A CLASS OF TOEPLITZ OPERATORS ON BALLS IN $\mathbb{C}^n$

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by

Gerard McDonald

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STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

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Abstract of the Dissertation

A Class of Toeplitz Operators on Balls in $\mathbb{C}^n$

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We obtain necessary and sufficient conditions for the Fredholmness of Toeplitz operators on the unit ball B^{2n} in $\mathbb{C}^n$ with symbols in the closed subalgebra $\mathfrak{A}$ of $L^\infty(B^{2n})$ generated by the bounded holomorphic functions on B^{2n} and the continuous functions on $\bar{B}^{2n}$. Using a Hölder norm bound on solutions of the equation $\bar{\partial}u = f$ on B^{2n} due to Kerzman, we obtain a factorization result for functions in $\mathfrak{A}$. We show that a Toeplitz operator with symbol in $\mathfrak{A}$ is Fredholm if and only if its symbol is bounded away from zero in a neighborhood of the boundary of B^{2n}. If the operator is Fredholm and n is greater than 1, it has index zero.

We also show that the closed subalgebra generated by Toeplitz operators with symbols in $\mathfrak{A}$ modulo the compacts is naturally isometrically isomorphic to $\mathfrak{A}$ modulo the ideal of functions in $C(\bar{B}^{2n})$ which vanish on the boundary of B^{2n}.

The maximal ideal spaces of these algebras are shown to be naturally homeomorphic to the maximal ideal space of the bounded holomorphic functions on B^{2n} minus the evaluation functionals.

Dedication

I give this thesis as a small gift to my parents, family, and friends, for their great love. Amor vincit omnia!

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CHAPTER 0. INTRODUCTION.

In this thesis we will study a class of Toeplitz operators on the Hilbert space $H^2(B^{2n})$ of square integrable holomorphic functions on the unit ball B^{2n} in $\mathbb{C}^n$. Specifically we will consider Toeplitz operators T_φ with symbol φ in the closed subalgebra of $L^\infty(B^{2n})$ generated by $C(\bar{B}^{2n})$ and $H^\infty(B^{2n})$, respectively the continuous functions on $\bar{B}^{2n}$ and the bounded holomorphic functions on B^{2n}.

Let $\mathfrak{A}$ be the above closed subalgebra, $\mathcal{L}(H^2(B^{2n}))$ the bounded linear operators on $H^2(B^{2n})$ and $\mathcal{K}$ the ideal of compact operators. We will answer two main questions:

1) If $\varphi \in \mathfrak{A}$ when is T_φ Fredholm?

2) If $\mathfrak{J}(\mathfrak{A})$ is the closed subalgebra of $\mathcal{L}(H^2(B^{2n}))$ generated by $\{T_\varphi : \varphi \in \mathfrak{A}\}$, can $\mathfrak{J}(\mathfrak{A})/\mathcal{K}$ be identified with $\mathfrak{A}$ modulo some ideal?

For Toeplitz operators on the unit circle $\mathbb{T}$ questions 1) and 2) have been answered very neatly for the subalgebras $C(\mathbb{T})$ and $H^\infty+C(\mathbb{T})$ of $L^\infty(\mathbb{T})$. The methods used often involved properties of holomorphic functions on the disc peculiar to (or at least apparently peculiar to) the one variable case. Recently complex analysts have been obtaining very delicate bounds, in various norms, for solutions of the equation $\bar{\partial}u = f$ on strongly pseudoconvex domains in $\mathbb{C}^n$. These results allow one to avoid the above mentioned difficulties and answer 1) and 2) for $C(\bar{B}^{2n})$ and, as we will show in this paper, for $\mathfrak{A}$.

Venugopalkrishna was the first to note the connection between some of these $\bar{\partial}$-results and Toeplitz operators with symbols in $C(\bar{B}^{2n})$. In [17] he answered one half of 1) for functions continuous on the closure of any strongly pseudoconvex domain. Coburn ([3]) answered the other half of 1) and question 2) for $\bar{B}^{2n}$. His work involves properties of the Bergman kernel for B^{2n} and so the results are not immediately extendable to arbitrary strongly pseudoconvex domains. We will have more to say about this in the conclusion.

The similarity of the answers for $C(\mathbb{T})$ and $C(\bar{B}^{2n})$ led us to believe that the answers for $H^{\infty}+C(\mathbb{T})$ and $\mathfrak{A}$ would also be similar to each other and in fact they are.

In Chapter I we will present the relevant facts concerning Toeplitz operators on the circle, in particular those concerning Toeplitz operators with symbols in $C(\mathbb{T})$ and $H^{\infty}+C(\mathbb{T})$. We will also consider the results of Venugopalkrishna and Coburn for Toeplitz operators with symbols in $C(\bar{B}^{2n})$.

Chapter II will contain the necessary facts about forms on B^{2n}, a statement of the relevant $\bar{\partial}$ result and some corollaries. Our main result, that if φ is in $\mathfrak{A}$ then T_{φ} is Fredholm if and only if φ is bounded away from zero on a neighborhood of the boundary of the ball, is contained in Chapter III.

Chapters IV and V are more or less corollaries to III. We show that $\mathfrak{J}(\mathfrak{A})/\mathcal{K}$ is naturally isometrically isomorphic to $\mathfrak{A}$ modulo the ideal of those functions in $C(\bar{B}^{2n})$ which are identically zero on the boundary. In V we compute the maximal ideal space of this quotient algebra, and, among other things, show that it is connected. In Chapter VI we consider matrices of Toeplitz operators with symbols in $\mathfrak{A}$. We show that such operators are Fredholm if and only if the determinant of their matrix symbol is bounded away from zero on a neighborhood of the boundary.

We now establish some notation. As we previously said $H^2(B^{2n})$ is the complex Hilbert space of all square-integrable holomorphic functions on the open unit ball B^{2n} in $\mathbb{C}^n$. The bounded holomorphic functions on B^{2n} will be written $H^\infty(B^{2n})$ and the bounded continuous functions on B^{2n} will be written $\mathcal{B}(B^{2n})$. The closed subalgebra of $L^\infty(B^{2n})$ generated by $H^\infty(B^{2n})$ and $C(\bar{B}^{2n})$ (or equivalently by $H^\infty(B^{2n}) + C(\bar{B}^{2n}) = \{f+g : f \in H^\infty(B^{2n}), g \in C(\bar{B}^{2n})\}$) will be denoted by $\mathfrak{A}$. Clearly $\mathfrak{A}$ is contained in $\mathcal{B}(B^{2n})$, so it makes sense to talk about the value of an element of $\mathfrak{A}$ at a point in B^{2n}. We will let $\mathcal{J}_0$ be the ideal in $\mathcal{B}(B^{2n})$ of all continuous functions on $\bar{B}^{2n}$ which are identically zero on the boundary.

The norm we use on $\mathbb{C}^n$ will be the euclidean norm, i.e. for $\zeta = (\zeta_1,\ldots,\zeta_n)$ in $\mathbb{C}^n$

$$|\zeta|^2 = \Sigma|\zeta_i|^2.$$

In terms of this norm, $B^{2n} = \{\zeta \in \mathbb{C}^n : |\zeta| < 1\}$. The boundary of B^{2n} will be written ∂B^{2n}. On the unit circle in $\mathbb{C}$, the

Hardy spaces H^2 and H^∞ are defined

$$H^2 = \{f \in L^2(\mathbb{T}) : \int_0^{2\pi} f(e^{i\theta})e^{in\theta}d\theta = 0,\ n > 0\}$$

and

$$H^\infty = \{f \in L^\infty(\mathbb{T}) : \int_0^{2\pi} f(e^{i\theta})e^{in\theta}d\theta = 0,\ n > 0\}.$$

Alternatively we can write

$$H^\infty = \{f \in L^\infty(B^2) : f \text{ holomorphic on } B^2\}.$$

The collection of bounded linear operators on $H^2(B^{2n})$ and H^2 will be written respectively $\mathcal{L}(H^2(B^{2n}))$ amd $\mathcal{L}(H^2)$. The compact operators on both spaces will be denoted by $\mathcal{K}$. Elements of the quotient algebra $\mathcal{L}(H^2(B^{2n}))/\mathcal{K}$ will be written in the form $S+\mathcal{K}$ where S is in $\mathcal{L}(H^2(B^{2n}))$. Elements of the other quotient algebras we consider will be written in the same manner. All norms on quotient algebras will be quotient norms.

CHAPTER I. KNOWN RESULTS ON $\mathbb{T}$ AND B^{2n}.

In this section we will consider Toeplitz operators on $H^2(B^{2n})$ with symbols in $C(\bar{B}^{2n})$ and show that such an operator is Fredholm if and only if its symbol is bounded away from zero on a neighborhood of the boundary of B^{2n}. Before doing this we will briefly consider Toeplitz operators on $\mathbb{T}$. For a more detailed discussion see Douglas [5].

Recall that for any φ belonging to $L^\infty(T)$, the Toeplitz operator T_φ is defined on the Hardy space H^2 by

$$T_\varphi f = P\varphi f$$

where P is the orthogonal projection of $L^2(T)$ onto the closed subspace H^2.

<u>Proposition 1.0</u>. The map $\tau : L^\infty(T) \to \mathcal{L}(H^2)$ defined by

$$\tau(\varphi) = T_\varphi$$

is a *-isometric linear map. Restricted to H^∞, τ is multiplicative. If φ belongs to $L^\infty(T)$ and f to $C(T)$ then $\tau(\varphi)\tau(f) - \tau(\varphi f)$ is compact.

If S is a closed subalgebra of $L^\infty(T)$ then one defines $\mathcal{J}(S)$ to be the closed subalgebra of $\mathcal{L}(H^2)$ generated by the set $\{T_\varphi : \varphi \in S\}$.

<u>Definition 1.1</u>. Let H be a Hilbert space and let T belong to $\mathcal{L}(H)$. Then T is said to be Fredholm if $T+\mathcal{K}$ is invertible

in $\mathcal{L}(H)/\mathcal{K}$ where $\mathcal{K}$ is the closed ideal of compact operators on H. Equivalently T is Fredholm if dim ker T and dim ker T* are finite and the range of T is closed. The index of a Fredholm operator, denoted j(T), is defined

$$j(T) = \dim \ker T - \dim \ker T^*.$$

For certain subalgebras S one can reduce questions of the Fredholmness of T_φ, φ belonging to S, to questions about the behaviour of the function φ. We next consider two such subalgebras.

Suppose first that $S = C(\mathbb{T})$. We note that for f and g in $C(\mathbb{T})$

$$\tau(f)\tau(g) - \tau(g)\tau(f) \in \mathcal{K},$$

and hence $\mathcal{T}(C(\mathbb{T}))/\mathcal{K}$ is a commutative C*-algebra. It can be shown ([1] and [2]) that $\mathcal{T}(C(\mathbb{T}))$ is *-isometrically isomorphic to $C(\mathbb{T})$ via the map

$$T_f + \mathcal{K} \to f.$$

Thus in particular the operator T_f is Fredholm if and only if f is nonzero on $\mathbb{T}$. Furthermore, one can show that if T_f is Fredholm the index of T_f is equal to minus the winding number of the curve traced out by f with respect to the origin.

For our second example we consider

$$H^{\infty}+C(\mathbb{T}) = \{f+g : f \in H^{\infty}, g \in C(\mathbb{T})\}.$$

It can be shown that $H^{\infty}+C(\mathbb{T})$ is a closed subalgebra of $L^{\infty}(\mathbb{T})$ ([4]). Douglas showed that $\mathfrak{J}(H^{\infty}+C(\mathbb{T}))/\mathcal{K}$ is isometrically isomorphic to $H^{\infty}+C(T)$ where the map is again

$$T_{\varphi}+\mathcal{K} \to \varphi.$$

He also showed that T_{φ} is Fredholm if and only if φ is invertible in $H^{\infty}+C(\mathbb{T})$ if and only if the harmonic extension of φ to the open disc is bounded away from zero on a neighborhood of the boundary. Furthermore he was able to obtain a formula for the index of a Fredholm operator in terms of its harmonic extension. The analysis of this example is made more difficult than the first because $H^{\infty}+C(T)$ is not a C*-algebra.

