arrow[r] & 0
\end{tikzcd}$$

It follows that $H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+E}) \to H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}})$ is surjective.

Look at the following exact sequence

$$0 \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}))$$

$$\to H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \simeq \mathbb{C} \to 0.$$

Let $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)$ correspond to a line bundle L over $(|E|, \mathcal{O}_E)$. There exists $f \in H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}))$ such that the image of f in $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$ viewed as a section of line bundle L is nowhere zero. So L is a trivial bundle.

" $\Longrightarrow$ " Suppose conversely that $H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+E})$ $\to H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}})$ is surjective and $\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$ is the sheaf of germs of sections of a trivial line bundle over $(|E|, \mathcal{O}_E)$. Then $H^o(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \simeq \mathbb{C} \simeq$ $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$. The exact sequence

$$0 \simeq H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to$$

$$H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-E)) \to 0$$

shows that $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-E)) \simeq \mathbb{C}$ is an isomorphism. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-E)) \arrow[r] \arrow[d] & H^0(M,\mathcal{O}) \arrow[r] \arrow[d] & H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+E}) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \arrow[r] & H^0(M,\mathcal{O}) \arrow[r] & H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-E)) \arrow[r] \arrow[d] & H^1(M,\mathcal{O}) \arrow[r] \arrow[d] & H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+E}) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \arrow[r] & H^1(M,\mathcal{O}) \arrow[r] & H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \arrow[r] & 0
\end{tikzcd}$$

Since $H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}+E}) \to H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}})$ is surjective, $H^0(M,\mathcal{O}) \to H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}})$ is also surjective. So $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \simeq \mathbb{C} \to H^1(M,\mathcal{O})$ is injective. Since $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i}-E)) \simeq \mathbb{C}$, the usual long cohomology exact sequence argument shows that

$$H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \cong \mathbb{C} \to H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E))$$

is an isomorphism. Look at the following commutative diagram with exact rows

$$\begin{tikzcd}
H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1})) \arrow[r, "\alpha"] \arrow[d, "\beta"] & H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \cong \mathbb{C} \arrow[r] \arrow[d] & 0 \\
H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \cong \mathbb{C} \arrow[r] \arrow[d, "\gamma"] & H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \cong \mathbb{C} \arrow[r] & 0 \\
H^1(M, \mathcal{O}) & &
\end{tikzcd}$$

There exists $\lambda \in H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - Z_{k-1}))$ such that $\alpha(\lambda) \neq 0 \neq \gamma \cdot \beta(\lambda)$. Use the notation of Lemma 3.8 of [24]. $\mathcal{U}$ is a Leray cover for A_1. So there exists $\{\lambda_{oi} = y_1^{\sum_{j=0}^{\ell} B_j z_1 + e_1 - 1} f_i(x_1)$ + higher power of y_1 }, $f_i(x_1)$ is holomorphic for $r \leq |x_1| \leq R$ such that $\text{cls}[\lambda] \neq 0$ in $H^1(M', \mathcal{O})$. Let ω be the element such that $<\lambda, \omega> \neq 0$. Then on p_1 by Lemma 3.8 of [24], we know that $w_1 > \sum_{i=0}^{\ell} {}_{B_i} z_1 + e_1 - 1$ where w_1 is the order of pole of ω on A_1. $(\omega) = [\omega] + D$ where D is a positive divisor which does not involve any A_i and $[\omega] = -\sum_i w_i A_i$. For any $A_i \subseteq A$, $-A_i \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E) = A_i \cdot (\omega) = A_i \cdot [\omega] + A_i \cdot D$,

$$\Longrightarrow \quad A_i \cdot ([\omega] + \sum_{i=0}^{\ell} Z_{B_i} + E + D) = 0 \qquad \text{for all } A_i \subseteq A$$

$$\Longrightarrow \quad A_i \cdot (\sum_{i=0}^{\ell} Z_{B_i} + E + [\omega]) \leq 0 \qquad \text{for all } A_i \subseteq A$$

Let $Y = \sum_{i=0}^{\ell} Z_{B_i} + E + [\omega] = \Sigma y_i A_i$. We have $y_i \leq 0$ and $A_i \cdot Y \leq 0$ for all $A_i \subseteq A$. By the proof of Theorem 3.11, this is possible only if $Y = 0$. It follows easily that $D = 0$ and $(\omega) = - \sum_{i=0}^{\ell} Z_{B_i} - E$. So ω has no zeros in a neighborhood of A, i.e., ${}_V\mathcal{O}_p$ is Gorenstein. Q.E.D.