We now consider Toeplitz operators on $H^2(B^{2n})$. First we need a definition.

Definition 1.2. A normed vector space V of functions on a set S is called a proper functional space if for every s in S there is a constant C(s) depending only on s such that

$$|f(s)| \leq C(s)\ \|f\|$$

for all f in V.

Of course $H^2(B^{2n})$ is a proper functional space, with reproducing kernel

$$K(\lambda,\zeta) = \frac{n!}{\pi^n}(1-\bar{\lambda}\cdot\zeta)^{-n-1}$$

where $|\zeta| \leq 1$, $|\lambda| < 1$, $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$ and $\lambda \cdot \zeta = \bar{\lambda}_1 \zeta_1 + \ldots + \bar{\lambda}_n \zeta_n$. For details see [15]. Specifically we have for each φ in $H^2(B^{2n})$ and ζ in B^{2n}

$$\varphi(\zeta) = (\varphi, K(\zeta, \cdot)),$$

where the inner product is the usual L^2-inner product. Since $H^2(B^{2n})$ is a closed subspace of $L^2(B^{2n})$ there is an orthogonal projection of $L^2(B^{2n})$ onto $H^2(B^{2n})$ which we will designate by P. If ζ in B^{2n} is fixed then $K(\zeta, \cdot)$ belongs in $H^2(B^{2n})$ and so for φ in $L^2(B^{2n})$

$$(\varphi, K(\zeta, \cdot)) = (P\varphi, K(\zeta, \cdot)) = P\varphi(\zeta).$$

Definition 1.3. For φ in $L^\infty(B^{2n})$ the Toeplitz operator T_φ with symbol φ is defined

$$T_\varphi f = P\varphi f, \qquad f \text{ in } H^2(B^{2n}).$$

For S a closed subalgebra of $L^\infty(B^{2n})$, $\mathcal{T}(S)$ is the closed subalgebra of $\mathcal{L}(H^2(B^{2n}))$ generated by $\{T_\varphi : \varphi \in S\}$.

As in the circle case, one can define a map

$$\tau : L^\infty(B^{2n}) \to \mathcal{L}(H^2(B^{2n}))$$

by

$$\tau(\varphi) = T_\varphi.$$

Proposition 1.4. The map τ is a *-linear map. Restricted to $H^\infty(B^{2n})$, τ is multiplicative.

Proof. The first statement is proved in the same manner as on the circle ([5; p. 178]) and is trivial. Since the product of two holomorphic functions is again a holomorphic function we have for φ and ψ in $H^\infty(B^{2n})$

$$\begin{aligned} T_\varphi T_\psi f &= T_\varphi P\psi f \\ &= T_\varphi \psi f \\ &= P\varphi\psi f \\ &= \varphi\psi f \\ &= T_{\varphi\psi} f, \end{aligned}$$

and thus the second statement is proved. □

The above proof actually shows that for any φ in $L^\infty(B^{2n})$ and ψ in $H^\infty(B^{2n})$

$$T_\varphi T_\psi = T_{\varphi\psi},$$
$$T_{\bar\psi} T_\varphi = T_{\bar\psi\varphi}.$$

Venugopalkrishna [17] showed that if φ and ψ are smooth (C^∞)-functions on a strongly pseudoconvex domain Ω in $\mathbb{C}^n$ which are continuous on $\bar\Omega$ then $T_\varphi T_\psi - T_{\varphi\psi}$ is compact. We will define strong pseudoconvexity and show that B^{2n} is such a domain in the next section. The basic idea of his proof is that for any φ, ψ in $L^\infty(\Omega)$

$$\begin{aligned} T_{\varphi\psi} - T_\varphi T_\psi &= (PM_\varphi M_\psi - PM_\varphi PM_\psi)|H^2(\Omega) \\ &= PM_\varphi(I-P)M_\psi|H^2(\Omega) \end{aligned}$$

where M_φ and M_ψ are the usual multiplication operators on $L^2(\Omega)$. One then shows that $(I-P)M_\psi|H^2(\Omega)$ is a compact operator if ψ is continuous on $\bar{B}^{2n}$ and smooth on the interior. This depends on a result of Kohn on the solution of the $\bar{\partial}$-Neumann problem. For details see Folland and Kohn [7]. The fact that φ is smooth is used nowhere in this proof. We will have more to say about the $\bar{\partial}$ operator in the next chapter.

Proposition 1.5. If φ is in $L^\infty(B^{2n})$ and f in $C(\bar{B}^{2n})$ then

$$T_\varphi T_f - T_{\varphi f} \in \mathcal{K}.$$

Proof. Choose a sequence $\{f_m\}$ of functions in $C(\bar{B}^{2n})$ which are smooth on B^{2n} and such that f_m converges to f in the sup norm. By the preceeding remarks there exists for each m a compact operator K_m such that

$$K_m = T_\varphi T_{f_m} - T_{\varphi f_m} \to T_\varphi T_f - T_{\varphi f}$$

in norm. Since $\mathcal{K}$ is closed the result follows. ▫

The above result is of course true for any strongly pseudoconvex domain. We have now shown that τ defined on $L^\infty(B^{2n})$ has the properties of τ on $L^\infty(\mathbb{T})$, as given in Proposition 1.0, except the property of being an isometry.

We now look at $\mathcal{T}(C(\bar{B}^{2n}))$.

Lemma 1.6. Let B_ε be the open ball of radius ε about the origin and let φ in $L^\infty(B^{2n})$ be zero on $B^{2n}\setminus\bar{B}_\varepsilon$. Then T_φ is compact.

Proof. Consider the map

$$R_\varepsilon : H^2(B^{2n}) \to L^2(B^{2n})$$

defined by

$$R_\varepsilon(f) = f\chi_\varepsilon,$$

where χ_ε is the characteristic function for $\bar{B}_\varepsilon$. Using the fact that $\sup_{\bar{B}_\varepsilon}|f(\zeta)| \leq C\|f\|_2$ for f in $H^2(B^{2n})$ where C is independent of f, and applying the Arzela-Ascoli theorem, it is easy to see that R_ε is compact (see [9; p. 12] and [7; p. 81] for details). If φ vanishes off $\bar{B}_\varepsilon$ we have

$$\begin{aligned} T_\varphi f &= PM_\varphi f \\ &= PM_\varphi R_\varepsilon f. \end{aligned}$$

Thus $T_\varphi = PM_\varphi R_\varepsilon$ and so T_φ is compact. ∎

The next result is the main result in this section and is due to Venugopalkrishna [17] and Coburn [3].

Proposition 1.7. The ideal $\mathcal{K}$ is contained in $\mathfrak{J}(C(\bar{B}^{2n}))$. The C*-algebra $\mathfrak{J}(C(\bar{B}^{2n}))/\mathcal{K}$ is commutative and is *-isometrically isomorphic to $C(\partial B^{2n})$. The isometry is the map

$$\varphi \to T_{\tilde{\varphi}} + \mathcal{K},$$

where $\tilde{\varphi}$ is any continuous extension of φ to all of $\bar{B}^{2n}$.

We only sketch the proof here which can be found in [3]. Coburn showed that $\mathcal{K}$ is contained in $\mathcal{J}(C(\bar{B}^{2n}))$ by showing that $\mathcal{J}(C(\bar{B}^{2n}))$ is irreducible and contains at least one compact operator. From 1.5 it follows that $\mathcal{J}(C(\bar{B}^{2n}))/\mathcal{K}$ is commutative. He shows that if φ is in $C(\bar{B}^{2n})$ then T_φ is compact if and only if $\varphi|\partial B^{2n} = 0$ and so $\varphi \to T_{\tilde{\varphi}} + \mathcal{K}$ is a well-defined one-one map. The map is clearly a *-homomorphism and so is an isometry. Finally the map is onto since its range contains the dense set $\{T_\psi + \mathcal{K} : \psi \in C(\bar{B}^{2n})\}$. This completes the proof.

Since $\mathcal{J}(C(\bar{B}^{2n}))/\mathcal{K}$ is a self-adjoint subalgebra of $\mathcal{L}(H^2(B^{2n}))/\mathcal{K}$, $T_\varphi + \mathcal{K}$ is invertible in $\mathcal{J}(C(\bar{B}^{2n}))/\mathcal{K}$ if and only if it is invertible in $\mathcal{L}(H^2(B^{2n}))/\mathcal{K}$. Thus it follows from Proposition 1.7 that for φ in $C(\bar{B}^{2n})$, T_φ is Fredholm if and only if $\varphi|\partial B^{2n}$ is nonzero. Venugopalkrishna [17] showed that if T_φ is Fredholm then the index of T_φ is equal to minus the winding number of the curve traced out by $\varphi|\partial B^{2n}$ with respect to the origin if $n = 1$, and is zero if $n > 1$.

Before proceeding to the next section we obtain a corollary to Lemma 1.6 which we will need later.

<u>Corollary 1.8</u>. $\|T_\varphi + \mathcal{K}\| \le \lim\limits_{\varepsilon \to 0} \sup\limits_{1 > |\zeta| > 1-\varepsilon} |\varphi(\zeta)|$ for all φ in $L^\infty(B^{2n})$.

<u>Proof</u>. Let χ_ε be as in 1.6. Then

$$\|T_\varphi + \mathcal{K}\| \le \|T_\varphi - T_{\varphi\chi_\varepsilon}\|$$

$$\le \|\varphi - \varphi\chi_\varepsilon\|_\infty$$

$$= \sup_{1>|\zeta|1-\varepsilon} |\varphi(\zeta)|. \qquad \blacksquare$$

Of course if φ is in $C(\bar{B}^{2n})$, Proposition 1.7 shows that

$$\|T_\varphi + \mathcal{K}\| = \lim_{\varepsilon\to 0} \sup_{1>|\zeta|>1-\varepsilon} |\varphi(\zeta)|.$$

CHAPTER II. $\bar\partial$ RESULTS.

We will begin this section with some basic definitions and then state a result on the $\bar\partial$ operator (2.4) due to Kerzman [14]. The rest of the section will be devoted to corollaries to this result. These corollaries will form the basis for our work in the remaining chapters.

Throughout this paper z_i will denote the i-th coordinate map on $\mathbb{C}^n$, i.e., for $\zeta = (\zeta_1,\ldots,\zeta_n)$ in $\mathbb{C}^n$, $z_i(\zeta) = \zeta_i$. If α is a polyindex, $\alpha = (\alpha_1,\ldots,\alpha_n)$ where each α_i is a nonnegative integer, z^α will denote the function $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. The collection of all polynomials in $z_1,\ldots,z_n,\bar z_1,\ldots,\bar z_n$ will be written $\mathcal{P}(z,\bar z)$ where $\bar z_j = x_j - iy_j$, x_j and y_j the underlying real coordinates. We also write

$$2\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}, \quad 2\frac{\partial}{\partial \bar z_j} = \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}.$$

<u>Definition 2.1</u>. A relatively compact open set Ω in $\mathbb{C}^n$ is said to be strongly pseudoconvex (with smooth C^4 boundary) if there exists an open neighborhood U of $\partial\Omega$ and a C^4 function $\lambda : U \to R$ such that:

a) $\Omega \cap U = \{\zeta \in U : \lambda(\zeta) < 0\}$

b) $\nabla\lambda(\zeta) \neq 0$ for $\zeta \in U$ where $\nabla\lambda = \left(\frac{\partial\lambda}{\partial x_1},\frac{\partial\lambda}{\partial y_1},\ldots,\frac{\partial\lambda}{\partial x_n},\frac{\partial\lambda}{\partial y_n}\right)$

c) λ is strictly pluri-subharmonic, i.e.

$$\sum_{i,j=1}^{n} \frac{\partial^2\lambda}{\partial z_i \partial \bar z_j}(\zeta)\mu_i\bar\mu_j \geq L(\zeta)|\mu|^2, \quad \zeta \in U, \ \mu \in \mathbb{C}^n,$$

where L is a (strictly) positive function on U.

This definition can be found in [13]. Of course B^{2n} is strongly pseudoconvex. All we have to do is set $\lambda(\zeta) = |\zeta|^2-1$ and $U = \mathbb{C}^n\backslash\{0\}$. Up until 2.8 the reader can substitute any strongly pseudoconvex Ω for B^{2n}.

We now consider very briefly forms on B^{2n}. Details can be found in Gunning and Rossi [9].

Definition 2.2. A smooth $(0,p)$-form on B^{2n} is an object of the form

$$\varphi = \Sigma\varphi_I d\bar{z}^I$$

where $I = (i_1,\ldots,i_p)$ and $d\bar{z}^I = d\bar{z}^{i_1}\wedge\ldots\wedge d\bar{z}^{i_p}$, and where each φ_I is a C^∞ function on B^{2n}.

A smooth $(0,0)$-form on B^{2n} is simply a smooth function. We can define a map $\bar{\partial}$ from smooth $(0,p)$-forms to smooth $(0,p+1)$-forms by

$$\bar{\partial}\varphi = \sum_{I,k}(\partial\varphi_I/\partial\bar{z}_k)d\bar{z}^k \wedge d\bar{z}^I.$$

As usual $\bar{\partial}^2 = \bar{\partial}\bar{\partial} = 0$. A smooth $(0,p)$-form φ is called closed if $\bar{\partial}\varphi = 0$. It follows from the Cauchy-Riemann equations in several variables that the closed $(0,0)$-forms are precisely the holomorphic functions on B^{2n}.

Definition 2.3. A smooth $(0,1)$-form φ on B^{2n}, $\varphi = \sum_{i=1}^{n} \varphi_i d\bar{z}^i$, is said to be bounded if $\|\varphi_i\|_\infty < \infty$ for each i.

The following result is due to Kerzman [13].

Proposition 2.4. For every bounded smooth (o,1)-form ϕ on B^{2n} which is closed there is a continuous function u on $\bar{B}^{2n}$ which is smooth on B^{2n} and satisfies $\bar{\partial}u = \phi$.

The solution u can be written in the form

$$u(w) = \int_{B^{2n}} \Omega(\zeta,w)\wedge\phi(\zeta)$$

where the kernel Ω can be explicitly determined. For n = 1 the above reduces to

$$u(w) = \frac{1}{2\pi i}\int_{B^2} \frac{\phi(\zeta)}{\zeta-w}\, d\zeta\, d\bar{\zeta},$$

that is,

$$\Omega(\zeta,w) = \frac{1}{2\pi i}\,\frac{d\zeta}{\zeta-w}.$$

Thus Ω can be considered a generalization of the Cauchy kernel to higher dimensions.

One final comment should be made about 2.4. It is well-known that the equation $\bar{\partial}u = \phi$ has a smooth solution on B^{2n} for any closed form ϕ ([9; p. 63]). The important thing about 2.4 is that for forms satisfying the added boundedness hypothesis a solution can be found which is continuous on the closure.

Recall that we defined $\mathfrak{A}$ to be the smallest closed subalgebra of $L^\infty(B^{2n})$ containing $H^\infty(B^{2n}) + C(\bar{B}^{2n})$. Since there is no reason to believe that $H^\infty(B^{2n}) + C(\bar{B}^{2n})$ is an algebra, one cannot conclude immediately that $H^\infty(B^{2n}) + C(\bar{B}^{2n})$ is dense in $\mathfrak{A}$.

<u>Corollary 2.5</u>. Let $\mathfrak{B} = \{\varphi \in C^{\infty}(B^{2n}) : \varphi$ and $\bar{\partial}\varphi$ are bounded$\}$. Then we have

1) $\mathfrak{B}$ is a subalgebra of $L^{\infty}(B^{2n})$;

2) $\mathfrak{B}$ is contained in $H^{\infty}(B^{2n}) + C(\bar{B}^{2n})$;

3) $\operatorname{clos} \mathfrak{B} = \mathfrak{A}$.

Thus in particular $H^{\infty}(B^{2n}) + C(\bar{B}^{2n})$ is dense in $\mathfrak{A}$.

<u>Proof</u>. Clearly $\mathfrak{B}$ is closed under addition and scalar multiplication. If φ and ψ belong to $\mathfrak{B}$ then $\varphi\psi$ also belongs to $\mathfrak{B}$ since

$$\bar{\partial}(\varphi\psi) = \psi\bar{\partial}\varphi + \varphi\bar{\partial}\psi.$$

Again suppose φ is in $\mathfrak{B}$. Since $\bar{\partial}^2 = 0$ we see that $\bar{\partial}\varphi$ is a closed bounded (0,1)-form. Therefore, by 2.4 there is a function u in $C(\bar{B}^{2n})$ which is smooth on B^{2n}, for which $\bar{\partial}u = \bar{\partial}\varphi$. This implies that $\varphi-u$ is holomorphic, and in fact $\varphi-u$ is in $H^{\infty}(B^{2n})$ since φ and u are both bounded. We can write

$$\varphi = (\varphi-u) + u,$$

and so φ is in $H^{\infty}(B^{2n}) + C(\bar{B}^{2n})$. This proves 2).

Finally, we have $H^{\infty}(B^{2n}) + \mathcal{P}(z,\bar{z}) \subset \mathfrak{B}$, and so

$$\operatorname{clos}[H^{\infty}(B^{2n}) + \mathcal{P}(z,\bar{z})] \subset \operatorname{clos} \mathfrak{B}.$$

The term on the left is equal to $\operatorname{clos}[H^{\infty}(B^{2n}) + C(\bar{B}^{2n})]$, and so by 2)

$$\operatorname{clos} \mathfrak{B} = \operatorname{clos}[H^\infty(B^{2n}) + C(\overline{B}^{2n})].$$

Since clos $\mathfrak{B}$ is an algebra, we conclude that clos $\mathfrak{B} = \mathfrak{A}$. ∎

Definition 2.6. A set $U \subset B^{2n}$ is a neighborhood of $\lambda \in \partial B^{2n}$ in B^{2n} if there exists an open set V in $\mathbb{C}^n$ such that

1) $U = V \cap B^{2n}$, and
2) $\lambda \in V$.

We will say that a function φ which is continuous on B^{2n} is continuously extendable to $B^{2n} \cup \{\lambda\}$ with value α at λ if for every sequence ξ_k in B^{2n} converging to λ, $f(\xi_k)$ converges to α.

Corollary 2.7. Let $\lambda \in \partial B^{2n}$ and f be in $H^\infty(B^{2n})$. If f is bounded away from zero on a neighborhood of λ in ∂B^{2n} then there exists g in $H^\infty(B^{2n})$ such that fg is continuously extendable to $B^{2n} \cup \{\lambda\}$ with value 1 at λ.

Proof. Suppose U is a neighborhood of λ in B^{2n} on which f is bounded away from zero. Let ψ be a smooth function which is identically -1 on a neighborhood of λ in $\mathbb{C}^n$ and such that supp $\psi \cap B^{2n}$ is contained in U. Then $\frac{1}{f}\psi$ is defined and bounded on B^{2n} and we have

$$\bar\partial(\tfrac{1}{f}\psi) = \tfrac{1}{f}\bar\partial\psi.$$

Thus $\bar\partial(\frac{1}{f}\psi)$ is a bounded closed (0,1)-form on B^{2n}. By Proposition 2.4 there is a continuous function φ on $\overline{B}^{2n}$

such that

$$\bar{\partial}\varphi = \bar{\partial}(\tfrac{1}{f}\psi).$$

By subtracting $\varphi(\lambda)$ from φ if necessary, we may assume $\varphi(\lambda) = 0$. Let $g = \varphi - \frac{1}{f}\psi$. Since $\bar{\partial}g = 0$ and since g is the difference of two bounded functions, we have that g is in $H^{\infty}(B^{2n})$. Finally

$$fg = f(\varphi - \tfrac{1}{f}\psi) = f\varphi - \psi,$$

and so $fg = f\varphi + 1$ in a neighborhood of λ in B^{2n}. Since $\varphi(\lambda) = 0$ and since f is bounded, it follows that $f\varphi(\xi_k)$ converges to 0 for any sequence ξ_k converging to λ. ∎

It is a well-known fact, due to Sarason [16], that $H^{\infty}+C(\mathbb{T}) = \operatorname{clos}\{\psi\bar{\chi}_n = \psi \in H^{\infty},\ n \geq 0\}$, where $\chi_n(e^{i\theta}) = e^{in\theta}$. Our next corollary will show that a result like this is true for $\mathfrak{A}$ on B^{2n}. First we suppose $n = 1$. We have just shown that

$$\mathfrak{A} = \operatorname{clos}[H^{\infty}(B^2) + \mathcal{P}(z,\bar{z})].$$

Let f belong to $H^{\infty}(B^2)$ and let p be a polynomial in z and $\bar{z}$, $p = \sum_{i,j} \alpha_{ij} z^i \bar{z}^j$. Since $z\bar{z}-1$ belongs to $\mathcal{J}_0$ we can write

$$p = \bar{z}^m q + h,$$

where m is the largest value of j in the previous expression of p, h belongs to $\mathcal{J}_0$, and q is a polynomial in z only. Thus

$$f+p = \bar{z}^m(fz^m+q) + h_1,$$

where

$$h_1 = f(1-\bar{z}^m z^m) + h.$$

Thus h_1 is in $\mathcal{J}_0$, and we have for $n = 1$,

$$\mathfrak{A} = \operatorname{clos}\{f\bar{z}^m + h : f \in H^\infty(B^2),\ m \geq 0,\ h \in \mathcal{J}_0\}.$$

For $n \geq 2$ the above proof falls apart, primarily because each of the coordinate functions z_i has zeros on the boundary. We can obtain a weaker result which will, however, be sufficient for our purposes.

Corollary 2.8. Let n be greater than 1. Suppose φ in $\mathfrak{B}$ is bounded away from zero on a neighborhood of ∂B^{2n}. Then there exists f in $H^\infty(B^{2n})$, g in $C(\bar{B}^{2n})$, and h in $\mathcal{J}_0$ such that

$$\varphi = fg + h.$$

Proof. Let B_ε be the ball of radius ε about 0 in $\mathbb{C}^n$. Since $n > 1$, $B^{2n}\setminus\bar{B}_\varepsilon$ is simply connected for $\varepsilon < 1$. Thus if ε is chosen sufficiently close to 1 so that φ is bounded away from zero on $B^{2n}\setminus\bar{B}_\varepsilon$, there exists a smooth function u on $B^{2n}\setminus\bar{B}_\varepsilon$ such that $\varphi = e^u$. By choosing $\varepsilon' > \varepsilon$ we can construct a smooth function $\tilde{u}$ on B^{2n} agreeing with u on $B^{2n}\setminus\bar{B}_{\varepsilon'}$. Let $\tilde{\varphi} = e^{\tilde{u}}$. Since $\bar{\partial}\tilde{\varphi} = e^{\tilde{u}}\bar{\partial}\tilde{u}$, we have

$$\bar{\partial}\tilde{u} = \frac{1}{\tilde{\varphi}}\,\bar{\partial}\tilde{\varphi}.$$

On $B^{2n}\setminus\bar{B}_{\varepsilon'}$, $\bar{\partial}\tilde{u} = \frac{1}{\varphi}\bar{\partial}\tilde{\varphi}$ and so $\bar{\partial}\tilde{u}$ is a bounded $(0,1)$-form. By Proposition 2.4 there exists a smooth function ψ in $C(\bar{B}^{2n})$

such that $\tilde{u}-\psi$ is holomorphic. It may not be the case that $\tilde{u}-\psi$ is bounded. However we can write

$$\tilde{\varphi} = e^{\tilde{u}} = e^{\tilde{u}-\psi}e^{\psi},$$

and hence

$$e^{\tilde{u}-\psi} = \frac{1}{e^{\psi}}\,\tilde{\varphi}.$$

Since ψ is continuous on $\bar{B}^{2n}$ we see that $\frac{1}{e^{\psi}}$ is bounded and hence $e^{\tilde{u}-\psi}$ is in $H^{\infty}(B^{2n})$. Let $f = e^{\tilde{u}-\psi}$, $g = e^{\psi}$, and $h = \varphi-\tilde{\varphi}$. We have

$$\begin{aligned}\varphi &= \tilde{\varphi}+(\varphi-\tilde{\varphi})\\ &= fg + h.\end{aligned}$$

Since h belongs to $\mathcal{I}_0$ we are done. ▪

We point out that f and g are both bounded away from zero on B^{2n}.