Theorem 6.2: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space V with p as its only weakly elliptic singularity. Suppose K' exists. Let $Z_{B_0} = Z, \ldots, Z_{B_\ell}, Z_E$ be the elliptic sequence. Then p is a maximally elliptic singularity if and only if $H^0(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i} + E}) \to H^0(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}})$

is surjective for $0 \leq j \leq \ell$ and $\mathcal{O}(- \sum_{i=0}^{j} Z_{B_i}) / \mathcal{O}(- \sum_{i=0}^{j} Z_{B_i} - E)$ is the sheaf of germs of sections of a trivial line bundle over $(|E|, \mathcal{O}_E)$ for $0 \leq j \leq \ell$.

Proof: Let us first prove that $\dim H^0(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i} + E}) - 1 \leq$

$\dim H^0(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}}) \leq \dim H^0(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i} + E})$. We recall that

$\chi(\sum_{i=0}^{j} Z_{B_i} + E) = 0 = \chi(\sum_{i=0}^{j} Z_{B_i})$. The exact sequence

$$H^1(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i} + E}) \to H^1(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}}) \to 0$$

shows that

$$\dim H^{o}(M, \mathcal{O}_{\sum\limits_{i=0}^{j} Z_{B_i}}) = \dim H^{1}(M, \mathcal{O}_{\sum\limits_{i=0}^{j} Z_{B_i}})$$

$$\lesssim \dim H^{1}(M, \mathcal{O}_{\sum\limits_{i=0}^{j} Z_{B_i}+E})$$

$$= \dim H^{o}(M, \mathcal{O}_{\sum\limits_{i=0}^{j} Z_{B_i}+E})$$

By the proof of previous theorem, we know that either $H^{o}(M, \mathcal{O}(-\sum\limits_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum\limits_{i=0}^{j} Z_{B_i}-E)) \simeq 0 \simeq H^{1}(M, \mathcal{O}(-\sum\limits_{i=0}^{j} Z_{B_i})/\mathcal{O}(\sum\limits_{i=0}^{j} Z_{B_i}-E))$ or $H^{o}(M, \mathcal{O}(-\sum\limits_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum\limits_{i=0}^{j} Z_{B_i}-E)) \simeq \mathbb{C} \simeq H^{1}(M, \mathcal{O}(-\sum\limits_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum\limits_{i=0}^{j} Z_{B_i}-E))$.

The exact sequence

$$0 \to H^{o}(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-E)) \to H^{o}(M, \mathcal{O}_{\sum\limits_{i=0}^{j} Z_{B_i}+E})$$

$$\to H^{o}(M, \mathcal{O}_{\sum\limits_{i=0}^{j} Z_{B_i}}) \to H^{1}(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-E))$$

shows that $\dim H^{o}(M, \mathcal{O}_{\sum\limits_{i=0}^{j} Z_{B_i}}) \geq \dim H^{1}(M, \mathcal{O}_{\sum\limits_{i=0}^{j} Z_{B_i}+E}) - 1$. Choose a computation sequence for $Z_{B_{j+1}}$ as follows $Z_o = 0, Z_1, \ldots, Z_k = E, \ldots, Z_\ell = Z_{B_{j+1}}$. Consider the following sheaf exact sequence.

$$0 \to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{k+1})/\mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i}) \to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E)/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{B_{j+1}})$$

$$\to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E)/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{k+1}) \to 0$$

$$0 \to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{k+2})/\mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i}) \to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{k+1})/\mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})$$

$$\to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{k+1})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{k+2}) \to 0$$

.

$$0 \to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{\ell-1})/\mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i}) \to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{\ell-2})/\mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})$$

$$\to \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{\ell-2})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - Z_{\ell-1}) \to 0.$$