CHAPTER III. MAIN THEOREM.

In this chapter we obtain our main result (Theorem 3.8) that for φ in $\mathfrak{A}$, T_φ is Fredholm if and only if φ is bounded away from zero on a neighborhood of ∂B^{2n}. As a corollary to this result we show that the essential spectrum of T_φ is connected. We first obtain some results for Toeplitz operators with holomorphic symbols.

Proposition 3.1. If φ is in $H^\infty(B^{2n})$ then T_φ is invertible if and only if φ is bounded away from zero on B^{2n}, i.e.

$$\sigma(T_\varphi) = \operatorname{clos}[\varphi(B^{2n})].$$

Proof. If $\lambda \notin \operatorname{clos}[\varphi(B^{2n})]$ then $(\varphi-\lambda)^{-1}$ belongs to $H^\infty(B^{2n})$ and so by Proposition 1.4,

$$T_{(\varphi-\lambda)}T_{(\varphi-\lambda)^{-1}} = T_{(\varphi-\lambda)(\varphi-\lambda)^{-1}} = I,$$

and

$$T_{(\varphi-\lambda)^{-1}}T_{(\varphi-\lambda)} = T_{(\varphi-\lambda)^{-1}(\varphi-\lambda)} = I.$$

On the other hand, if λ belongs to $\varphi(B^{2n})$ then $\varphi-\lambda$ has a zero in B^{2n}, say at ζ. Since

$$\operatorname{ran}(T_{\varphi-\lambda}) = \{(\varphi-\lambda)f : f \in H^2(B^{2n})\},$$

it follows that for every g in $\operatorname{ran}(T_{\varphi-\lambda})$, $g(\zeta) = 0$. Since $H^2(B^{2n})$ is a proper functional space we have

$$\operatorname{ran}(T_{\varphi-\lambda}) \neq H^2(B^{2n}),$$

and hence

$$\varphi(B^{2n}) \subset \sigma(T_\varphi) \subset \mathrm{clos}[\varphi(B^{2n})].$$

Since the spectrum is always closed, the result follows. ∎

Corollary 3.2. If φ belongs to $H^\infty(B^{2n})$ and $n \geq 2$, then T_φ is invertible if and only if φ is bounded away from zero on a neighborhood of ∂B^{2n}.

Proof. Suppose φ is bounded away from zero on some neighborhood of ∂B^{2n}. Then there exists $\varepsilon < 1$ such that 0 does not belong to $\mathrm{clos}[\varphi(B^{2n} \backslash \bar{B}_\varepsilon)]$. By Hartog's theorem (see [11; pp. 20-21]) this implies φ is nonzero on all of B^{2n}. ∎

Lemma 3.3. If $\varphi \in \mathfrak{A}$ and $\psi \in L^\infty(B^{2n})$ then $T_\psi T_\varphi - T_{\psi\varphi}$ is compact.

Proof. Since $H^\infty(B^{2n}) + C(\bar{B}^{2n})$ is dense in $\mathfrak{A}$ and since $\mathcal{K}$ is norm closed, it suffices to show that $T_\psi T_{(f+g)} - T_{\psi(f+g)}$ is compact for f in $H^\infty(B^{2n})$ and g in $C(\bar{B}^{2n})$. In this case we have

$$T_\psi T_{(f+g)} - T_{\psi(f+g)} = (T_\psi T_f - T_{\psi f}) + (T_\psi T_g - T_{\psi g}),$$

and the result follows from Proposition 1.5 and the remark after Proposition 1.4. ∎

Corollary 3.4. If φ and ψ are functions in $\mathfrak{A}$ then $T_\psi T_\varphi - T_\varphi T_\psi$ is compact and hence $\mathfrak{J}(\mathfrak{A})/\mathcal{K}$ is abelian.

Proof. Since $\mathcal{K}$ is contained in $\mathfrak{J}(C(\bar{B}^{2n}))$ it must be contained in the larger algebra $\mathfrak{J}(\mathfrak{A})$, and so $\mathfrak{J}(\mathfrak{A})/\mathcal{K}$ is a Banach algebra. The rest follows from 3.3 and the definition of $\mathfrak{J}(\mathfrak{A})$. ∎

We now begin the proof of one half of our main result, i.e. if φ in $\mathfrak{A}$ is bounded away from zero on a neighborhood of ∂B^{2n} then T_φ is Fredholm. The following elementary lemma outlines our approach.

Lemma 3.5. Let A be a Banach algebra and let a in A be left invertible. If $\{\varphi_n\}$ is a sequence of invertible elements in A converging to a in norm then a is invertible.

Proof. Suppose a is not invertible. The set of all left invertible elements which are not invertible is open ([5; p. 35]). Thus the sequence $\{\varphi_n\}$ cannot converge to a. ∎

Lemma 3.6. If φ in $\mathfrak{A}$ is bounded away from zero on a neighborhood of ∂B^{2n} then $T_\varphi + \mathcal{K}$ is left invertible in $\mathcal{L}(H^2(B^{2n}))/\mathcal{K}$.

Proof. If φ is bounded away from zero on a neighborhood of ∂B^{2n} we can choose an ε-ball B_ε about the origin such that φ is bounded away from zero on $B^{2n}\backslash\bar{B}_\varepsilon$. By choosing $\varepsilon' > \varepsilon$ we can find a function ψ in $\mathcal{B}(B^{2n})$ such that

$$\psi|(B^{2n}\backslash\bar{B}_{\varepsilon'}) = \frac{1}{\varphi}.$$

Thus $1-\psi\varphi$ is in $\mathcal{J}_0$ and so by Lemma 3.3

$$(T_\psi+\mathcal{K})(T_\varphi+\mathcal{K}) = T_{\psi\varphi}+\mathcal{K}$$
$$= I+\mathcal{K}. \qquad \square$$

Theorem 3.7. Suppose $n \geq 2$. If φ in $\mathfrak{A}$ is bounded away from zero on a neighborhood of ∂B^{2n} then T_φ is Fredholm with index 0.

Proof. By Corollary 2.4 there exists a sequence $\{\varphi_n\}$ in $\mathfrak{B}$ such that φ_k converges to φ in the sup norm. Passing to a subsequence if necessary, we may assume that each φ_k is bounded away from zero on some neighborhood of ∂B^{2n}. Therefore Corollary 2.8 applies to each φ_k, and there exists functions f_k in $H^\infty(B^{2n})$ and g_k in $C(\bar{B}^{2n})$, both bounded away from zero on B^{2n}, and h_k in $\mathcal{J}_0$ such that $\varphi_k = f_k g_k+h_k$. Thus for each k, $T_{\varphi_k} = T_{f_k}T_{g_k} + K_k$, for some compact K_k. The operator T_{f_k} is invertible by Proposition 3.1. The operator T_{g_k} is Fredholm with index 0 by Coburn's result. Thus T_{φ_k} is Fredholm with index 0. From Lemmas 3.5 and 3.6 we conclude T_φ is Fredholm. Since the index is norm continuous ([5; p. 138]), we see that T_φ has index 0. $\square$

In the case $n = 1$ we know that for any φ in $\mathfrak{A}$ there is a sequence $\{\varphi_k\}$ converging to φ in norm where

$$\varphi_k = f_k\bar{z}^{m_k}+h_k$$

for some function f_k in $H^\infty(B^2)$ and h_k in $\mathcal{J}_0$. If φ is bounded away from zero on a neighborhood of ∂B^2 we can assume that each f_k is bounded away from zero on a neighborhood of ∂B^2 (but not necessarily on all of B^2). Thus each f_k has a finite number of zeros in the disc, say at $\lambda_{1_k},\ldots,\lambda_{i_k}$, and we can write

$$f_k = g_k \prod_{j_k=1_k}^{i_k} (z-\lambda_{j_k})^{p_{j_k}},$$

where g_k is in $H^\infty(B^2)$ and is bounded away from zero on B^2 ([10; p. 66]). By 3.1 and a remark after 1.7, T_{f_k} is Fredholm and has index equal to $\Sigma\, p_{jk}$. Since $T_{\varphi_k} = T_{f_k} T_{\bar{z}^{m_k}} + K_k$ for some compact K_k, it follows that each T_{φ_k} is Fredholm with index $\Sigma\, p_{j_k} - m_k$. Again by Lemmas 3.5 and 3.6, T_φ is Fredholm, although of course its index need not be zero.

<u>Theorem 3.8</u>. If φ belongs to $\mathfrak{A}$ then T_φ is Fredholm if and only if φ is bounded away from zero on a neighborhood of ∂B^{2n}.

Unlike the "if" part, the "only if" part of this theorem is proved the same way for $n = 1$ as for $n > 1$. First we state some immediate corollaries to 3.8 and then we will devote the rest of the chapter to the proof of the theorem.

Recall that for a bounded linear operator T on a Hilbert space H, the essential spectrum of T, written ess $\sigma(T)$, is defined to be the set of all α in $\mathbb{C}$ such that $T-\lambda$ is not Fredholm. The spectrum of T will be written $\sigma(T)$.

Corollary 3.9. If $\varphi \in \mathfrak{A}$ then ess $\sigma(T_\varphi)$ is connected.

Proof. Let $S = \bigcap_{m=1}^{\infty} \operatorname{clos}\{\varphi(B^{2n} \backslash B_{1-1/m})\}$. Since φ is continuous each $\operatorname{clos}\{\varphi(B^{2n} \backslash B_{1-1/m})\}$ is connected and hence S is connected, being a nested intersection of compact connected sets. We will show ess $\sigma(T_\varphi) = S$. If $\alpha \notin$ ess $\sigma(T_\varphi)$ then $\varphi - \alpha$ is bounded away from zero on a neighborhood of ∂B^{2n} and so $\alpha \notin S$. This last argument is reversible and the result follows. ∎

Corollary 3.10. If $n \geq 2$ and $\varphi \in H^\infty(B^{2n})$ then T_φ is Fredholm if and only if T_φ is invertible.

Proof. Combine Corollary 3.2 and Theorem 3.8.

The following definition can be found in Coburn [3].

Definition 3.11. For any sequence $\{\lambda_m\}$ in B^{2n} with $\lambda_m \to \lambda \in \partial B^{2n}$ we associate a sequence $\{f_m\}$ in $H^2(B^{2n})$ by defining

$$f_m(\zeta) = K(\lambda_m, \zeta) K(\lambda_m, \lambda_m)^{-1/2}.$$

From the definition of $K(\lambda_m, \zeta)$ it is clear that $\{f_m\}$ converges uniformly to zero in the complement of any open neighborhood of λ. Of course this implies $f_m \to 0$ pointwise on B^{2n}.

Lemma 3.12. For all m, $\|f_m\|_2 = 1$. The sequence $\{f_m\}$ contains no norm convergent subsequence.

Proof. The first statement is immediate because

$$\begin{aligned}\|f_m\|_2^2 &= (f_m, f_m) \\ &= K(\lambda_m,\lambda_m)^{-1}(K(\lambda_m,\zeta),K(\lambda_m,\zeta)) \\ &= 1,\end{aligned}$$

since K is a reproducing kernel.

Suppose there is a subsequence $\{f_{m_k}\}$ converging in the L^2-norm to some g in $H^2(B^{2n})$. Since each f_{m_k} is a unit vector we must have $\|g\|_2 = 1$. On the other hand since $H^2(B^{2n})$ is a proper functional space we have that $f_{m_k}(\zeta) \to g(\zeta)$ for each ζ in B^{2n}. We have already seen that $f_{m_k}(\zeta)$ converges to 0 for every ζ in B^{2n} and so $g = 0$. This is, of course, impossible. ∎

Definition 3.13. Let H be a separable Hilbert space and let T belong to $\mathcal{L}(H)$. We will call α in $\mathbb{C}$ a π-ess eigenvalue for T if there exists a sequence of unit vectors $\{h_m\}$ in H such that

1) $\{h_m\}$ has no norm convergent subsequence;
2) $\lim \|(T-\alpha)h_m\| = 0$.

For each T the collection of all such α will be denoted π-ess $\sigma(T)$.

Actually π-ess $\sigma(T)$ is precisely the left essential spectrum of T ([6]). We want to emphasize the role of the $\{h_m\}$ and so we will not make this identification. However, we do need the following inclusion and will prove it directly.

Lemma 3.14. For H and T as in 3.13 we have π-ess $\sigma(T) \subset$ ess $\sigma(T)$.

Proof. Since H is separable the unit ball in H is weakly sequentially compact. Let α be in π-ess $\sigma(T)$ and let $\{h_m\}$ be a sequence satisfying 1) and 2) of 3.13. Since the unit ball of H is weakly sequentially compact there exists g in $H^2(B^{2n})$ such that for some subsequence $\{h_{m_k}\}$, h_{m_k} converges to g weakly. Suppose there exists an S in $\mathfrak{L}(H)$ such that $S(T-\alpha)+K = I$ for some compact operator K. Since K is compact $Kh_{m_k} \to Kg$ in norm. By hypothesis $(T-\alpha)h_{m_k} \to 0$ in norm and hence so does $S(T-\alpha)h_{m_k}$. Thus $h_{m_k} \to Kg$ in norm, which is a contradiction.

Lemma 3.15. If φ is in $C(\overline{B}^{2n})$ and $\varphi(\lambda) = \alpha$, where $\lambda \in \partial B^{2n}$, then for any sequence $\{\lambda_m\}$ in B^{2n} converging to λ,

$$\|(T_\varphi-\alpha)f_m\|_2 \to 0.$$

The proof can be found in [3; p. 435]. Since

$$\begin{aligned}\|(T_\varphi-\alpha)f_m\|_2^2 &= \|P(M_\varphi-\alpha)f_m\|_2^2 \\ &= \|(M_\varphi-\alpha)f_m\|_2 \\ &= \int_{B^{2n}} |\varphi f_m-\varphi(\lambda)f_m|^2,\end{aligned}$$

the result follows almost immediately from the continuity of φ at λ. We now extend this result, not to $\mathfrak{A}$ but to

$$\tilde{\mathfrak{A}} = \{\varphi \in \mathcal{B}(B^{2n}) : \tilde{\varphi} \in \mathfrak{A}\}.$$

Lemma 3.16. If φ is in $\mathfrak{A}$ and φ is not bounded away from α on a neighborhood of ∂B^{2n} then $\alpha \in \pi\text{-ess } \sigma(T_\varphi)$.

Proof. Since $\bar{B}^{2n}$ is compact we can find a sequence $\{\lambda_m\}$ in B^{2n} such that $\lambda_m \to \lambda$ for some $\lambda \in \partial B^{2n}$ and $\varphi(\lambda_m) \to \alpha$. Let $\{f_m\}$ be the sequence associated with $\{\lambda_m\}$. We want to show $\|(T_\varphi-\alpha)f_{m_k}\|_2 \to 0$ for some subsequence $\{f_{m_k}\}$.

First we assume $\bar{\varphi} \in H^\infty(B^{2n})$. For w in B^{2n} fixed we have

$$\begin{aligned}
[T_\varphi K(w,\cdot)](\zeta) &= (\varphi K(w,\cdot),K(\zeta,\cdot)) \\
&= (K(w,\cdot),\bar{\varphi}K(\zeta,\cdot)) \\
&= \overline{(\bar{\varphi}K(\zeta,\cdot),K(w,\cdot))} \\
&= \overline{(\bar{\varphi}(w)K(\zeta,w))} \\
&= \varphi(w)K(w,\zeta).
\end{aligned}$$

The above calculations involve only elementary properties of the kernel function. It follows then for each m that $T_\varphi f_m = \varphi(\lambda_m)f_m$. Therefore we have:

$$\begin{aligned}
\|(T_\varphi-\alpha)f_m\|_2 &= \|\varphi(\lambda_m)f_m-\alpha f_m\|_2 \\
&= |\varphi(\lambda_m)-\alpha|,
\end{aligned}$$

which goes to zero.

Next let us assume $\varphi = f+g$ where $\bar{f}$ belongs to $H^\infty(B^{2n})$ and g belongs to $C(\bar{B}^{2n})$. Let $\beta = g(\lambda) = \lim g(\lambda_m)$. Thus $\lim f(\lambda_m) = \alpha-\beta$ and we have

$$\|(T_\varphi-\alpha)f_m\|_2 \le \|(T_f-(\alpha-\beta))f_m\|_2 + \|(T_g-\beta)f_m\|_2.$$

The terms on the right hand size go to zero by Lemma 3.15 and the preceeding part of the proof.

Finally, let φ be an arbitrary function in $\mathfrak{A}$. Since $H^\infty(B^{2n}) + C(\bar{B}^{2n})$ is dense in $\mathfrak{A}$, we can find a function ψ of the form just considered such that $\|\varphi-\psi\|_\infty < \varepsilon/4$. Since ψ is bounded we can assume there exists a γ in $\mathbb{C}$ such that $\psi(\lambda_m) \to \gamma$ (passing to a subsequence if necessary). By the preceeding inequality we have $|\alpha-\gamma| < \varepsilon/3$. Thus

$$\begin{aligned}\|(T_\varphi-\alpha)f_m\|_2 &\leq \|T_\varphi f_m - T_\psi f_m\|_2 + \|T_\psi f_m - \gamma f_m\|_2 + \|\gamma f_m - \alpha f_m\|_2 \\ &\leq \|\varphi-\psi\|_\infty + \|(T_\psi-\gamma)f_m\|_2 + |\gamma-\alpha| \\ &\leq 2\varepsilon/3 + \|(T_\psi-\gamma)f_m\|_2,\end{aligned}$$

and the result follows by the preceeding paragraph. ∎

Proof of Theorem 3.8. We have to show that if φ in $\mathfrak{A}$ is not bounded away from zero on a neighborhood of ∂B^{2n} then T_φ is not Fredholm. However, if φ is not bounded away from zero on some neighborhood then 0 is in ess $\sigma(T_{\bar\varphi})$, and since $T_{\bar\varphi} = T_\varphi^*$, we are done. ∎

We point out that we have shown for φ in $\mathfrak{A}$ that π-ess $\sigma(T_{\bar\varphi})$ = ess $\sigma(T_{\bar\varphi})$. In particular T_φ is Fredholm if and only if it is right Fredholm.

CHAPTER IV. THE ALGEBRA $\mathfrak{T}(\mathfrak{A})/\mathcal{K}$.

We now consider the algebra $\mathfrak{T}(\mathfrak{A})/\mathcal{K}$. In this section we show that $\mathfrak{T}(\mathfrak{A})/\mathcal{K}$ is isometrically isomorphic to $\mathfrak{A}/\mathcal{I}_0$ and, using this identification, in the next section we compute its maximal ideal space. The map

$$\mu : \mathfrak{A}/\mathcal{I}_0 \to \mathfrak{T}(\mathfrak{A})/\mathcal{K}$$
$$\varphi+\mathcal{I}_0 \to T_\varphi+\mathcal{K},$$

will be our isometry. Most of the present section is devoted to showing that the norms of $\varphi+\mathcal{I}_0$ and $T_\varphi+\mathcal{K}$ are equal.

Lemma 4.1. If $\varphi \in \mathfrak{A}$ then $r_{ess}\ \sigma(T_\varphi) = \|T_\varphi+\mathcal{K}\| = \lim_{\varepsilon\to 0} \sup_{1>|\zeta|>1-\varepsilon} |\varphi(\zeta)|$.

Proof. If $a \in A$, an arbitrary Banach algebra, we know that $r_\sigma(a) \le \|a\|$ (where $r_\sigma(a)$ is the spectral radius). Thus we have

$$r_{ess}\ \sigma(T_\varphi) \le \|T_\varphi+\mathcal{K}\| \le \lim_{\varepsilon\to 0} \sup_{1>|\zeta|>1-\varepsilon} |\varphi(\zeta)|.$$

Suppose $k < \lim_{\varepsilon\to 0} \sup_{1>|\zeta|>1-\varepsilon} |\varphi(\zeta)|$. We can find a sequence $\{\zeta_m\}$ in B^{2n} converging to a $\lambda \in \partial B^{2n}$ such that

$$k < |\varphi(\zeta_m)| \le \|\varphi\|_\infty.$$

By passing to a subsequence if necessary we can assume there exists an α in $\mathbb{C}$ such that $\varphi(\zeta_m) \to \alpha$. Thus $k \le |\alpha|$, and by Theorem 3.8 $\alpha \in r_{ess}\ \sigma(T_\varphi)$. Since k was arbitrary

$$\lim_{\varepsilon\to 0} \sup_{1>|\zeta|>1-\varepsilon} |\varphi(\zeta)| \le r_{ess}\ \sigma(T_\varphi),$$

and the result follows. ▫

Lemma 4.2. If $\varphi \in \mathfrak{A}$ then $\varphi+\mathcal{J}_0$ is invertible in $\mathfrak{A}/\mathcal{J}_0$ if and only if φ is bounded away from zero on a neighborhood of ∂B^{2n}.

Proof. Suppose φ is bounded away from zero on a neighborhood of ∂B^{2n}. By Corollary 2.8 and the remarks following Theorem 3.7 we can find a sequence of functions $\{\varphi_m\}$ in $\mathcal{B}(B^{2n})$ of the form

$$\varphi_m = f_m g_m + h_m,$$

where f_m is in $H^\infty(B^{2n})$ and is bounded away from zero on B^{2n}, g_m is in $C(\bar{B}^{2n})$ and is bounded away from zero on a neighborhood of ∂B^{2n}, h_m is in $\mathcal{J}_0$, such that φ_m converges to φ in norm. Thus $\varphi_m+\mathcal{J}_0$ converges to $\varphi+\mathcal{J}_0$ in the quotient norm. Since

$$\varphi_m+\mathcal{J}_0 = (f_m+\mathcal{J}_0)(g_m+\mathcal{J}_0)$$

it is clear that $(\varphi_m+\mathcal{J}_0)^{-1}$ exists in $\mathfrak{A}/\mathcal{J}_0$. In the proof of Lemma 3.6 we showed that $\varphi+\mathcal{J}_0$ is invertible in $\mathcal{B}(B^{2n})/\mathcal{J}_0$. Since the inverse operation is continuous we have

$$(\varphi_m+\mathcal{J}_0)^{-1} \to (\varphi+\mathcal{J}_0)^{-1}$$

in norm. Thus $(\varphi+\mathcal{J}_0)^{-1}$ is in $\mathfrak{A}/\mathcal{J}_0$ since $\mathfrak{A}/\mathcal{J}_0$ is a closed subalgebra of $\mathcal{B}(B^{2n})/\mathcal{J}_0$.