By the usual long cohomology exact sequence argument,

$$H^0(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E)/\mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})) \cong 0 \cong H^1(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E)/\mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})).$$

The following two exact sequences

$$0 \to H^0(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E)/\mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})) \to H^0(M, \mathcal{O}_{\sum_{i=0}^{j+1} Z_{B_i}})$$

$$\to H^0(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i} + E}) \to H^1(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E)/\mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i}))$$

$$\to H^1(M, \mathcal{O}_{\sum_{i=0}^{j+1} Z_{B_i}}) \to H^1(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i} + E}) \to 0$$

$$0 \to H^0(M, \mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})) \to H^0(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E))$$

$$\to H^0(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E) / \mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})) \to H^1(M, \mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i}))$$

$$\to H^1(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E)) \to H^1(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E) / \mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})) \to 0$$

shows that $H^d(M, \mathcal{O}_{\sum_{i=0}^{j+1} Z_{B_i}}) \to H^d(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i} + E})$ and $H^d(M, \mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})) \to H^d(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E))$ are the isomorphisms for $d = 0, 1$.

Suppose p is a maximally elliptic singularity. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \arrow[r] \arrow[d] & H^0(M, \mathcal{O}) \arrow[r] \arrow[d] & H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) \arrow[d] \\
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \arrow[r] \arrow[d, dash, dotted] & H^0(M, \mathcal{O}) \arrow[r] \arrow[d, dash, dotted] & H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \arrow[d, dash, dotted] \\
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})) \arrow[r] \arrow[d] & H^0(M, \mathcal{O}) \arrow[r] \arrow[d] & H^0(M, \mathcal{O}_{\sum_{i=0}^{j+1} Z_{B_i}}) \arrow[d] \\
0 \arrow[r] & H^0(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})) \arrow[r] \arrow[d, dash, dotted] & H^0(M, \mathcal{O}) \arrow[r] \arrow[d, dash, dotted] & H^0(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}}) \arrow[d, dash, dotted] \\
0 \arrow[r] & H^0(M, \mathcal{O}(-Z)) \arrow[r] & H^0(M, \mathcal{O}) \arrow[r] & H^0(M, \mathcal{O}_Z)
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) \arrow[r] \arrow[d] & H^1(M, \mathcal{O}) \arrow[r] \arrow[d] & H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i})) \arrow[r] \arrow[d, dotted] & H^1(M, \mathcal{O}) \arrow[r] \arrow[d, dotted] & H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i}}) \arrow[r] \arrow[d, dotted] & 0 \\
{} \arrow[r] & H^1(M, \mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})) \arrow[r] \arrow[d] & H^1(M, \mathcal{O}) \arrow[r] \arrow[d] & H^1(M, \mathcal{O}_{\sum_{i=0}^{j+1} Z_{B_i}}) \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & H^1(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})) \arrow[r] \arrow[d, dotted] & H^1(M, \mathcal{O}) \arrow[r] \arrow[d, dotted] & H^1(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}}) \arrow[r] \arrow[d, dotted] & 0 \\
{} \arrow[r] & H^1(M, \mathcal{O}(-Z)) \arrow[r] & H^1(M, \mathcal{O}) \arrow[r] & H^1(M, \mathcal{O}_Z) \arrow[r] & 0
\end{tikzcd}$$

Since $H^1(M, \mathcal{O}(-\sum_{i=0}^{\ell} Z_{B_i} - E)) = 0$, $\dim H^0(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) = \dim H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E})$ $= \dim H^1(M, \mathcal{O}) = \ell + 2$. It follows that $H^0(M, \mathcal{O}_{\sum_{i=0}^{j+1} Z_{B_i}}) \to H^0(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}})$ is surjective for all $0 \leq j \leq \ell$ and $\dim H^0(M, \mathcal{O}_{\sum_{i=0}^{j+1} Z_{B_i}}) = \dim H^0(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}}) + 1$.

Moreover, $H^0(M,\mathcal{O}(-\sum_{i=0}^{j+1} Z_{B_i})) \to H^0(M,\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}))$ is not an isomorphism for all $0 \leq j \leq \ell$. The exact sequence

(6.1)

$$0 \to H^0(M,\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-E)) \to H^0(M,\mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}+E})$$

$$\to H^0(M,\mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}}) \to 0$$

shows that $H^0(M,\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-E)) \simeq \mathbb{C}$. We have the following exact sequence

$$0 \to H^0(M,\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-E)) \to H^0(M,\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}))$$

$$\to H^0(M,\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-E)) \simeq \mathbb{C} \to 0.$$

It follows readily that $\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-E)$ is a sheaf of germs of sections of a trivial line bundle over $(|E|,\mathcal{O}_E)$ for $0 \leq j \leq \ell$.

Conversely, suppose $H^0(M,\mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}+E}) \to H^0(M,\mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}})$ is surjective for $0 \leq j \leq \ell$, and $\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}-E)$ is the sheaf of germs of sections of a trivial line bundle over $(|E|,\mathcal{O}_E)$

for $0 \leq j \leq \ell$. From (6.1) since $H^o(M, \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i})/\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - E)) \simeq \mathbb{C}$, $\dim H^o(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i} + E}) = \dim H^o(M, \mathcal{O}_{\sum_{i=0}^{j} Z_{B_i}}) + 1$. By induction,

$\dim H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) = \ell + 1 + \dim H^o(M, \mathcal{O}_Z) = \ell + 2$. However,

$\dim H^1(M, \mathcal{O}) = \dim H^1(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) = \dim H^o(M, \mathcal{O}_{\sum_{i=0}^{\ell} Z_{B_i} + E}) = \ell + 2$.

So p is a maximally elliptic singularity.

Corollary 6.3: Let π: M → V be the minimal good resolution of normal two dimensional Stein spaces with p as its only weakly elliptic singularity. Suppose p is an almost minimally elliptic singularity but not a minimally elliptic singularity and K' exists. Then ${}_V\mathcal{O}_p$ is Gorenstein if and only if $H^o(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) \simeq \mathbb{C}$.

§2 Existence Theorem for Almost Minimally Quasi-Simple Elliptic Singularities

Definition 6.4: Let π: M → V be the minimal good resolution of a normal two dimensional Stein space with p as its only weakly elliptic singularity. If the minimally elliptic cycle $E = A_1$ is a nonsingular elliptic curve, we say that p is a quasi-simple elliptic singularity.

Theorem 6.5: Let $\pi: M \to V$ be the minimal good resolution of normal two dimensional Stein space with p as its only quasi-simple elliptic singularity. Let Γ denote the weighted dual graph along with the genera of the A_i. Suppose K' exists and $Z_{B_o}=Z$, $Z_{B_1}, \ldots, Z_{B_{\ell+1}}=Z_E=A_1$ is the elliptic sequence. Let L_j be the line bundle corresponding to $\mathcal{O}(-\sum_{i=0}^{j} Z_{B_i}) \otimes \mathcal{O}(-\sum_{i=0}^{j} Z_{B_i} - A_1)$ $0 \leq j \leq \ell$ over the elliptic curve A_1. Then we can deform a suitably large infinitisimal neighborhood B of the exceptional set of p such that L_j are trivial bundles over A_1 simultaneously and that Γ is preserved.