On the other hand if φ is not bounded away from zero on any neighborhood of ∂B^{2n} we can find a sequence $\{\zeta_m\}$ in

B^{2n} converging to some λ on ∂B^{2n} such that $\varphi(\zeta_m)$ converges to zero. If $\varphi+\mathcal{J}_0$ were invertible in $\mathfrak{A}/\mathcal{J}_0$ there would exist a ψ in $\mathfrak{A}$ such that $\varphi\psi-1$ belongs to $\mathcal{J}_0$. This is impossible since $(\varphi\psi-1)(\zeta_m)$ converges to 1. ∎

Notice that the above proof shows $\varphi+\mathcal{J}_0$ is invertible in $\mathfrak{A}/\mathcal{J}_0$ if and only if it is invertible in $\mathcal{B}(B^{2n})/\mathcal{J}_0$.

<u>Lemma 4.3</u>. If φ is in $\mathfrak{A}$ then

$$\|\varphi+\mathcal{J}_0\| = r_{\sigma_{\mathfrak{A}/\mathcal{J}_0}}(\varphi+\mathcal{J}_0) = \lim_{\varepsilon\to 0} \sup_{1>|\zeta|>1-\varepsilon} |\varphi(\zeta)|.$$

<u>Proof</u>. Since $\mathfrak{A}/\mathcal{J}_0$ is a subalgebra of $\mathcal{B}(B^{2n})/\mathcal{J}_0$ we have

$$r_{\sigma_{\mathfrak{A}/\mathcal{J}_0}}(\varphi+\mathcal{J}_0) = r_{\sigma_{\mathcal{B}(B^{2n})/\mathcal{J}_0}}(\varphi+\mathcal{J}_0).$$

Because $\mathcal{B}(B^{2n})/\mathcal{J}_0$ is a C*-algebra, the term on the left is equal to $\|\varphi+\mathcal{J}_0\|$ (see [5; p. 92]) and hence the first equality of the lemma is established.

By Lemma 4.2 and the last half of the proof of 4.1, we have

$$\lim_{\varepsilon\to 0} \sup_{1>|\zeta|>1-\varepsilon} |\varphi(\zeta)| \le r_{\sigma_{\mathfrak{A}/\mathcal{J}_0}}(\varphi+\mathcal{J}_0).$$

If $|\alpha| > \lim_{\varepsilon\to 0} \sup_{1>|\zeta|>1-\varepsilon} |\varphi(\zeta)|$, then $\varphi-\alpha$ is bounded away from zero on a neighborhood of ∂B^{2n} and so by the previous result α is not in $\sigma_{\mathfrak{A}/\mathcal{J}_0}(\varphi+\mathcal{J}_0)$. This establishes the second equality. ∎

<u>Theorem 4.4</u>. The map $\mu(\varphi+\mathcal{J}_0) = T_\varphi+\mathcal{K}$ is an isometrical isomorphism from $\mathfrak{A}/\mathcal{J}_0$ onto $\mathcal{T}(\mathfrak{A})/\mathcal{K}$.

<u>Proof</u>. If f is in $\mathcal{J}_0$ then T_f is compact and so μ is well-defined and is obviously linear. By Proposition 1.4 μ is multiplicative. Since $\|\varphi+\mathcal{J}_0\| = \|T_\varphi+\mathcal{K}\|$, μ is an isometrical isomorphism. The range of μ is $\{T_\varphi+\mathcal{K} : \varphi \in \mathfrak{A}\}$. By Lemma 3.3 and the definition of $\mathcal{T}(\mathfrak{A})$, this set is dense in $\mathcal{T}(\mathfrak{A})/\mathcal{K}$. The range of an isometry is closed, hence μ is onto. ∎

<u>Corollary 4.5</u>. $\mathcal{T}(\mathfrak{A})/\mathcal{K} = \{T_\varphi+\mathcal{K} : \varphi \in \mathfrak{A}\}$.

<u>Corollary 4.6</u>. If φ belongs to $\mathfrak{A}$ then $T_\varphi+\mathcal{K}$ is invertible in $\mathcal{T}(\mathfrak{A})/\mathcal{K}$ if and only if T_φ is Fredholm.

This last corollary could have been proved directly in the last section. Remember that $\mathcal{T}(\mathfrak{A})/\mathcal{K}$ is not a self-adjoint subalgebra of $\mathcal{L}(H^2(B^{2n}))/\mathcal{K}$. If it were the above result would of course be trivial ([5; p. 92]).

CHAPTER V. THE MAXIMAL IDEAL SPACE OF $\mathfrak{J}(\mathfrak{A})/\mathcal{K}$.

Having identified $\mathfrak{J}(\mathfrak{A})/\mathcal{K}$ with $\mathfrak{A}/\mathfrak{I}_0$ we will compute the maximal ideal space $M_{\mathfrak{J}(\mathfrak{A})/\mathcal{K}}$ by computing $M_{\mathfrak{A}/\mathfrak{I}_0}$. This is done in Theorem 5.8. Before doing this we need some basic results about $M_{H^\infty(B^{2n})}$. Most of these results are proved in the same way as they are for $n = 1$. For a general reference see Hoffman [10].

Recall first that for a Banach algebra A, its maximal ideal space M_A is the collection of all multiplicative linear functionals on A. The topology on M_A is the w*-topology, i.e. if $\{m_\alpha\}$ is a net in M_A then $m_\alpha \to m$ in M_A if and only if $m_\alpha(f) \to m(f)$ for all f in A.

Let $\mathcal{P}(\bar{B}^{2n})$ denote the uniform closure of the (holomorphic) polynomials on $\bar{B}^{2n}$. It is well-known ([9; p. 58]) that $M_{\mathcal{P}(\bar{B}^{2n})} = \bar{B}^{2n}$ in the sense that every multiplicative linear functional on $\mathcal{P}(\bar{B}^{2n})$ is an evaluation at ζ, denoted e_ζ, for some ζ in $\bar{B}^{2n}$. The map $e_\zeta \to \zeta$ is a homeomorphism from $M_{\mathcal{P}(\bar{B}^{2n})}$ onto $\bar{B}^{2n}$.

We can define a map

$$p : M_{H^\infty(B^{2n})} \to \bar{B}^{2n}$$

by

$$m \to m|\mathcal{P}(\bar{B}^{2n}) \to \zeta,$$

i.e. $p(m) = \zeta$ where m restricted to $\mathcal{P}(\bar{B}^{2n})$ is e_ζ.

Lemma 5.1. Suppose f is in $H^\infty(B^{2n})$ and that $f(\lambda) = 0$ for some λ in B^{2n}. Then there exist $f_1,\ldots,f_n$ in $H^\infty(B^{2n})$ such that

$$f = (z_1-\lambda_1)f_1 + \ldots + (z_n-\lambda_n)f_n$$

where $\lambda = (\lambda_1,\ldots,\lambda_n)$.

Proof. Let D be a unit ball containing, but not centered at, the origin in $\mathbb{C}^n$. Using the holomorphic change of variables $\varphi(\zeta) = \zeta+\lambda$, it suffices to show that if $f \in H^\infty(D)$ and $f(0) = 0$ then there exist $f_1,\ldots,f_n$ in $H^\infty(D)$ such that

$$f = z_1f_1 + \ldots + z_nf_n.$$

Clearly D is strongly pseudoconvex. Let $A(\bar{D})$ be the collection of all functions continuous on $\bar{D}$ and holomorphic on D and let I_0 be the ideal of all functions h in $A(\bar{D})$ such that $h(0)$. Kerzman and Nagel have shown ([14; p. 215]) that for any strongly pseudoconvex domain (and so in particular for D) that h in I_0 can be written

$$h = z_1h_1 + \ldots + z_nh_n,$$

where $h_1,\ldots,h_n$ are in $A(\bar{D})$. Thus the map $\sigma : \sum_n \oplus A(\bar{D}) \to I_0$ given by

$$\sigma(g_1,\ldots,g_n) = z_1g_1 + \ldots + z_ng_n$$

is onto.

For ζ in D and $0 < \nu < 1$ define $f_\nu(\zeta) = f(\nu\zeta)$. Since

D is star shaped with respect to the origin these functions are well-defined, and f_ν is in I_0 for each ν. By the open mapping theorem there exists a constant C, independent of ν, and $f_\nu^1,\ldots,f_\nu^n$ in $A(\bar{D})$ such that

$$f_\nu = z_1 f_\nu^1 + \ldots + z_n f_\nu^n,$$

and

$$\|f_\nu^i\|_\infty \le C\|f_\nu\|_\infty \le C\|f\|_\infty.$$

By Vitali's theorem [9; p. 11] we can find a sequence $\nu_k \to 1$ such that for all i, $f_{\nu_k}^i$ converges to some holomorphic function f_i, uniformly on compact subsets of D. Thus each f_i is in $H^\infty(D)$. Since $f_{\nu_k} \to f$ pointwise on D we have

$$f = z_1 f_1 + \ldots + z_n f_n. \quad \blacksquare$$

The idea of using the open mapping theorem was suggested by Professor Rossi.

Proposition 5.2. The map $p : M_{H^\infty(B^{2n})} \to \bar{B}^{2n}$ is a continuous onto mapping. The map p^{-1} is a well-defined map on B^{2n} and is a homeomorphism onto an open subset of $M_{H^\infty(B^{2n})}$.

Proof. Let $\{m_\alpha\}$ be a net in $M_{H^\infty(B^{2n})}$ converging to m, i.e. $m_\alpha(f) \to m(f)$ for all f in $H^\infty(B^{2n})$ and hence, in particular, for all f in $\mathcal{P}(\bar{B}^{2n})$. Therefore $p(m_\alpha)$ converges to p(m) in $\bar{B}^{2n}$, and we conclude p is continuous.

If λ is in B^{2n} then e_λ is in $M_{H^\infty(B^{2n})}$, and clearly $p(e_\lambda) = \lambda$. Thus we have

$$B^{2n} \subset p(M_{H^\infty(B^{2n})}) \subset \bar{B}^{2n}.$$

Since p is continuous and $M_{H^\infty(B^{2n})}$ is compact, $p(M_{H^\infty(B^{2n})})$ is compact and hence closed. Thus p is onto.

Suppose m in $M_{H^\infty(B^{2n})}$ is such that $p^{-1}(\lambda) = m$, i.e. $m|P(B^{2n}) = e_\lambda$ for some λ in B^{2n}. If f in $H^\infty(B^{2n})$ is zero at λ then by the preceeding lemma $m(f) = 0$. Thus for any g in $H^\infty(B^{2n})$

$$m(g) - g(\lambda) = m(g-g(\lambda)) = 0.$$

Thus m is e_λ, and we see that p^{-1} is well-defined on B^{2n}.

Since p is continuous, $p^{-1}(B^{2n})$ is an open subset of $M_{H^\infty(B^{2n})}$. Suppose $\zeta_\alpha \to \zeta$ in B^{2n}. Then $f(\zeta_\alpha)$ converges to $f(\zeta)$ for all f in $H^\infty(B^{2n})$ and so e_{ζ_α} converges to e_ζ in the topology of $M_{H^\infty(B^{2n})}$. Since $p^{-1}(\zeta_\alpha) = e_{\zeta_\alpha}$ and $p^{-1}(\zeta) = e_\zeta$ it follows that p^{-1} is continuous. For more details in the case $n = 1$ see [10]. ∎

Using the map p^{-1} we can identify B^{2n} with an open subset of $M_{H^\infty(B^{2n})}$. We will eventually show that $M_{\mathfrak{A}/\mathfrak{I}_0} = M_{H^\infty(B^{2n})} \backslash B^{2n}$. For λ on ∂B^{2n} we set $F_\lambda = p^{-1}(\{\lambda\})$. Since p is continuous each F_λ is compact. Furthermore we have

$$M_{H^\infty(B^{2n})} \backslash B^{2n} = \bigcup_{\lambda \in \partial B^{2n}} F_\lambda,$$

as sets, and the F_λ are disjoint.