Proof: Let $Z_{B_i} = \sum_h {}_{B_i}z_h A_h$. Then ${}_{B_o}z_1 = {}_{B_1}z_1 = \cdots = {}_{B}z_1 = e_1 = 1$ where $E = \sum e_h A_h$. For $0 \leq j \leq \ell$, L_j are line bundles of chern class zero over the elliptic curve A_1. Let N be the normal bundle of A_1 in M. Let $Z_{B_i} = A_1 + \sum_{j=2}^{n} {}_{B_1}z_j A_j + D_i$ where $A_1, \ldots, A_n$ are distinct, $A_1 \cdot A_j = 1$ $2 \leq j \leq n$ and D_i is a positive cycle which does not involve $A_1, \ldots, A_n$ and $A_1 \cdot D_i = 0$. Then $L_j = N^{-j} \xi_{p_2}^{-\sum_{i=0}^{j} {}_{B_i}z_2} \cdots \xi_{p_n}^{-\sum_{i=0}^{j} {}_{B_i}z_n}$ where $p_i = A_1 \cap A_i$ $2 \leq i \leq n$. We want to show that by varying the point of intersection in $A_1 \cap A_i$, $2 \leq i \leq n$, we can vary L_j in the Picard variety of A_1 and make L_j trivial simultaneously for all $0 \leq j \leq \ell$. Since $c(N^{-1}) = -A_1 \cdot A_1$, $c(N^{-1}\xi_{p_2}^{1+A_1 \cdot A_1}) = 1$. By Riemann Roch Theorem, there is a nontrivial section $f \in \Gamma(A_1, \mathcal{O}(N^{-1}\xi_{p_2}^{1+A_1 \cdot A_1}))$. $\mathcal{V}(f) = 1 \cdot q$ for some $q \varepsilon A_1$. It is then clear that $N^{-1}\xi_{p_2}^{1+A_1\cdot A_1} = \xi_q$; for if $f_1 \varepsilon \Gamma(A_1, \mathcal{O}(\xi_q))$ is a nontrivial section of the point bundle so that f_1 vanishes precisely at q, then $f/f_1 \varepsilon H^o(A_1, \mathcal{O}(N^{-1}\xi_{p_2}^{1+A_1\cdot A_1}\xi_q^{-1})$ is a holomorphic nowhere-vanishing section, so necessarily $N^{-1}\xi_{p_2}^{1+A_1\cdot A_1}\xi_q^{-1} = 1$. Consequently $N^{-1} = \xi_{p_2}^{-1-A_1\cdot A_1}\xi_q$ and hence $L_j = \xi_{p_2}^{-j-jA_1\cdot A_1}\ \xi_q^{\ j}\ \xi_{p_2}^{-\sum_{i=0}^{j} B_i z_2} \cdots \xi_{p_n}^{-\sum_{i=0}^{j} B_i z_n}$. Since

$A_1 \cdot \sum_{i=0}^{j} Z_{B_i} = 0$, we have $j(A_1\cdot A_1) + \sum_{i=0}^{j} B_i z_2 + \ldots + \sum_{i=0}^{j} B_i z_n = 0$.

Hence

$$L_j = (\xi_{p_2}^{-j}\ \xi_q^{j})(\xi_{p_2}\ \xi_{p_3}^{-1})^{\sum_{i=0}^{j} B_i z_3} \ldots\ldots(\xi_{p_2}\ \xi_{p_n}^{-1})^{\sum_{i=0}^{j} B_i Z_n}$$

$$= (\xi_{p_2}^{-1}\ \xi_q)^{j}\ (\xi_{p_2}\xi_q^{-1}\ \xi_q\ \xi_{p_3}^{-1})^{\sum_{i=0}^{j} B_i z_3} \ldots.(\xi_{p_2}\xi_q^{-1}\xi_q\xi_{p_n}^{-1})^{\sum_{i=0}^{j} B_i z_n}$$

$$= (\xi_{p_2}\xi_q^{-1})^{\sum_{h=3}^{n}\sum_{i=0}^{j} B_i z_h - j}\ (\xi_{p_3}\ \xi_q^{-1})^{\sum_{i=0}^{j} B_i z_3} \ldots(\xi_{p_n}\xi_q)^{\sum_{i=0}^{j} B_i z_n}$$

Now Abel's Theorem says that by varying the point of intersection in $A_1 \cap A_i = \{p_i\}$, $2 \leq i \leq n$, we can vary $\xi_{p_i} \xi_q^{-1}$ in the Picard variety of A_1 and make $\xi_{p_i} \xi_q^{-1}$ trivial so that L_j is a trivial line bundle simultaneously for all $0 \leq j \leq \ell$.

Corollary 6.6: The hypothesis is the same as Theorem 6.5. If Γ is also an almost minimally elliptic singularity. Then we can deform a suitable large infinitesimal neighborhood B of the exceptional set of p such that ${}_V\mathcal{O}_p$ is Gorenstein.

Proof: Trivial consequence of Corollary 6.4 and Theorem 6.5.

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