The following theorem can be proved in the case $n = 1$ by using inner functions ([10; p. 162]).

Proposition 5.3. Let f be a function in $H^\infty(B^{2n})$, let λ be on ∂B^{2n} and α be in $\mathbb{C}$. There exists an m in F_λ such that $m(f) = \alpha$ if and only if there exists a sequence $\{\lambda_k\}$ in B^{2n} converging to λ such that $f(\lambda_k)$ converges to α.

Proof. Suppose $\lambda_k \to \lambda$ and $f(\lambda_k) \to \alpha$. Let

$$J = \{\varphi \in H^\infty(B^{2n}) : \varphi(\lambda_k) \to 0\}.$$

Clearly J is a proper ideal of $H^\infty(B^{2n})$ and so is contained in some maximal ideal J_0. Let m be the element in $M_{H^\infty(B^{2n})}$ having kernel J_0. If $\lambda = (\lambda^1,\ldots,\lambda^n)$ then

$$f-\alpha \in J,$$
$$z_i-\lambda^i \in J,\ i = 1,\ldots,n.$$

Since $m(z_i) = \lambda^i$, m is in F_λ and $m(f) = \alpha$. This argument can be found in [10].

Suppose now that there is no sequence $\lambda_k \to \lambda$ such that $f(\lambda_k) \to \alpha$. It is clearly sufficient to prove the proposition for $\alpha = 0$. Thus f is bounded away from zero on a neighborhood of λ in B^{2n}. By Corollary 2.7 there exists g in $H^\infty(B^{2n})$ such that fg is continuously extendable to $B^{2n} \cup \{\lambda\}$ with $fg(\lambda) = 1$. Before completing the proof we need a lemma.

Lemma 5.4. Let φ be in $H^\infty(B^{2n})$ and λ be on ∂B^{2n}. There exists a complex number β such that $m(\varphi) = \beta$ for all m in F_λ if and only if φ is continuously extendable to $B^{2n} \cup \{\lambda\}$ with $\varphi(\lambda) = \beta$.

One half of this lemma follows from what has already been proved in 5.3. So suppose φ is continuously extendable to $B^{2n} \cup \{\lambda\}$ with $\varphi(\lambda) = \beta$, and again we can assume $\beta = 0$. Let $h(\zeta) = \frac{1}{2}(1+\bar{\lambda}\cdot\zeta)$. This function is continuous on $\bar{B}^{2n}$ and has its maximum 1 at λ. Since φ is continuous and zero at λ we have

$$(1-h^k)\varphi \to \varphi, \text{ as } k \to \infty,$$

in the sup norm. Because h is in $P(\bar{B}^{2n})$ we have for each m in F_λ

$$m((1-h^k)\varphi) = 0,$$

for all k and so $m(\varphi) = 0$. Again see [10; p. 161].

Returning to 5.3, we see that $m(fg) = 1$ for all m in F_λ. Since g is in $H^\infty(B^{2n})$ we have $m(fg) = m(f)m(g)$ and so $m(f) \neq 0$. ∎

The following corollary is trivial but we single it out because it is the key to identifying $M_{\mathfrak{A}}$ and $M_{H^\infty(B^{2n})}$.

Corollary 5.5. For f in $H^\infty(B^{2n})$ and λ on ∂B^{2n} we have

$$\sup_{m \in F_\lambda} |m(f)| \leq \sup_{u \in U} |f(u)|,$$

where U is any neighborhood of λ in B^{2n}.

Lemma 5.6. If m belonging to $M_{H^\infty(B^{2n})}$ is in F_λ for some λ on ∂B^{2n}, then m can be extended to a multiplicative linear functional on $\mathfrak{A}$.

Proof. Let $\mathfrak{w} = \{ \sum_{|\beta|\le k} \varphi_\beta \bar{z}^\beta$: β a polyindex, k ranges over positive integers, $\varphi_\beta \in H^\infty(B^{2n})\}$. $\mathfrak{w}$ is obviously an algebra and is dense in $\mathfrak{A}$ by the Stone-Weierstrass theorem. Define $\tilde{m}$ on $\mathfrak{w}$ by

$$\tilde{m}(\psi) = m(\sum_{|\beta|\le k} \varphi_\beta \bar{\lambda}^\beta)$$

where

$$\psi = \sum_{|\beta|\le k} \varphi_\beta \bar{z}^\beta.$$

Since the representation of ψ in the above form is not necessarily unique, we must show that $\tilde{m}$ is well-defined. To do this it clearly suffices to show that $\sum_{|\beta|\le k} \varphi_\beta \bar{z}^\beta$ identically zero on B^{2n} implies $m(\Sigma\varphi_\beta\bar{\lambda}^\beta) = 0$. But in general by the continuity of the $\bar{z}^\beta$ at λ and the boundedness of the φ_β, given $\varepsilon > 0$ we can find a neighborhood U of λ in B^{2n} such that

$$\sup_U |\Sigma\varphi_\beta\bar{\lambda}^\beta| < \varepsilon + \sup_U |\Sigma\varphi_\beta\bar{z}^\beta|.$$

Since $\Sigma\varphi_\beta\bar{\lambda}^\beta \in H^\infty(B^{2n})$ we have

$$|m(\Sigma\varphi_\beta\bar{\lambda}^\beta)| \le \|\Sigma\varphi_\beta\bar{z}^\beta\|_\infty.$$

Thus $\tilde{m}$ is well-defined. Since $\tilde{m}$ is obviously linear and multiplicative, this last inequality also says that $\tilde{m}$ is bounded on $\mathfrak{w}$. Hence $\tilde{m}$ can be extended to be a multiplicative linear functional on clos $\mathfrak{w} = \mathfrak{A}$. ∎

The following lemma will be used to show that any m in $M_{\mathfrak{A}}$ is completely determined by its behaviour on $H^\infty(B^{2n})$.

Lemma 5.7. Let m belong to $M_{\mathfrak{A}}$. Then $m(\bar{z}_i) = \overline{m(z_i)}$ for $i = 1,\ldots,n$.

Proof. If x_i and y_i are respectively the real and imaginary parts of the coordinate function z_i, we have

$$m(\bar{z}_i) = m(x_i) - im(y_i), \overline{m(z_i)} = \overline{m(x_i)+im(y_i)}.$$

All we need show now is that $m(x_i)$ and $m(y_i)$ are real. We know that the spectrum of x_i as an element of $C(\bar{B}^{2n})$ is $[0,1]$ and that its spectrum as an element of $\mathfrak{B}(B^{2n})$ is $[0,1]$ as well. Thus $\sigma_{\mathfrak{A}}(x_i) = [0,1]$ and similarly $\sigma_{\mathfrak{A}}(y_i) = [0,1]$. Since $\mathfrak{A}$ is a commutative Banach algebra the range of the Gelfand transform of any element is equal to the spectrum of that element ([5; p. 45]), so both $m(x_i)$ and $m(y_i)$ are in $[0,1]$ and hence real. ∎

Proposition 5.8. $M_{\mathfrak{A}}$ is naturally homeomorphic to $M_{H^\infty(B^{2n})}$.

Proof. Define the map $\eta : M_{\mathfrak{A}} \to M_{H^\infty(B^{2n})}$ by

$$\eta(m) = m|H^\infty(B^{2n}).$$

This map is of course well-define and continuous. Since the spaces involved are compact Hausdorff spaces, to show η is a homeomorphism all we need show is that it is one-one and onto.

If ζ is in B^{2n} then e_ζ, evaluation at ζ, is an element of $M_{\mathfrak{A}}$, and of course $\eta(e_\zeta) = \zeta$, and so

$$\eta(M_{\mathfrak{A}}) \supset B^{2n}.$$

Now if m belongs to $M_{H^\infty(B^{2n})} \backslash B^{2n}$ then m is in F_λ for some λ on ∂B^{2n} and hence by 5.6 m is in the range of η and so η is onto.

To show η is one-to-one let us suppose m_1, m_2 are in $M_{\mathfrak{A}}$ and $\eta(m_1) = \eta(m_2)$, i.e., $m_1|H^\infty(B^{2n}) = m_2|H^\infty(B^{2n})$. By the previous lemma $m_1(\bar{z}_i) = m_2(\bar{z}_i)$. Thus by the Stone-Weierstrass theorem $m_1|C(\bar{B}^{2n}) = m_2|C(\bar{B}^{2n})$, and hence $m_1 = m_2$. ◻

Using the map η^{-1} we can identify B^{2n} and $M_{H^\infty(B^{2n})} \backslash B^{2n}$?? as disjoint subsets of $M_{\mathfrak{A}}$ and of course

$$M_{\mathfrak{A}} = M_{H^\infty(B^{2n})} \backslash B^{2n} \cup B^{2n}.$$

If m belongs to $\eta^{-1}(M_{H^\infty(B^{2n})})$ then $m|H^\infty(B^{2n})$ is in F_λ for some λ on ∂B^{2n}, and so by 5.7, $m|C(\bar{B}^{2n}) = e_\lambda$. If m belongs to $\eta^{-1}(B^{2n})$ then $m|H^\infty(B^{2n}) = e_\zeta$ for some ζ in B^{2n} and hence $m|C(\bar{B}^{2n}) = e_\zeta$.

Theorem 5.9. The maximal ideal space $M_{\mathfrak{A}/\mathcal{J}_0}$ is naturally homeomorphic to $M_{H^\infty(B^{2n})} \backslash B^{2n}$.

Proof. Define the map $\mu : M_{H^\infty(B^{2n})} \backslash B^{2n} \to M_{\mathfrak{A}\backslash\mathcal{J}_0}$ by

$$\mu(m)(f+\mathcal{J}_0) = m(f),$$

for f in $\mathfrak{A}$. From the preceeding remark we see that

$$M_{H^\infty(B^{2n})} \backslash B^{2n} = \{m \in M_{\mathfrak{A}} : m(\varphi) = 0 \text{ for all } \varphi \text{ in } \mathcal{J}_0\}.$$

That μ is a homeomorphism now follows from a standard result on the maximal ideal space of quotient algebras [8; p. 13]. ∎

Thus we have identified $M_{\mathfrak{J}(\mathfrak{A})/\mathcal{K}}$ and $M_{H^\infty(B^{2n})} \backslash B^{2n}$. One can see immediately that if m is in $M_{H^\infty(B^{2n})} \backslash B^{2n}$, and $\tilde{m}$ is the element in $M_{\mathfrak{J}(\mathfrak{A})/\mathcal{K}}$ with which it is identified then

$$\tilde{m}(T_\varphi + \mathcal{K}) = m(\varphi),$$

for φ in $H^\infty(B^{2n})$, and

$$\tilde{m}(T_f + \mathcal{K}) = f(\lambda),$$

for f in $C(\bar{B}^{2n})$ and m in F_λ.

Proposition 5.10. The maximal ideal space $M_{\mathfrak{J}(\mathfrak{A})/\mathcal{K}}$ is connected.

Proof. Due to the fact that $M_{\mathfrak{J}(\mathfrak{A})/\mathcal{K}} = M_{H^\infty(B^{2n})} \backslash B^{2n}$, it suffices to show that the latter space is connected. This fact is well-known for $n = 1$ (see [10]), and the proof of the general result is exactly the same. We only outline the proof. Since $p(M_{H^\infty(B^{2n})} \backslash B^{2n}) = \partial B^{2n}$ and ∂B^{2n} is connected, a standard argument ([10; p. 188]) reduces the proof to showing that each F_λ is connected.

Recall that for f in $H^\infty(B^{2n})$, the Gelfand transform of f, denoted $\hat{f}$, is the continuous function on $M_{H^\infty(B^{2n})}$ defined by $\hat{f}(m) = m(f)$. Denote by $\hat{H}^\infty_\lambda$ the algebra obtained by restricting each $\hat{f}$ to F_λ. Thus $\hat{H}^\infty_\lambda$ is a subalgebra of $C(F_\lambda)$. Since F_λ is a peak set for $\hat{H}^\infty = \{\hat{f} : f \in H^\infty(B^{2n})\}$ (use the function h defined in Lemma 5.4), we know that $\hat{H}^\infty_\lambda$ is a function algebra (see Gamelin [8; p. 57] for details), and hence

$$M_{\hat{H}^\infty_\lambda} = F_\lambda.$$

Since F_λ is a maximal ideal space, F_λ will be connected if $\hat{H}^\infty_\lambda$ contains no non-trivial idempotents, by Shilov's theorem [8; p. 88]). That $\hat{H}^\infty_\lambda$ contains no idempotents follows from Lemma 5.4, for if $\hat{g}$ were an idempotent in $\hat{H}^\infty_\lambda$ we would have

$$\hat{g}^2 - \hat{g} \equiv 0$$

on F_λ. Thus by 5.4, g must be zero or one on small neighborhoods of λ in B^{2n}. Since g is continuous it must be identically one or the other on some neighborhood of λ in B^{2n} and so, again by 5.4, $\hat{g} \equiv 0$ or $\hat{g} \equiv 1$. ∎

CHAPTER VI. THE MATRIX CASE.

In this chapter we consider the Fredholmness of matrices of Toeplitz operators with symbols in $\mathfrak{A}$. We write

$$H_k^2(B^{2n}) = \sum_k \oplus H^2(B^{2n}), \quad L_k^2(B^{2n}) = \sum_k \oplus L^2(B^{2n}).$$

Thus $H_k^2(B^{2n})$ is a closed subspace of the Hilbert space $L_k^2(B^{2n})$. Let $\mathfrak{A}_k$ be the collection of $k\times k$ matrices (φ_{ij}) where each φ_{ij} is in $\mathfrak{A}$. This collection is a Banach algebra and can be identified with $\mathfrak{A} \otimes M_k$, where M_k is the collection of $k\times k$ matrices with complex entries. If P_k denotes the orthogonal projection from $L_k^2(B^{2n})$ onto $H_k^2(B^{2n})$ then we can define the Toeplitz operator T_φ on $H_k^2(B^{2n})$ by

$$T_\varphi f = P_k(\varphi f)$$

for $\varphi = (\varphi_{ij})$ in $\mathfrak{A}_k$. Obviously T_φ can be written as the $k\times k$ matrix operator $(T_{\varphi_{ij}})$

Let $\mathcal{K}$ denote the ideal of compact operators in $\mathcal{L}(H_k^2(B^{2n}))$. The elements of $\mathcal{K}$ are precisely those operator matrices of the form (K_{ij}) where each K_{ij} is a compact operator on $H^2(B^{2n})$. Thus the following lemma follows immediately from Lemma 3.3.

Lemma 6.1. If φ and ψ are in $\mathfrak{A}_k$ then $T_\varphi T_\psi - T_{\varphi\psi}$ is compact.

Proposition 6.2. If φ belongs to $\mathfrak{A}_k$ then T_φ is Fredholm if and only if $\det \varphi$ is bounded away from zero on a neighborhood of ∂B^{2n}.

Proof. Since det $\varphi(\zeta) = \det(\varphi_{ij}(\zeta))$, we see that det φ is an element of $\mathfrak{A}$. For the remainder of this proof we will write d for det φ.

Suppose d is bounded away from zero on a neighborhood of ∂B^{2n}. Then by Lemma 4.2 there exists a g in $\mathfrak{A}$ such that $gd \equiv 1$ on ∂B^{2n}. By Cramer's rule we know that if A belongs to M_k then there exists a B in M_k whose entries are polynomials in the entries of A and such that

$$AB = BA = \det A \cdot I,$$

where I is the identity matrix. Thus we can find a ψ in $\mathfrak{A}_k$ such that

$$\varphi\psi = \psi\varphi = d\cdot I.$$

We write $g\psi = (g\cdot I)\psi = (g\psi_{ij})$. By Lemma 6.1 we have

$$\begin{aligned} T_\varphi T_{g\psi} &= T_{\varphi(g\psi)} + K \\ &= T_{(gd\cdot I)} + K, \end{aligned}$$

for some compact K, and hence

$$T_{(gd\cdot I)} - I = \begin{pmatrix} T_{gd-1} & & 0 \\ & \ddots & \\ 0 & & T_{gd-1} \end{pmatrix}.$$

Since T_{gd-1} is a compact operator on $H^2(B^{2n})$ by Proposition 1.7 it follows that $T_{(gd\cdot I)} - I$ is a compact operator on $H^2_k(B^{2n})$ and so

$$(T_\varphi + \mathcal{K})(T_{g\psi} + \mathcal{K}) = I + \mathcal{K}.$$

Similarly

$$(T_{g\psi}+\mathcal{K})(T_{\varphi}+\mathcal{K}) = I+\mathcal{K},$$

and so T_φ is Fredholm.

On the other hand let us suppose d is not bounded away from zero on a neighborhood of ∂B^{2n}. Let $\psi = \varphi^*$, the adjoint of φ. Thus det ψ is not bounded away from zero on a neighborhood of ∂B^{2n}, and if $\psi = (\psi_{ij})$ then $\overline{\psi}_{ij} \in \mathfrak{A}$ for each i and j. We can find a sequence $\{\lambda_m\}$ in B^{2n} such that λ_m converges to some λ on the boundary of B^{2n} and such that det $\psi(\lambda_m) \to 0$. By passing to a subsequence if necessary we can assume that there are a_{ij} in $\mathbb{C}$ such that

$$\psi_{ij}(\lambda_m) \to a_{ij},$$

for all i,j. Let $a = (a_{ij})$. Since det is continuous we have det $a = 0$. Therefore there is a nonzero vector $(\zeta_1,\ldots,\zeta_k)$ in $\mathbb{C}^k$ such that $a(\zeta_1,\ldots,\zeta_k) = 0$, and so for each m

$$a(\zeta_1 f_m,\ldots,\zeta_k f_m) = 0$$

where $\{f_m\}$ is as defined in 3.11. Thus we have

$$\|T_\psi(\zeta_1 f_m,\ldots,\zeta_k f_m)\| = \|(T_\psi - a)(\zeta_1 f_m,\ldots,\zeta_k f_m)\|$$

$$= \left\| \begin{pmatrix} \sum_{\mu=1}^{k} (T_{\psi_{1\mu}} - a_{1\mu})\zeta_\mu f_m \\ \vdots \\ \sum_{\mu=1}^{k} (T_{\psi_{k\mu}} - a_{k\mu})\zeta_\mu f_m \end{pmatrix} \right\|$$

$$\leq \sum_{\mu} |\zeta_\mu| \left\| \begin{matrix} (T_{\psi_{1\mu}} - a_{1\mu})f_m \\ \vdots \\ (T_{\psi_{k\mu}} - a_{k\mu})f_m \end{matrix} \right\|,$$

where the norms are all the norm on $H^2_k(B^{2n})$. It follows now from the proof of 3.16 that $\|T_\psi(\zeta_1 f_m, \ldots, \zeta_k f_m)\|$ converges to zero. Since

$$\|(\zeta_1 f_m, \ldots, \zeta_k f_m)\| = |(\zeta_1, \ldots, \zeta_k)|,$$

(remember f_m is a unit vector in $H^2(B^{2n})$), we see that 0 is in π-ess $\sigma(T_\psi)$. Therefore since $T_\psi = (T_\varphi)^*$, it follows that T_φ is not Fredholm. □

We have not been able to find any method for computing the index of T_φ if it is Fredholm. Venugopalkrishna [17] has obtained an index result for φ in $C(\overline{B}^{2n}) \otimes M_k$ where $k \geq n$. If T_φ is Fredholm in this case then $\varphi|\partial B^{2n}$ determines an element of the homotopy group $\pi_{2n-1}(GL(k,\mathbb{C}))$. Since $k \geq n$, $\pi_{2n-1}(GL(k,C))$ is isomorphic to $\mathbb{Z}$ by the Bott periodicity theorem, and Venugopalkrishna shows that the index of T_φ is the integer given by the Bott isomorphism (see also [7; p. 81]).

In this case we conjecture that for φ in $\mathfrak{A}_k$ with T_φ Fredholm and φ bounded away from zero on $B^{2n}\backslash B_\varepsilon$, then

$$j(T_\varphi) = j(T_{\varphi_\varepsilon})$$

where

$$\varphi_\varepsilon(\zeta) = \varphi(\tfrac{1}{\varepsilon}\zeta).$$

Thus φ_ε is continuous on $\overline{B}^{2n}$ and nonzero on the boundary ∂B^{2n} and so T_{φ_ε} is Fredholm with an index which can be computed.

Further results on the index of Toeplitz operators with continuous matrix-valued symbols can be found in [12].

CHAPTER VII. CONCLUSION.

We would like to make a few general remarks before ending this paper. The algebra $\mathfrak{A}$ can of course be defined on any strongly pseudoconvex domain D. Can our results be extended to the arbitrary case? For any strongly pseudoconvex domain D it is true that

$$1) \quad T_\varphi T_\psi = T_{\varphi\psi}, \ \varphi \in L^\infty(B^{2n}), \ \psi \in \mathfrak{A},$$

and

$$2) \quad \varphi \in \mathcal{J}_0 \Rightarrow T_\varphi \ \text{compact}.$$

Thus if ψ is bounded away from zero on a neighborhood of ∂D then T_ψ is left Fredholm for ψ in $\mathfrak{A}$. If $\psi+\mathcal{J}_0$ were invertible in $\mathfrak{A}/\mathcal{J}_0$, then of course T_ψ would be Fredholm. So we now want to know whether ψ bounded away from zero on a neighborhood of the boundary of a strongly pseudoconvex domain implies $\psi+\mathcal{J}_0$ is invertible in $\mathfrak{A}/\mathcal{J}_0$. If all the results in Chapter V were correct for the domain the answer would be yes. Our proof of 5.1 depended on B^{2n} being star-shaped with respect to any of its points. But we really only use this result to show that fibers in $M_{\mathfrak{A}}$ over B^{2n} are trivial. Since these fibers are eliminated in $M_{\mathfrak{A}/\mathcal{J}_0}$, it doesn't make much difference whether they were trivial or not from the point of view of $M_{\mathfrak{A}/\mathcal{J}_0}$. We have used several other facts in V which might not be true for arbitrary D but will be true for domains with additional convexity requirements, for example, polynomial convexity. The problem here is that we may add too many little requirements and end up with results that only apply to domains more or less like the ball. For instance, proofs of the results of V which wind up excluding multiply connected domains ought to be avoided and so one would have to check that the convexity requirements had not eliminated such domains.

Now suppose ψ in $\mathfrak{A}$ is not bounded away from zero on any neighborhood of the boundary ∂D. Does this imply T_ψ is not Fredholm? This question, like the index question, appears much more difficult than the preceeding one and has not been answered fully even for ψ belonging to $C(\bar{D})$. Our method of solving it on the ball depends on the explicit determination of the Bergman kernel for $H^2(B^{2n})$. As more is learned about these kernels we should be able to tell whether our methods can be generalized here as well.

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